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AMBARZUMYAN–TYPE THEOREMS ON STAR GRAPHS

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Abstract. The so-called Ambarzumyan theorem states that if the Neumann eigenvalues of the Sturm-Liouville operator $-\frac{d^2}{dx^2}+q$ with an integrable real-valued potential q on $[0,\pi]$ are $\{n^2 : n \geqslant 0\}$, then $q=0$ for almost all $x\in[0,\pi]$. In this work, the classical Ambarzumyan theorem is extended to star graphs with Dirac operators on its edges. We prove that if the spectrum of Dirac operator on star graphs coincides with for the unperturbed case, then the potential is identically zero.

1. Introduction

Quantum graphs have proved an important model in the study of semiclassical systems whose classical analogues are chaotic [16, 17]. Recently, there has been increasing interest in spectral theory of differential operator on graphs [2, 5, 10, 13, 15, 16, 17, 18, 20, 27]. The Dirac operator on a graph was considered previously, e.g., by Bulla and Trenkler [5] as an alternative model of a simple scattering system. Self-adjoint realizations of the Dirac operator on graphs were considered by Bulla and Trenkler [5], and Bolte and Harrison [2].

Ambarzumyan [1] in 1929 proved that for an integrable real-valued potential q on $[0,\pi]$ if the eigenvalues of a Sturm-Liouville differential expression

$$-y''(x)+q(x)y(x)=\lambda y(x) \text{ on } [0,\pi]$$

subject to the Neumann boundary conditions $y'(0)=y'(\pi)=0$ are equal to n^2 for $n=0,1,2,\cdots$, then $q(x)=0$ for almost all $x\in[0,\pi]$. This is an exceptional situation since in general additional information is needed in order to reconstruct the potential $q(x)$ uniquely [3, 4, 21]. Since then there have appeared many generalizations of the result of Ambarzumyan in various directions [1, 6, 7, 8, 9, 11, 12, 14, 19, 23, 24, 25, 26]. The main aim of this work is to prove several generalizations of the well-known Ambarzumyan theorem for Dirac operators on star graphs. New Ambarzumyan-type results for Dirac operators on star graphs are obtained by applying the method of [25].

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2. Main Results

In this work, we consider Dirac systems on star-shaped graphs consisting of d segments of equal length π, $d \geqslant 2$:

$$L_j y_j \overset{def}{=} \left\{ B_0 \frac{d}{dx} + V_j(x) + \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix} \right\} y_j = \lambda y_j,\ y_j = \begin{pmatrix} y_{j,1} \\ y_{j,2} \end{pmatrix},\ j = \overline{1,d}. \quad (2.1)$$

Here λ is a spectral parameter, $m > 0$ is the mass of the described particle and

$$B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ V_j(x) = \begin{pmatrix} p_j(x) & q_j(x) \\ q_j(x) & -p_j(x) \end{pmatrix},$$

and $p_j(x)$ and $q_j(x)$ belong to $AC([0,\pi])$ (absolutely continuous functions on $[0,\pi]$).

Systems (2.1) are subject to the boundary conditions

$$y_{j,1}(0) = 0, \quad j = \overline{1,d} \quad (2.2)$$

or

$$y_{j,2}(0) = 0, \quad j = \overline{1,d} \quad (2.3)$$

at the pendant vertices 0, and

$$y_{1,1}(\pi) = y_{2,1}(\pi) = \cdots = y_{d,1}(\pi), \quad (2.4)$$

$$\sum_{j=1}^{d} y_{j,2}(\pi) = 0 \quad (2.5)$$

at the central vertex π.

Equations (2.1), (2.2), (2.4) and (2.5) or (2.1), (2.3), (2.4) and (2.5) can be rewritten in the following form

$$LY \overset{def}{=} \left\{ B \frac{d}{dx} + V(x) + m \begin{pmatrix} I_d & 0 \\ 0 & -I_d \end{pmatrix} \right\} Y = \lambda Y,\ Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \quad (2.6)$$

subject to the boundary conditions

$$Y_1(0) = 0,\ A_1 Y_1(\pi) + A_2 Y_2(\pi) = 0 \quad (2.7)$$

or

$$Y_2(0) = 0,\ A_1 Y_1(\pi) + A_2 Y_2(\pi) = 0, \quad (2.8)$$

where the values of $Y_1 = (y_{1,1}, \cdots, y_{d,1})^t$ and $Y_2 = (y_{1,2}, \cdots, y_{d,2})^t$ are vectors of length d, and A^t denotes the transpose of a matrix A, B and $V(x)$ are given by

$$B = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix},\quad V(x) = \begin{pmatrix} P(x) & Q(x) \\ Q(x) & -P(x) \end{pmatrix},$$

where

$$P(x) = \mathrm{diag}[p_1(x), p_2(x), \cdots, p_d(x)], \quad Q(x) = \mathrm{diag}[q_1(x), q_2(x), \cdots, q_d(x)]$$

are matrix-valued functions, and, finally, A_1 and A_2 are the following $d \times d$ matrices:

$$A_1 = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 1 & \cdots & 1 \end{pmatrix}. \tag{2.9}$$

For convenience, we denote by $T_1(V), T_2(V)$ the operators acting in Hilbert space $H = \overset{d}{\underset{1}{\oplus}} L_2(0,\pi)$ for the problem (2.6)*,* (2.7) *or* (2.6)*,* (2.8)*, respectively.* We notice that the domain of an operator $T_j(V)$, $j = 1,2$, is

$$D(T_j(V)) = \left\{ \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \mid Y_j(0) = 0,\ A_1Y_1(\pi) + A_2Y_2(\pi) = 0,\ Y_1, Y_2 \in W_2^1[0,\pi] \right\}.$$

We denote by $W_2^1[0,\pi]$ the space of vector-valued functions $F(x), x \in [0,\pi]$, such that the vector-valued function $F(x)$ is absolutely continuous and $F'(x) \in \overset{d}{\underset{1}{\oplus}} L_2(0,\pi)$. It is easy to verify that the operators $T_1(V)$ and $T_2(V)$ are both self-adjoint, and have real discrete spectra [2]. Denote by $\sigma(T_i(V))$ the spectrum of self-adjoint operator $T_i(V)$, $i = 1,2$, respectively.

In the case $V(x) = 0$ in (2.6), we can calculate the eigenvalues of the operators $T_1(0)$ and $T_2(0)$. The spectrum $\sigma(T_1(0))$ of the operator $T_1(0)$ consists of

$$\begin{aligned} \lambda_n &= \sqrt{m^2 + (n + (1/2))^2}, \quad \sqrt{m^2 + n^2}, \\ \lambda_{-n} &= -\sqrt{m^2 + (n + (1/2))^2}, \quad -\sqrt{m^2 + n^2}, \quad n \in \mathbf{N} \bigcup \{0\}. \end{aligned} \tag{2.10}$$

Each of the eigenvalues $\pm\sqrt{m^2 + (n + (1/2))^2}$ is simple, and $\pm\sqrt{m^2 + n^2}$ is of multiplicity $d - 1$.

The spectrum $\sigma(T_2(0))$ of the operator $T_2(0)$ consists of

$$\begin{aligned} \lambda_n &= \sqrt{m^2 + (n + (1/2))^2}, \quad \sqrt{m^2 + n^2}, \\ \lambda_{-n} &= -\sqrt{m^2 + (n + (1/2))^2}, \quad -\sqrt{m^2 + n^2}, \quad n \in \mathbf{N} \bigcup \{0\}. \end{aligned} \tag{2.11}$$

Each of the eigenvalues $\pm\sqrt{m^2 + n^2}$ is simple, and each of the eigenvalues $\pm\sqrt{m^2 + (n + (1/2))^2}$ is of multiplicity $d - 1$.

In this work we obtain the following results.

THEOREM 2.1. *Let $tr \int_0^\pi P(x)dx = 0$, and $Q(0) = Q(\pi) = 0$. If $\sigma(T_1(V)) = \sigma(T_1(0))$ (including multiplicities), then $V(x) = 0$ on $[0,\pi]$.*

THEOREM 2.2. *Let $tr \int_0^\pi P(x)dx = 0$, and $Q(0) = Q(\pi) = 0$. If $\sigma(T_2(V)) = \sigma(T_2(0))$ (including multiplicities), then $V(x) = 0$ on $[0,\pi]$.*

3. Proofs

In the proofs of the main results, we shall use the second power L^2 of the Dirac operator L, which is a $2d$-dimensional vectorial Sturm-Liouville operator generated by the differential expression

$$-Y'' + \Omega(x)Y,\ x \in (0,\pi). \tag{3.1}$$

Here $\Omega(x)$ is a $2d \times 2d$ self-adjoint matrix of the form

$$\Omega(x) = \begin{pmatrix} (P(x)+m)^2 + Q^2(x) + Q'(x) & -P'(x) \\ -P'(x) & (P(x)+m)^2 + Q^2(x) - Q'(x) \end{pmatrix}. \tag{3.2}$$

Let $\Phi_1(x,\lambda)$ satisfy the matrix differential equation

$$\begin{cases} -Y'' + \Omega(x)Y = \lambda Y \\ Y(0) = I_{2d},\ Y'(0) = 0, \end{cases} \tag{3.3}$$

then, by [9, 22], the solution $\Phi_1(x,\lambda)$ can be expressed as

$$\Phi_1(x,\lambda) = \cos(\sqrt{\lambda}x)I_{2d} + \int_0^x K(x,t)\cos(\sqrt{\lambda}t)dt, \tag{3.4}$$

where $K(x,t)$ is a symmetric matrix-valued function whose entries are continuously differentiable in both of its variables.

Similarly, let $\Phi_2(x,\lambda)$ satisfy the matrix differential equation

$$\begin{cases} -Y'' + \Omega(x)Y = \lambda Y \\ Y(0) = 0,\ Y'(0) = I_{2d}, \end{cases} \tag{3.5}$$

then the solution $\Phi_2(x,\lambda)$ can be expressed as

$$\Phi_2(x,\lambda) = \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}I_{2d} + \int_0^x L(x,t)\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}dt, \tag{3.6}$$

where $L(x,t)$ is a symmetric matrix-valued function whose entries are continuously differentiable in both of its variables. Furthermore, the kernels $K(x,t)$, $L(x,t)$ satisfy [9, 22]

$$\begin{aligned} K(x,x) &= \tfrac{1}{2}\textstyle\int_0^x \Omega(x)dx,\ K_t'(x,0) = 0; \\ L(x,x) &= \tfrac{1}{2}\textstyle\int_0^x \Omega(x)dx,\ L(x,0) = 0. \end{aligned} \tag{3.7}$$

Thus $(\Phi_1(x,\lambda),\Phi_2(x,\lambda))$ is the fundamental matrix of solutions of the equation $-Y'' + \Omega(x)Y = \lambda Y$ [25] and

$$\Phi_1(\pi,\lambda) = \cos(\sqrt{\lambda}\pi)I_{2d} + K(\pi,\pi)\frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}} - \frac{1}{\sqrt{\lambda}}\int_0^\pi K_t'(\pi,t)\sin(\sqrt{\lambda}t)dt, \tag{3.8}$$

and

$$\Phi_1'(\pi,\lambda) = -\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)I_{2d} + K(\pi,\pi)\cos(\sqrt{\lambda}\pi) + \int_0^\pi K_x'(\pi,t)\cos(\sqrt{\lambda}t)dt. \quad (3.9)$$

Similarly,

$$\Phi_2(\pi,\lambda) = \frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}I_{2d} - \frac{\cos(\sqrt{\lambda}\pi)}{\lambda}L(\pi,\pi) + \frac{1}{\lambda}\int_0^\pi L_t'(\pi,t)\cos(\sqrt{\lambda}t)dt, \quad (3.10)$$

and

$$\Phi_2'(\pi,\lambda) = \cos(\sqrt{\lambda}\pi)I_{2d} + \frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}L(\pi,\pi) + \int_0^\pi L_x'(\pi,t)\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}dt. \quad (3.11)$$

Here

$$K(\pi,\pi) = L(\pi,\pi) = \frac{1}{2}\int_0^\pi \Omega(x)dx \overset{def}{=} \begin{pmatrix} K_1 & P(0)-P(\pi) \\ P(0)-P(\pi) & K_1 \end{pmatrix}, \quad (3.12)$$

where the $d\times d$ matrix K_1 satisfies

$$\begin{aligned} K_1 &= \tfrac{1}{2}\textstyle\int_0^\pi [(P(x)+m)^2 + Q^2(x)]dx \\ &= \tfrac{1}{2}\mathrm{diag}[\textstyle\int_0^\pi((p_1(x)+m)^2+q_1^2(x))dx, \cdots, \int_0^\pi((p_d(x)+m)^2+q_d^2(x))dx] \\ &\overset{def}{=} \mathrm{diag}\,[k_1,\cdots,k_d]. \end{aligned} \quad (3.13)$$

Denote the $2d\times 2d$ matrices

$$\Phi_i(x,\lambda) \overset{def}{=} \begin{pmatrix} \Phi_{i1}(x,\lambda) & \Phi_{i2}(x,\lambda) \\ \Phi_{i3}(x,\lambda) & \Phi_{i4}(x,\lambda) \end{pmatrix}, \quad i=1,2$$

and

$$C_i = (c_{(i-1)d+1},\cdots,c_{(i-1)d+d})^t, \quad i=1,2,3,4.$$

Therefore, general solutions of the equation $-Y''+\Omega(x)Y=\lambda Y$ have the form

$$\begin{aligned} Y(x,\lambda) &= (\Phi_1(x,\lambda),\Phi_2(x,\lambda))C \\ &= \begin{pmatrix} \Phi_{11}(x,\lambda)C_1+\Phi_{12}(x,\lambda)C_2+\Phi_{21}(x,\lambda)C_3+\Phi_{22}(x,\lambda)C_4 \\ \Phi_{13}(x,\lambda)C_1+\Phi_{14}(x,\lambda)C_2+\Phi_{23}(x,\lambda)C_3+\Phi_{24}(x,\lambda)C_4 \end{pmatrix}, \end{aligned} \quad (3.14)$$

where $C=(c_1,c_2,\cdots,c_{4d})^t$, $c_k\in\mathbf{C}$, $k=\overline{1,4d}$.

Now we can prove theorems in this work.

The proof of Theorem 2.1. First, we study the second power operator $T_1^2(V)$ of Dirac operator $T_1(V)$ defined by (2.6) and (2.7), where the domain

$$D(T_1^2(V)) = \{Y \mid Y\in D(T_1(V)),\ T_1(V)(Y)\in D(T_1(V))\}.$$

By the assumption $Q(0)=Q(\pi)=0$ and a direct calculation, the operator $T_1^2(V)$ can be rewritten as

$$T_1^2(V)Y=-Y''+\Omega(x)Y=\lambda Y,\quad Y=\begin{pmatrix}Y_1\\Y_2\end{pmatrix},\ \lambda\in\mathbf{C},\tag{3.15}$$

subject to the boundary conditions (i.e., the domain $D(T_1^2(V))$)

$$\begin{cases}Y_1(0)=Y_2'(0)=0\\ A_1Y_1(\pi)+A_2Y_2(\pi)=0,\ \ A_1Y_2'(\pi)-A_2Y_1'(\pi)=0.\end{cases}\tag{3.16}$$

Substituting (3.14) into (3.16), we have

$$\begin{cases}C_1=0\\ C_4=0\\ (A_1\Phi_{11}(\pi,\lambda)+A_2\Phi_{13}(\pi,\lambda))C_1+(A_1\Phi_{12}(\pi,\lambda)+A_2\Phi_{14}(\pi,\lambda))C_2\\ \qquad +(A_1\Phi_{21}(\pi,\lambda)+A_2\Phi_{23}(\pi,\lambda))C_3+(A_1\Phi_{22}(\pi,\lambda)+A_2\Phi_{24}(\pi,\lambda))C_4=0\\ (A_1\Phi_{13}'(\pi,\lambda)-A_2\Phi_{11}'(\pi,\lambda))C_1+(A_1\Phi_{14}'(\pi,\lambda)-A_2\Phi_{12}'(\pi,\lambda))C_2\\ \qquad +(A_1\Phi_{23}'(\pi,\lambda)-A_2\Phi_{21}'(\pi,\lambda))C_3+(A_1\Phi_{24}'(\pi,\lambda)-A_2\Phi_{22}'(\pi,\lambda))C_4=0.\end{cases}\tag{3.17}$$

Denote the matrix

$$W_1(\lambda)=\begin{pmatrix}I_d&0&0&0\\0&I_d&0&0\\S_1(\lambda)&S_2(\lambda)&S_3(\lambda)&S_4(\lambda)\\S_5(\lambda)&S_6(\lambda)&S_7(\lambda)&S_8(\lambda)\end{pmatrix},\tag{3.18}$$

where

$$\begin{aligned}&S_1(\lambda)=A_1\Phi_{11}(\pi,\lambda)+A_2\Phi_{13}(\pi,\lambda),\ \ S_2(\lambda)=A_1\Phi_{22}(\pi,\lambda)+A_2\Phi_{24}(\pi,\lambda),\\ &S_3(\lambda)=A_1\Phi_{12}(\pi,\lambda)+A_2\Phi_{14}(\pi,\lambda),\ \ S_4(\lambda)=A_1\Phi_{21}(\pi,\lambda)+A_2\Phi_{23}(\pi,\lambda),\\ &S_5(\lambda)=A_1\Phi_{13}'(\pi,\lambda)-A_2\Phi_{11}'(\pi,\lambda),\ \ S_6(\lambda)=A_1\Phi_{24}'(\pi,\lambda)-A_2\Phi_{22}'(\pi,\lambda),\\ &S_7(\lambda)=A_1\Phi_{14}'(\pi,\lambda)-A_2\Phi_{12}'(\pi,\lambda),\ \ S_8(\lambda)=A_1\Phi_{23}'(\pi,\lambda)-A_2\Phi_{21}'(\pi,\lambda).\end{aligned}$$

From (3.17), we see that if $Y(x,\lambda)=(\Phi_1(x,\lambda),\Phi_2(x,\lambda))C$ is a nontrivial solution of the problem (3.15) and (3.16), then there exists a non-vanishing vector C satisfying the matrix equation

$$W_1(\lambda)C=0.$$

Therefore, λ is an eigenvalue of the operator $T_1^2(V)$ (the problem (3.15) and (3.16)) if and only if the matrix $W_1(\lambda)$ is singular. Furthermore, the multiplicity of λ is equal to $4d-\operatorname{rank}W_1(\lambda)$.

For $d\times d$ matrix A, denote by $(A)_d$ the last row vector in A and $(A)_{1:d-1}$ the $(d-1)\times d$ matrix consisting of the first $(d-1)$ rows of A.

The spectral mapping theorem for self-adjoint operators implies that the point spectrum $\sigma(T_1^2(V))$ of the operator $T_1^2(V)$ is the square $(\sigma(T_1(V)))^2$ of the point spectrum of an operator $T_1(V)$; moreover, the multiplicity of an eigenvalue λ^2 of $T_1^2(V)$ equals the sum of multiplicities of λ and $-\lambda$ as eigenvalues of $T_1(V)$.

Since by assumption $\{\pm\sqrt{m^2+n^2} : n=0,1,2,\cdots\} \subset \sigma(T_1(V))$, and each of the eigenvalues is of multiplicity $d-1$, by the spectral mapping theorem for self-adjoint operators we know that the sequence $\{m^2+n^2 : n=0,1,2,\cdots\}$ consists of the eigenvalues for the problem (3.15) and (3.16), and each of the eigenvalues is of multiplicity $2d-2$.

From (3.18) it follows that (for brevity we set $\lambda_n = \sqrt{m^2+n^2}$)

$$\operatorname{rank} W_1(\lambda_n^2) = 2d+2.$$

Using Riemann-Lebesgue Lemma, from (3.8)–(3.11), we obtain

$$\begin{aligned}
\cos(\lambda_n\pi) &= (-1)^n \cos\tfrac{m^2\pi}{\lambda_n+n} = (-1)^n + o(1/n),\\
\sin(\lambda_n\pi) &= (-1)^n \sin\tfrac{m^2\pi}{\lambda_n+n} = (-1)^n \tfrac{m^2\pi}{\lambda_n+n} + o(1/n^2),\\
\Phi_{14}(\pi,\lambda_n^2) &= \cos(\lambda_n\pi) I_d + K_4(\pi,\pi)\tfrac{\sin(\lambda_n\pi)}{\lambda_n} + o(1/n),\\
\Phi_{21}(\pi,\lambda_n^2) &= \tfrac{\sin(\lambda_n\pi)}{\lambda_n} I_d - \tfrac{\cos(\lambda_n\pi)}{\lambda_n^2} K_1 + o(1/n^2)\\
&= (-1)^n \tfrac{m^2\pi}{\lambda_n(\lambda_n+n)} I_d + \tfrac{(-1)^{n+1}}{\lambda_n^2} K_1 + o(1/n^2),
\end{aligned} \tag{3.19}$$

which yield, together with (2.9) and (3.18),

$$\begin{aligned}
(S_3(\lambda_n^2))_d &= (A_1\Phi_{12}(\pi,\lambda_n^2) + A_2\Phi_{14}(\pi,\lambda_n^2))_d = (A_2\Phi_{14}(\pi,\lambda_n^2))_d\\
&= (\underbrace{(-1)^n + o(1/n),\cdots,(-1)^n + o(1/n)}_{d}),\\
(S_3(\lambda_n^2))_{1:d-1} &= O(1/n^2),\\
(S_4(\lambda_n^2))_{1:d-1} &= (A_1\Phi_{21}(\pi,\lambda_n^2) + A_2\Phi_{23}(\pi,\lambda_n^2))_{1:d-1} = (A_1\Phi_{21}(\pi,\lambda_n^2))_{1:d-1}\\
&= \left((-1)^n \tfrac{m^2\pi}{\lambda_n(\lambda_n+n)} A_1 + \tfrac{(-1)^{n+1}}{\lambda_n^2} A_1 K_1 + o(1/n^2)\right)_{1:d-1},\\
(S_4(\lambda_n^2))_d &= o(1/n),\\
S_7(\lambda_n^2) &= A_1\Phi_{14}'(\pi,\lambda_n^2) - A_2\Phi_{12}'(\pi,\lambda_n^2) = O(1),\\
(S_8(\lambda_n^2))_d &= (A_1\Phi_{23}'(\pi,\lambda_n^2) - A_2\Phi_{21}'(\pi,\lambda_n^2))_d = (-A_2\Phi_{21}'(\pi,\lambda_n^2))_d\\
&= (\underbrace{(-1)^{n+1} + o(1/n),\cdots,(-1)^{n+1} + o(1/n)}_{d}),\\
(S_8(\lambda_n^2))_{1:d-1} &= o(1/n).
\end{aligned}$$

Denote

$$S \stackrel{def}{=} \begin{pmatrix} S_3(\lambda_n^2) & S_4(\lambda_n^2) \\ S_7(\lambda_n^2) & S_8(\lambda_n^2) \end{pmatrix} = \begin{pmatrix} (S_3(\lambda_n^2))_{1:d-1} & (S_4(\lambda_n^2))_{1:d-1} \\ (S_3(\lambda_n^2))_d & (S_4(\lambda_n^2))_d \\ (S_7(\lambda_n^2))_{1:d-1} & (S_8(\lambda_n^2))_{1:d-1} \\ (S_7(\lambda_n^2))_d & (S_8(\lambda_n^2))_d \end{pmatrix},$$

then

$$S = \begin{pmatrix} O(1/n^2) & \left(\frac{(-1)^n m^2\pi}{\lambda_n(\lambda_n+n)}A_1 + \frac{(-1)^{n+1}}{\lambda_n^2}A_1K_1 + o(\frac{1}{n^2})\right)_{1:d-1} \\ (-1)^n I_{1\times d} + o(1/n) & o(1/n) \\ O(1) & o(1/n) \\ O(1) & (-1)^{n+1}I_{1\times d} + o(1/n) \end{pmatrix},$$

where $I_{1\times d} = (\underbrace{1,\cdots,1}_{d})$.

The fact that $\operatorname{rank} W_1(\lambda_n^2) = 2d+2$ yields

$$\operatorname{rank} S = 2,$$

which implies that

$$\operatorname{rank} \begin{pmatrix} O(1) & \left(-\frac{m^2\pi}{2}A_1 + A_1K_1 + o(1)\right)_{1:d-1} \\ I_{1\times d} + o(1/n) & o(1/n) \\ O(1) & o(1/n) \\ O(1) & I_{1\times d} + o(1/n) \end{pmatrix} = 2,$$

i.e.,

$$\operatorname{rank} \begin{pmatrix} \left(-\frac{m^2\pi}{2}A_1 + A_1K_1 + o(1)\right)_{1:d-1} & O(1) \\ I_{1\times d} + o(1/n) & O(1) \\ o(1/n) & I_{1\times d} + o(1/n) \\ o(1/n) & O(1) \end{pmatrix} = 2.$$

From (3.13) and (2.9), we see that

$$\begin{pmatrix} \left(-\frac{m^2\pi}{2}A_1 + A_1K_1 + o(1)\right)_{1:d-1} \\ I_{1\times d} + o(1/n) \end{pmatrix}$$
$$= \begin{pmatrix} k_1 - \frac{m^2\pi}{2} & -k_2 + \frac{m^2\pi}{2} & 0 & \cdots & 0 & 0 \\ 0 & k_2 - \frac{m^2\pi}{2} & -k_3 + \frac{m^2\pi}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & k_{d-1} - \frac{m^2\pi}{2} & -k_d + \frac{m^2\pi}{2} \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix} + o(1).$$

Denote

$$\widehat{S} \overset{def}{=} \begin{pmatrix} \left(-\frac{m^2\pi}{2}A_1 + A_1K_1 + o(1)\right)_{1:d-1} & O(1) \\ I_{1\times d} + o(1/n) & O(1) \\ o(1/n) & I_{1\times d} + o(1/n) \\ o(1/n) & O(1) \end{pmatrix}.$$

Then rank $\widehat{S} = 2$ implies that for each 3×3 minor of $\widehat{S}$ vanishes. Thus, for all $n \in \mathbf{N}$ we obtain

$$\det \begin{pmatrix} k_1 - \frac{m^2\pi}{2} + o(1) & -k_2 + \frac{m^2\pi}{2} + o(1) & O(1) \\ 1 + o(1) & 1 + o(1) & O(1) \\ o(1/n) & o(1/n) & 1 + o(1/n) \end{pmatrix} = 0$$

and

$$\det \begin{pmatrix} k_1 - \frac{m^2\pi}{2} + o(1) & -k_2 + \frac{m^2\pi}{2} + o(1) & O(1) \\ o(1) & k_2 - \frac{m^2\pi}{2} + o(1) & O(1) \\ o(1/n) & o(1/n) & 1 + o(1/n) \end{pmatrix} = 0.$$

By calculation of two determinants above, we have for all $n \in \mathbf{N}$

$$k_1 + k_2 - m^2\pi + o(1) = 0 \quad \text{and} \quad \left(k_1 - \frac{m^2\pi}{2}\right)\left(k_2 - \frac{m^2\pi}{2}\right) + o(1) = 0,$$

letting $n \to \infty$, it yields that

$$k_1 = k_2 = \frac{m^2\pi}{2}. \tag{3.20}$$

Similarly, we obtain

$$k_j = \frac{m^2\pi}{2}, \quad j = \overline{1,d}, \tag{3.21}$$

therefore

$$K_1 = \frac{m^2\pi}{2} I_d.$$

From (3.13) it follows

$$\operatorname{diag}\left[\int_0^\pi ((p_1(x)+m)^2 + q_1^2(x))dx, \cdots, \int_0^\pi ((p_d(x)+m)^2 + q_d^2(x))dx\right] = m^2\pi I_d,$$

and from this, we obtain

$$\sum_{j=1}^d \int_0^\pi [(p_j(x)+m)^2 + q_j^2(x)]dx = m^2\pi d,$$

which implies, by the assumption $\operatorname{tr}\int_0^\pi P(x)dx = 0$,

$$\sum_{j=1}^{d} \int_0^\pi [p_j^2(x) + q_j^2(x)]dx = 0,$$

therefore,

$$p_j(x) = q_j(x) = 0, \quad j = \overline{1,d}.$$

The proof is finished. □

The proof of Theorem 2.2. The operator $T_2^2(V)$ can be rewritten as

$$T_2^2(V)Y = -Y'' + \Omega(x)Y = \lambda Y,\ \lambda \in \mathbf{C}, \tag{3.22}$$

subject to the boundary conditions

$$\begin{cases} Y_1'(0) = Y_2(0) = 0 \\ A_1Y_1(\pi) + A_2Y_2(\pi) = 0,\ \ A_1Y_2'(\pi) - A_2Y_1'(\pi) = 0. \end{cases} \tag{3.23}$$

Substituting (3.14) into (3.23), we have

$$W_2(\lambda)\,(C_1, C_2, C_3, C_4)^t = 0, \tag{3.24}$$

where the matrix

$$W_2(\lambda) = \begin{pmatrix} I_d & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ S_3(\lambda) & S_4(\lambda) & S_1(\lambda) & S_2(\lambda) \\ S_7(\lambda) & S_8(\lambda) & S_5(\lambda) & S_6(\lambda) \end{pmatrix}. \tag{3.25}$$

For brevity we set $\lambda_n = \sqrt{m^2 + (n + (1/2))^2}$, by assumption

$$\{\pm\sqrt{m^2 + (n + (1/2))^2} : n = 0, 1, 2, \cdots\} \subset \sigma(T_2(V)),$$

and each of the eigenvalues is of multiplicity $d-1$, from (3.25) it follows that

$$\operatorname{rank} W_2(\lambda_n^2) = 2d + 2.$$

A simple calculation implies

$$\begin{aligned} (S_1(\lambda_n^2))_{1:d-1} &= \left((-1)^{n+1}\tfrac{m^2\pi}{\lambda_n + n + (1/2)}A_1 + \tfrac{(-1)^n}{\lambda_n}A_1K_1 + o(1/n^2)\right)_{1:d-1}, \\ (S_1(\lambda_n^2))_d &= O(1/n), \\ (S_2(\lambda_n^2))_d &= \left(\tfrac{(-1)^n}{\lambda_n}A_2 + o(1/n)\right)_d, \\ (S_2(\lambda_n^2))_{1:d-1} &= ((-1)^{n+1}\lambda_n A_2 + o(1))_d, \\ (S_5(\lambda_n^2))_{1:d-1} &= O(1/n), \qquad S_6(\lambda_n^2) = O(1/n). \end{aligned}$$

Denote

$$T \overset{def}{=} \begin{pmatrix} S_1(\lambda_n^2) & S_2(\lambda_n^2) \\ S_5(\lambda_n^2) & S_6(\lambda_n^2) \end{pmatrix} = \begin{pmatrix} (S_1(\lambda_n^2))_{1:d-1} & (S_2(\lambda_n^2))_{1:d-1} \\ (S_1(\lambda_n^2))_d & (S_2(\lambda_n^2))_d \\ (S_5(\lambda_n^2))_{1:d-1} & (S_6(\lambda_n^2))_{1:d-1} \\ (S_5(\lambda_n^2))_d & (S_6(\lambda_n^2))_d \end{pmatrix},$$

then

$$T = \begin{pmatrix} \left(\frac{(-1)^{n+1}m^2\pi}{\lambda_n+n+(1/2)}A_1 + \frac{(-1)^n}{\lambda_n}A_1K_1 + o(\frac{1}{n^2})\right)_{1:d-1} & o(\frac{1}{n^2}) \\ O(1/n) & \frac{(-1)^n}{n}I_{1\times d} + o(\frac{1}{n}) \\ O(1/n) & O(1/n) \\ n(-1)^{n+1}I_{1\times d} + o(1) & O(1/n) \end{pmatrix}.$$

The fact that $\operatorname{rank} W_2(\lambda_n^2) = 2d+2$ yields

$$\operatorname{rank} T = 2,$$

which implies that

$$\operatorname{rank} \begin{pmatrix} \left(-\frac{m^2\pi}{2}A_1 + A_1K_1 + o(1)\right)_{1:d-1} & o(1/n) \\ O(1) & I_{1\times d} + o(1) \\ O(1) & O(1) \\ I_{1\times d} + o(1) & o(1) \end{pmatrix} = 2,$$

i.e.,

$$\operatorname{rank} \begin{pmatrix} \left(-\frac{m^2\pi}{2}A_1 + A_1K_1 + o(1)\right)_{1:d-1} & o(1/n) \\ I_{1\times d} + o(1) & o(1) \\ O(1) & I_{1\times d} + o(1) \\ O(1) & O(1) \end{pmatrix} = 2.$$

Denote

$$\widehat{T} \overset{def}{=} \begin{pmatrix} \left(-\frac{m^2\pi}{2}A_1 + A_1K_1 + o(1)\right)_{1:d-1} & o(1/n) \\ I_{1\times d} + o(1) & o(1) \\ O(1) & I_{1\times d} + o(1) \\ O(1) & O(1) \end{pmatrix}.$$

Then $\operatorname{rank} \widehat{T} = 2$ implies that for each 3×3 minor of $\widehat{T}$ vanishes. Thus, for all $n \in \mathbf{N}$ we obtain

$$\det \begin{pmatrix} k_1 - \frac{m^2\pi}{2} + o(1) & -k_2 + \frac{m^2\pi}{2} + o(1) & o(1) \\ 1 + o(1) & 1 + o(1) & o(1) \\ O(1) & O(1) & 1 + o(1) \end{pmatrix} = 0$$

and

$$\det\begin{pmatrix} k_1-\frac{m^2\pi}{2}+o(1) & -k_2+\frac{m^2\pi}{2}+o(1) & o(1) \\ o(1) & k_2-\frac{m^2\pi}{2}+o(1) & o(1) \\ O(1) & O(1)) & 1+o(1) \end{pmatrix}=0.$$

By calculation of two determinants above, we have

$$k_1=k_2=\frac{m^2\pi}{2}.$$

Similarly, we obtain

$$k_1=k_2=\cdots=k_d=\frac{m^2\pi}{2}.$$

Therefore, the proof is completed. $\square$

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REFERENCES

[1] V. A. AMBARZUMYAN, *Über eine Frage der Eigenwerttheorie*, Z. Phys., **53** (1929), 690–695.
[2] J. BOLTE AND J. HARRISON, *Spectral statistics for the Dirac operator on graphs*, J. Phy. A: Math. Gen., **36** (2003), 2747–2769.
[3] G. BORG, *Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe, Bestimmung der Differentialgleichung durch die Eigenwerte*, Acta Math., **78** (1946), 1–96.
[4] G. BORG, *Uniqueness theorems in the spectral theory of* $y''+(\lambda-q(x))y=0$, in Proc. 11th Scandinavian Congress of Mathematicians (Oslo: Johan Grundt Tanums Forlag), pp. 276–287, 1952.
[5] W. BULLA AND T. TRENKLER, *The free Dirac operator on compact and noncompact graphs*, J. Math. Phys., **31** (1990), 1157–1163.
[6] R. CARLSON AND V. N. PIVOVARCHIK, *Ambarzumian's theorem for trees*, Electronic J. Diff. Equa. Vol. 2007, **142** (2007), 1–9.
[7] N. K. CHAKRAVARTY AND S. K. ACHARYYA, *On an extension of the theorem of V. A. Ambarzumyan*, Proc. Roy. Soc. Edinb. **A**, **110** (1988), 79–84.
[8] H. H. CHERN, C. K. LAW AND H. J. WANG, *Extension of Ambarzumyan's theorem to general boundary conditions*, J. Math. Anal. Appl., **309** (2005), 764–768.(corrigendum).
[9] H. H. CHERN AND C. L. SHEN, *On the n-dimensional Ambarzumyan's theorem*, Inverse Problems, **13** (1997), 15–18.
[10] S. CURRIE AND B. WATSON, *The M-matrix inverse problem for the Sturm-Liouville equation on graphs*, Proc. Roy. Soc. Edinburgh Sect. A, **139**, 4 (2009), 775–796.
[11] E. M. HARRELL, *On the extension of Ambarzunyan's inverse spectral theorem to compact symmetric spaces*, Amer. J. Math., **109** (1987), 787–795.
[12] M. HORVÁTH, *On a theorem of Ambarzumyan*, Proc. Roy. Soc. Edinb. **A**, **131** (2001), 899–907.
[13] R. KENYON, *The Laplacian and Dirac operators on critical planar graphs*, Invent. Math., **150** (2002), 409–439.

[14] M. KISS, *An n-dimensional Ambarzumyan type theorem for Dirac operators*, Inverse Problems, **20** (2004), 1593–1597.

[15] V. KOSTRYKIN AND R. SCHRADER, *Kirchoff's rule for quantumn wires*, J. Phys. A: Math. Gen., **32** (1999), 595–630.

[16] T. KOTTOS AND U. SMILANSKY, *Quantum chaos on graphs*, Phys. Rev. Lett. , **79** (1997), 4794–4797.

[17] T. KOTTOS AND U. SMILANSKY, *Periodic orbit theory and spectral statistics for quantum graphs*, Ann. Phys., **247** (1999), 76–124.

[18] P. KUCHMENT, *Quantum graphs: an introduction and a brief survey*, pp. 291–314, in *Analysis on Graphs and its Applications*, Proc. Symp. Pure. Math., AMS 2008.

[19] N. V. KUZNETSOV, *Generalization of a theorem of V. A. Ambarzumian*, Dokl. Akad. Nauk SSSR, **146** (1962), 1259–1262 (in Russian).

[20] C. K. LAW AND V. N. PIVOVARCHIK, *Characteristic functions of quantum graphs*, J. Phys. A: Math. Theor., **42** (2009), 035302(11pp).

[21] B. M. LEVITAN AND M. G. GASYMOV, *Determination of a differential equation by two of its spectra*, Usp. Mat. Nauk, **19** (1964), 3–63.

[22] B. M. LEVITAN AND I. S. SARGSJAN, *Sturm-Liouville and Dirac Operators (Russian)*, Nauka, Moscow 1988: English transl., Kluwer, Dordrecht, 1991.

[23] V. N. PIVOVARCHIK, *Ambarzumyan's theorem for a Sturm-Liouville boundary value problem on a star-shaped graph*, Funct. Anal. Appl., **39** (2005), 148–151.

[24] C. L. SHEN, *On some inverse spectral problems related to the Ambarzumyan problem and the dual string of the string equation*, Inverse Problems, **23** (2007), 2417–2436.

[25] C. F. YANG AND Z. Y. HUANG, *Inverse spectral problems for $2m$-dimensional canonical Dirac operators*, Inverse Problems, **23** (2007), 2565–2574.

[26] C. F. YANG AND X. P. YANG, *Some Ambarzumyan-type theorems for Dirac operators*, Inverse Problems, **25** (2009), 095012(13pp).

[27] V. A. YURKO, *Inverse spectral problems for Sturm-Liouville operators on graphs*, Inverse Problems, **21** (2005), 1075–1086.

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