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RELATIVELY SPECTRAL HOMOMORPHISMS AND K–INJECTIVITY

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Abstract. Let $\mathscr{A}$ and $\mathscr{B}$ be unital Banach algebras and $\phi\colon \mathscr{A} \to \mathscr{B}$ be a unital continuous homomorphism. We prove that if ϕ is relatively spectral (i.e., there is a dense subalgebra X of $\mathscr{A}$ such that $\mathrm{sp}_{\mathscr{B}}(\phi(a)) = \mathrm{sp}_{\mathscr{A}}(a)$ for every $a \in X$) and has dense range, then ϕ induces monomorphisms from $K_i(\mathscr{A})$ to $K_i(\mathscr{B})$, $i = 0,1$.

1. Introduction and Preliminaries

Let $\mathscr{A}$, $\mathscr{B}$ be unital Banach algebras and $\phi\colon \mathscr{A} \to \mathscr{B}$ be a unital homomorphism (i.e., $\phi(1) = 1$). If $\mathrm{sp}_{\mathscr{B}}(\phi(a)) = \mathrm{sp}_{\mathscr{A}}(a)$ for all $a \in \mathscr{A}$, we say that ϕ is spectral, here $\mathrm{sp}_{\mathscr{A}}(a)$ denotes the spectrum of a in $\mathscr{A}$. Recall from [3, Definition 10] that ϕ is said to be relatively spectral (to X) if there is a dense subalgebra X of $\mathscr{A}$ such that $\mathrm{sp}_{\mathscr{B}}(\phi(x)) = \mathrm{sp}_{\mathscr{A}}(x)$ for all $x \in X$. Furthermore, ϕ is said to be completely relatively spectral if $\phi_n\colon \mathrm{M}_n(\mathscr{A}) \to \mathrm{M}_n(\mathscr{B})$ is relatively spectral (to $\mathrm{M}_n(X)$) for each n, where $\phi_n((a_{ij})_{n\times n}) = (\phi(a_{ij}))_{n\times n}$, $(a_{ij})_{n\times n} \in \mathrm{M}_n(\mathscr{A})$.

It is known that if ϕ is spectral and has dense range, then ϕ induces an isomorphism $K_*(\mathscr{A}) \cong K_*(\mathscr{B})$ (cf. [2]). Recently, B. Nica shows that if ϕ is completely relatively spectral and $\mathrm{Ran}\,(\phi) = \phi(\mathscr{A})$ is dense in $\mathscr{B}$, then ϕ also induces an isomorphism $K_*(\mathscr{A}) \cong K_*(\mathscr{B})$ (cf. [3, Theorem 2]).

Since we do not know if a relatively spectral homomorphism is completely relatively spectral in general, except some special cases listed in [3], it is significant to investigate if the relatively spectral homomorphism ϕ with dense range could induce an isomorphism between $K_*(\mathscr{A})$ and $K_*(\mathscr{B})$.

In this short note, we prove following result which partially generalizes [3, Theorem 2].

THEOREM 1.1. *Let $\mathscr{A}$, $\mathscr{B}$ be two unital Banach algebras and ϕ be a unital continuous homomorphism from $\mathscr{A}$ to $\mathscr{B}$ with dense range. If ϕ is relatively spectral, then ϕ induces a monomorphism $K_*(\mathscr{A}) \to K_*(\mathscr{B})$.*

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We will prove this theorem in §2. Now we introduce some notations used in next section. For a Banach algebra $\mathscr{A}$ with unit 1, let $GL(\mathscr{A})$ (resp. $GL_0(\mathscr{A})$) denote the group of invertible elements in $\mathscr{A}$ (resp. the connected component of 1 in $GL(\mathscr{A})$). For a Banach algebra $\mathscr{A}$, we view $\mathscr{A}^n$ and $\mathrm{M}_n(\mathscr{A})$ as the set of all $n\times 1$ and $n\times n$ matrices over $\mathscr{A}$ respectively. The norm on $\mathscr{A}^n$ (resp. $\mathrm{M}_n(\mathscr{A})$) is given by $\|(a_1,\cdots,a_n)^T\| = \sum\limits_{i=1}^n \|a_i\|$ (resp. $\|(a_{ij})_{n\times n}\| = \sum\limits_{i,j=1}^n \|a_{ij}\|$). Set $GL_n(\mathscr{A}) = GL(\mathrm{M}_n(\mathscr{A}))$, $GL_n^0(\mathscr{A}) = GL_0(\mathrm{M}_n(\mathscr{A}))$.

Suppose $\mathscr{A}$ is unital and let $x\in GL(\mathscr{A})$. Denote by $[x]$ the equivalence class of x in $GL(\mathscr{A})/GL_0(\mathscr{A})$. The K_1–group of $\mathscr{A}$, denoted $K_1(\mathscr{A})$ is defined as $K_1(\mathscr{A}) = \bigcup\limits_{n=1}^\infty GL_n(\mathscr{A})/GL_n^0(\mathscr{A})$, where $GL_n(\mathscr{A})/GL_n^0(\mathscr{A}) \subset GL_{n+1}(\mathscr{A})/GL_{n+1}^0(\mathscr{A})$ in the sense that $[x]\mapsto[\mathrm{diag}\,(x,1)]$, $\forall x\in GL_n(\mathscr{A})/GL_n^0(\mathscr{A})$, $n=1,2,\cdots$. We can define the K_0–group of $\mathscr{A}$ by $K_0(\mathscr{A}) = K_1((S\mathscr{A})^+)$, where

$$(S\mathscr{A})^+ = \{f\in C([0,1],\mathscr{A})\,|\,f(0)=f(1)=\text{constant}\}.$$

More detailed information about $K_0(\mathscr{A})$ and $K_1(\mathscr{A})$ can be found in [1].

2. Proof of main theorem

In this section, we assume that $\mathscr{A}$ and $\mathscr{B}$ are unital Banach algebras and ϕ is a unital continuous homomorphism from $\mathscr{A}$ to $\mathscr{B}$.

Let M be a compact Hausdorff space and let $C(M,\mathscr{A})$ denote the Banach algebra consisting of all continuous maps $f\colon M\to\mathscr{A}$ with the norm $\|f\| = \sup\limits_{t\in M}\|f(t)\|$. When $M=\mathbf{S}^1$, we set $\Omega(\mathscr{A}) = C(\mathbf{S}^1,\mathscr{A})$. Let X be a dense subalgebra of $\mathscr{A}$ and put $M(X) = \{f\colon M\to X \text{ continuous}\}$. Define a homomorphism $\phi_M\colon C(M,\mathscr{A})\to C(M,\mathscr{B})$ by $\phi_M(f)(t) = \phi(f(t))$, $\forall f\in C(M,\mathscr{A})$ and $t\in M$.

LEMMA 2.1. *Let ϕ be a relatively spectral (to X) homomorphism with* $\mathrm{Ran}\,(\phi)$ *dense in $\mathscr{B}$. Then ϕ_M is a relatively spectral (to $M(X)$) homomorphism with* $\mathrm{Ran}\,(\phi_M)$ *dense in $C(M,\mathscr{B})$.*

Proof. Given $g\in C(M,\mathscr{B})$ and $\varepsilon>0$. Since g is continuous and M is compact, it follows that $g(M)$ is also compact in $\mathscr{B}$. Thus, there are $y_1,\cdots,y_n\in g(M)$ such that $g(M)\subset\bigcup\limits_{i=1}^n O(y_i,\varepsilon)$, where $O(y_i,\varepsilon) = \{y\in\mathscr{B}\,|\,\|y-y_i\|<\varepsilon\}$. Set $U_i = g^{-1}(O(y_i,\varepsilon))$, $i=1,\cdots,n$. Then $\{U_1,\cdots,U_n\}$ is an open cover of M. Choose a partition of unity $\{f_1,\cdots,f_n\}$ subordinate to this cover, so that each f_i is a continuous function from M to $[0,1]$ with support contained in U_i and $\sum\limits_{i=1}^n f_i(t) = 1$, $\forall t\in M$.

From $\overline{\phi(X)} = \mathscr{B}$, we can find $a_1,\cdots,a_n\in X$ such that $\|\phi(a_i)-y_i\|<\varepsilon$, $i=$

$1,\cdots,n$. Set $f_\varepsilon(t)=\sum_{i=1}^{n} a_i f_i(t)$, $\forall t\in M$. Then $f\in M(X)$ and

$$\|\phi(f_\varepsilon(t))-g(t)\|\leqslant\sum_{i=1}^{n}\|\phi(a_i)-y_i\|f_i(t)+\sum_{i=1}^{n}\|y_i f_i(t)-g(t)f_i(t)\|<2\varepsilon,$$

$\forall t\in M$. Thus, $\phi_M(M(X))$ is dense in $C(M,\mathscr{B})$.

If we set $\mathscr{B}=\mathscr{A}$ and $\phi=\mathrm{id}$ in above argument, then we have $\overline{M(X)}=C(M,\mathscr{A})$.

Now we show that ϕ_M is relatively spectral. But it is enough to prove that $\phi_M(f)$ is invertible in $C(M,\mathscr{B})$ for $f\in M(X)$ implies that f is invertible in $C(M,\mathscr{A})$. Since $\phi_M(f)$ is invertible in $C(M,\mathscr{B})$, it follows that $\phi(f(t))$ is invertible in $\mathscr{B}$, $\forall t\in M$. Thus, from the relatively spectral property of ϕ, we have $f(t)\in GL(\mathscr{A})$, $\forall t\in M$. This means that $f\in GL(C(M,\mathscr{A}))$. □

COROLLARY 2.2. *Let ϕ be a relatively spectral (to X) homomorphism with dense range. Suppose there is $a\in X$ such that $\|1-\phi(a)\|<1$. Then $a\in GL_0(\mathscr{A})$.*

Proof. Choose $x\in X$ such that $\|1-x\|\leqslant\frac{1}{1+\|\phi\|}$. Then $x\in GL_0(\mathscr{A})$ and $\|1-\phi(x)\|<1$. Put $f(t)=(1-t)x+ta$, $\forall t\in \mathrm{I}=[0,1]$. Then $f\in \mathrm{I}(X)$ and $\|1-\phi_I(f)\|<1$. So $f\in GL(C(I,\mathscr{A}))$ with $f_0=x$ and $f_1=a$ by Lemma 2.1, which means that $a\in GL_0(\mathscr{A})$. □

LEMMA 2.3. *Let ϕ be a relatively spectral (to X) homomorphism with dense range and let $z\in \mathrm{M}_n(X)$ with $\|1_n-\phi_n(z)\|<\frac{1}{3}$, where 1_n is the unit of $\mathrm{M}_n(\mathscr{A})$. Then for any $\varepsilon>0$, there is $z'\in \mathrm{M}_n(X)\cap GL_n^0(\mathscr{A})$ such that $\|z-z'\|<\varepsilon$.*

Proof. When $n=1$, the statement is true by Corollary 2.2. We assume that the statement is true for $1\leqslant n\leqslant m$. We now prove the argument is also true for $n=m+1$.

Let $y=\phi_{m+1}(z)\in\phi_{m+1}(\mathrm{M}_{m+1}(X))$ with $\|1_{m+1}-y\|<\frac{1}{3}$. Write $z=(z_{ij})_{m+1\times m+1}$ (resp. $y=(y_{ij})_{m+1\times m+1}$) as $z=\begin{pmatrix} z_1 & z_2\\ z_3 & z_4\end{pmatrix}$ (resp. $y=\begin{pmatrix} y_1 & y_2\\ y_3 & y_4\end{pmatrix}$), where $y_{ij}=\phi(z_{ij})$, $i,j=1,\cdots,m+1$, $z_1=(z_{ij})_{m\times m}\in \mathrm{M}_m(X)$ and

$$z_2=\begin{pmatrix} z_{1\,m+1}\\ \vdots\\ z_{m\,m+1}\end{pmatrix},\quad z_3=\begin{pmatrix} z_{m+1\,1} & \ldots & z_{m+1\,m}\end{pmatrix},\quad z_4=z_{m+1\,m+1}.$$

Then $\|1_m-y_1\|<\frac{1}{3}$, $\|y_2\|<\frac{1}{3}$, $\|y_3\|<\frac{1}{3}$ and $\|1-y_4\|<\frac{1}{3}$. By assumption, there is $z_1'\in GL_m^0(\mathscr{A})\cap \mathrm{M}_m(X)$ such that $\|z_1-z_1'\|<\frac{\varepsilon}{3(\|\phi_m\|+1)}$. Pick $x_1\in \mathrm{M}_m(X)$ such that

$$\|(z_1')^{-1}-x_1\|<\frac{\varepsilon}{3(\|z_1'\|+1)(\|z_3\|+1)(\|\phi_m\|+1)}. \tag{1}$$

Set

$$z' = \begin{pmatrix} 1_n & 0 \\ z_3x_1 & 1 \end{pmatrix} \begin{pmatrix} z_1' & z_2 \\ 0 & z_4 - z_3x_1z_2 \end{pmatrix} = \begin{pmatrix} z_1' & z_2 \\ z_3x_1z_1' & z_4 \end{pmatrix} \in \mathrm{M}_{m+1}(X).$$

Then $\|z - z'\| \leqslant \|z - z'\| + \|z_3\|\|1_m - x_1z_1'\| < \varepsilon$.

Note that $\|1_m - \phi_m(z_1)\| < \frac{1}{3}$ and $\|\phi_m(z_1) - \phi_m(z_1')\| < \frac{1}{3}$. So $\|1_m - \phi_m(z_1')\| < \frac{2}{3}$ and hence $\|(\phi_m(z_1'))^{-1}\| < 3$. Consequently, $\|\phi_m(x_1)\| < 4$ by (1). Therefore,

$$\|1 - \phi(z_4 - z_3x_1z_2)\| < \|1 - y_4\| + \|y_3\|\|\phi_m(x_1)\|\|y_2\| < 1. \tag{2}$$

Applying Corollary 2.2 to (2), we have $z_4 - z_3x_1z_2 \in GL_0(\mathscr{A})$. Thus, we can deduce that $\begin{pmatrix} z_1' & z_2 \\ 0 & z_4 - z_3x_1z_2 \end{pmatrix} \in GL^0_{m+1}(\mathscr{A})$. Since $\begin{pmatrix} 1_n & 0 \\ z_3x_1 & 1 \end{pmatrix} \in GL^0_{m+1}(\mathscr{A})$, it follows that $z' \in GL^0_{m+1}(\mathscr{A})$. This completes the proof. □

Now we give the proof of Theorem 1.1 as follows.

Proof. Let $G \in GL_n(\mathscr{A})$ such that $G_0 = \phi_n(G) \in GL^0_n(\mathscr{B})$ for some n. We will prove $G \in GL^0_n(\mathscr{A})$. Since X is dense in $\mathscr{A}$ and ϕ has dense range, we can find $A \in \mathrm{M}_n(X)$ with $\|A - G\|$ small enough so that $A \in GL_n(\mathscr{A})$ with $[A] = [G]$ in $GL_n(\mathscr{A})/GL^0_n(\mathscr{A})$ and $\phi_n(\mathscr{A}) \in GL^0_n(\mathscr{B})$. Noting that $\phi_n(A)$ can be written as $\phi_n(A) = e^{b_1} \cdots e^{b_s}$ for some $b_1, \cdots, b_s \in \mathrm{M}_n(\mathscr{B})$, we can find $a_1, \cdots, a_s \in \mathrm{M}_n(\mathscr{A})$ with $\|\phi_n(a_i) - b_i\|$ small enough, $i = 1, \cdots, s$, such that

$$\|\phi_n(A)^{-1} - \phi_n(e^{-a_1} \cdots e^{-a_s})\| < \frac{1}{6(\|\phi_n(A)\| + 1)}.$$

Choose $B_0 \in \mathrm{M}_n(X)$ such that

$$\|e^{-a_1} \cdots e^{-a_s} - B_0\| < \frac{1}{6(\|e^{a_1} \cdots e^{a_s}\| + 1)(\|\phi_n\| + 1)(\|\phi_n(A)\| + 1)}.$$

Then $B_0 \in GL^0_n(\mathscr{A})$ and

$$\|I_n - \phi_n(AB_0)\| \leqslant \|\phi_n(A)\|\|\phi_n(A)^{-1} - \phi_n(B_0)\| < \frac{1}{3}.$$

Therefore there exists $Z \in GL^0_n(\mathscr{A}) \cap \mathrm{M}_n(X)$ such that $\|AB_0 - Z\| < \frac{1}{\|(AB_0)^{-1}\|}$ by Lemma 2.3. So, $AB_0 \in GL^0_n(\mathscr{A})$ and hence $G \in GL^0_n(\mathscr{A})$.

Since $\phi_{\mathbf{S}^1}$ is relatively spectral and $\mathrm{Ran}\,(\phi_{\mathbf{S}^1})$ is dense in $\Omega(\mathscr{B})$ by Lemma 2.1, we get that the induced homomorphism $(\phi_{\mathbf{S}^1})_*\colon K_1(\Omega(\mathscr{A})) \to K_1(\Omega(\mathscr{B}))$ is injective by above argument. Thus, from the commutative diagram of split exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1(S\mathscr{B}) & \longrightarrow & K_1(\Omega(\mathscr{B})) & \longrightarrow & K_1(\mathscr{B}) & \longrightarrow & 0 \\ & & \phi_* \big\uparrow & & (\phi_{\mathbf{S}^1})_* \big\uparrow & & \phi_* \big\uparrow & & \\ 0 & \longrightarrow & K_1(S\mathscr{A}) & \longrightarrow & K_1(\Omega(\mathscr{A})) & \longrightarrow & K_1(\mathscr{A}) & \longrightarrow & 0, \end{array} \tag{3}$$

we get that $\phi_*\colon K_1(S\mathscr{A}) \to K_1(S\mathscr{B})$ is injective, that is, $\phi_*\colon K_0(\mathscr{A}) \to K_0(\mathscr{B})$ is injective. □

REMARK 2.4. If $\operatorname{csr}(C(\mathbf{S^1},\mathscr{B})) \leqslant \mathbf{2}$, where $\operatorname{csr}(\cdot)$ is the connected stable rank of Banach algebras introduced by Rieffel in [5], then ϕ_* is also surjective.

In fact, when $\operatorname{csr}(C(\mathbf{S}^1,\mathscr{B})) \leqslant 2$, we have $\operatorname{csr}(\mathscr{B}) \leqslant 2$ by [4, Proposition 8.4]. So the natural homomorphism $i_{\mathscr{B}}\colon GL(\mathscr{B})/GL_0(\mathscr{B}) \to K_1(\mathscr{B})$ is surjective by using the proof of Theorem 10.10 in [5]. On the other hand, if ϕ is relatively spectral and has dense range, then the induced homomorphism $\phi_*\colon GL(\mathscr{A})/GL_0(\mathscr{A}) \to GL(\mathscr{B})/GL_0(\mathscr{B})$ is injective by the proof of Theorem 1.1. To show ϕ_* is surjective, let $b \in GL(\mathscr{B})$ and pick $a \in X$ such that $\|\phi(a)-b\| < \dfrac{1}{\|b^{-1}\|}$. Then $\phi(a) \in GL(\mathscr{B})$ with $[b]=[\phi(a)]$ and hence $a \in GL(\mathscr{A})$. Thus, $\phi_*([a])=[b]$ and consequently, $\phi_*\colon K_1(\mathscr{A}) \to K_1(\mathscr{B})$ is surjective.

By the above argument, we also have $(\phi_{\mathbf{S}^1})_*$ is surjective. Thus, using the commutative diagram (3), we obtain that $\phi_*\colon K_1(S\mathscr{A}) \to K_1(S\mathscr{B})$ is surjective.

Especially, when $\mathscr{B}$ is of topological stable rank one, $\operatorname{csr}(\mathscr{B}) \leqslant 2$ and $\operatorname{csr}(C(\mathbf{S}^1,\mathscr{B})) \leqslant 2$. So ϕ_* is surjective.

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REFERENCES

[1] B. BLACKADAR, *K–Theory for Operator Algebras*, Springer–Verlag, 1986.
[2] J.-B. BOST, *Principe d'Oka, K–théorie et systémes dynamiques non commutatifs*, Invent. Math., **101**, 2 (1990), 261–333.
[3] B. NICA, *Relatively spectral morphisms and applications to K–theory*, J. Funct. Anal., **255** (2008), 3303–3328.
[4] B. NICA, *Homotopical stable ranks for Banach algebras*, http://arxiv.org/ abs/0911.2945v1.
[5] M.A. RIEFFEL, *Dimension and stable rank in the K–Theory of C^*–algebras*, Proc. London Math. Soc., **46**(1983), 301–333.

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