Operators
and
Matrices
Volume 5, Number 1 (2011), 157–171

STRONGLY SPLITTING WEIGHTED SHIFT OPERATORS ON BANACH SPACES AND UNICELLULARITY

M. T. KARAEV AND M. GÜRDAL

(*Communicated by D. Hadwin*)

Abstract. We introduce the notion of strong splitting operator on a separable Banach space, and prove a structure theorem for this operator. We consider the weighted shift operator T, $Te_n = \lambda_n e_{n+1}$, $n \geqslant 0$, acting in the Banach space X with basis $\{e_n\}_{n\geqslant 0}$. We give some sufficient conditions for X and for the weight sequence $\{\lambda_n\}_{n\geqslant 0}$ under which the operator is unicellular, that is, every nontrivial invariant subspace E of T has the form $E = X_i := \text{Span}\,\{e_k : k \geqslant i\}$ for some $i \geqslant 1$; and prove that the restricted operators $T|X_i$ $(i \geqslant 1)$ are strong splitting. Moreover, we describe in terms of so-called discrete Duhamel operator and diagonal operator all extended eigenvectors of the operators $T|X_i$ $(i \geqslant 1)$.

1. Introduction

Let X be a separable Banach space. If $(x_n)_{n\geqslant 1} \subset X$, we denote by Span $(x_n : n \geqslant 1)$ the closure of the linear hull generated by $(x_n)_{n\geqslant 1}$. The sequence $(x_n)_{n\geqslant 1}$ is called (see [1]):

- *complete* if Span $(x_n : n \geqslant 1) = X$;
- *minimal* if for all $n \geqslant 1$, $x_n \notin$ Span $(x_m : m \neq n)$;
- *uniformly minimal* if $\inf\limits_{n\geqslant 1} dist \left(\frac{x_n}{\|x_n\|}, \text{Span}\,(x_m : m \neq n)\right) > 0$;
- a *basis* in X if every element $x \in X$ can be uniquely decomposed in a convergent series $x = \sum_{n\geqslant 1} a_n x_n$.

Let $L(X)$ be the Banach algebra of all bounded linear operators on X and $A \in L(X)$. Following [2], we recall that an operator A is called a *splitting operator* in X if, for every $x \in X$ there exists a linear densely defined operator B_x (generally unbounded) such that

$$A^n x = B_x y_n \tag{1}$$

for each n, $n = 0, 1, 2, \ldots$, and for some complete system $\{y_n\}_{n\geqslant 0}$ of the space X.

Mathematics subject classification (2010): Primary 47B38, 47A15; Secondary 46B15.
Keywords and phrases: Banach space, weighted shift operator, strong splitting operator, invariant subspace, Duhamel product.

Paper OaM-05-11

An operator A is called *well splitting* if for every $x \in X$ the corresponding operators B_x in (1) are bounded in X. We say that the well splitting operator A is *strong splitting* if, for some $x_0 \in X$, the corresponding operator B_{x_0} in (1) is invertible. It is immediate from these definitions that a well splitting operator A is cyclic (i.e., there exists $x \in X$ such that Span $\{A^n x : n \geqslant 0\} = X$) if for some $x_0 \in X$ an operator B_{x_0} has dense range in X, and hence strong splitting operator is always cyclic.

It is easy to see that the concept of splitting operator is a generalization of the so-called basis operator introduced by Nikolski [3]:

Let A be a linear bounded operator acting in the space ℓ^p, $1 \leqslant p < \infty$. An operator A is called basis operator if it is cyclic and for every $x \in \ell^p$, $x \neq 0$, there exists (linear) isomorphism V of the space ℓ^p into itself, an integer i, $i \geqslant 0$, and a sequence $\{t_n\}_{n=0}^{\infty}$ $(t_n = t_n(x))$ of complex numbers such that

$$V\delta_{n+i} = t_n A^n x, \ \ n \geqslant 0,$$

where $\{\delta_n\}_{n=0}^{\infty}$ is the natural basis of the space ℓ^p : $\delta_n = \left\{\delta_{n,k}\right\}_{k=0}^{\infty}$, $\delta_{n,k}$ is the Kronecker symbol.

Here, in the short Section 2 of the present article, we give a structure theorem for the strong splitting operators on a Banach space X. Its proof uses the method of the paper [3]. In Section 3, we consider the weighted shift operator T, $Te_n = \lambda_n e_{n+1}$, $n \geqslant 0$, on the Banach space X with basis $\{e_n\}_{n\geqslant 0}$. We give some sufficient conditions for X and for the weights sequence $\{\lambda_n\}_{n\geqslant 0}$ under which the operator is unicellular, that is every nontrivial invariant subspace E of T has the form $E = X_i :=$ Span $\{e_k : k \geqslant i\}$ for some $i \geqslant 1$. We prove that the restricted operators $T|X_i$ $(i \geqslant 1)$ are strong splitting. In Section 4, we describe all so-called extended eigenvectors of the operators $T|E$, $E \in \mathit{Lat}(T)$. These results improve some results of the papers [2, 4, 5].

2. A structure theorem

Recall that operators $A_1 \in L(X_1)$ and $A_2 \in L(X_2)$ are called similar if there exists a linear isomorphism Q, $Q : X_1 \to X_2$, such that $A_2 = QA_1Q^{-1}$.

THEOREM 1. *Let X be a separable Banach space. An operator $A \in L(X)$ is a strong splitting operator if and only if it is similar to some strong splitting shift operator on X.*

Proof. First we note that an operator that is similar to a strong splitting operator is itself strong splitting. Indeed, let A_2 be a strong splitting operator on X. Then

$$A_2^n x = B_{2,x} y_n$$

for every $x \in X$ and $n \geqslant 0$, where an operator B_{2,x_2} is invertible in X for some $x_2 \in X$. Therefore, if A_1 is similar to A_2, that is

$$A_1 = QA_2Q^{-1}$$

for some invertible operator $Q \in L(X)$, then it is clear that

$$A_1^n x = QA_2^n Q^{-1}x = QB_{2,Q^{-1}x}y_n = B_{1,x}y_n,\ n \geqslant 0,$$

where $B_{1,x} := QB_{2,Q^{-1}x}$. Hence, A_1 is a splitting operator in X. On the other hand, if we denote $x_1 := Qx_2$, then we have

$$B_{1,x_1} = B_{1,Qx_2} = QB_{2,Q^{-1}Qx_2} = QB_{2,x_2},$$

and therefore B_{1,x_1} is invertible in X. Hence, A_1 is a strong splitting operator in X.

Let now A be a strong splitting operator on the space X. Denote by x_0 an element for which the operator B_{x_0} is invertible. Since $A^n x_0 = B_{x_0}y_n$ $(n \geqslant 0)$, where $\{y_n\}_{n\geqslant 0}$ is some complete system in X, by setting $A_0 := B_{x_0}^{-1}AB_{x_0}$ we obtain

$$A_0 y_n = B_{x_0}^{-1}AB_{x_0}y_n = B_{x_0}^{-1}AA^n x_0 = B_{x_0}^{-1}A^{n+1}x_0 = B_{x_0}^{-1}B_{x_0}y_{n+1} = y_{n+1}$$

for each $n \geqslant 0$, and thus $A_0 y_n = y_{n+1}$ $(n \geqslant 0)$, that is A_0 is a shift operator in X, and by virtue of above proved, this is a strong splitting operator. The theorem is proved. $\square$

3. Weighted shift operators on Banach spaces

In the following theorem, by using the discrete analog of the Duhamel product

$$(f \circledast g)(x) = \frac{d}{dx}\int_0^x f(x-t)\,g(t)\,dt,$$

we characterize the strong splitting property and unicellularity of some weighted shift operators on Banach space with basis $\{e_n\}_{n\geqslant 0}$. This improves some results of the papers [2, 4, 5]; see also [6, 7, 8]. But, as will be seen from its proof, the condition of basisity of $\{e_n\}_{n\geqslant 0}$ can be actually changed with $M-$basisity (Markushevich basis). Recall that (see, for instance [9]) a complete minimal system $\{e_n\}_{n\geqslant 0} \subset X$ with totally biorthogonal system $\{e_n^*\}_{n\geqslant 0} \subset X^*$ is called $M-$basis.

THEOREM 2. *Let T be a weighted shift operator, continuously acting in the Banach space X with basis $\{e_n\}_{n\geqslant 0}$, by the formula*

$$Te_n = \lambda_n e_{n+1},\ \ \lambda_n \neq 0,\ n \geqslant 0.$$

We put $X_i :=$ Span $\{e_k : k \geqslant i\}$ $(i = 0,1,2,...)$ and $w_n := \lambda_0\lambda_1...\lambda_{n-1}$, $w_0 := 1$. Suppose that:

(a) For every integer $i \geqslant 0$ there exists a number $N := N_i \geqslant i$ such that

$$\sum_{n,m\geqslant N}\left|\frac{w_{n+m-i}}{w_n w_m}\right| < \infty;$$

(b) $\|e_{n+m-i}\|_X \leqslant c_i\,\|e_n\|_X\,\|e_m\|_X$ for all $n,m \geqslant i$ $(i \geqslant 0)$ and for some $c_i > 0$.
Then we have:
(i) The operators $T_i := T|X_i$ $(i = 0,1,2,...)$ are strong splitting in X_i.
(ii) Lat $(T) = \{X_i : i = 1,2,...\}$, i.e., T is a unicellular operator on X.

Proof. (i) For arbitrary two elements $x=\sum\limits_{n\geqslant i}x_ne_n$ and $y=\sum\limits_{n\geqslant i}y_ne_n$ in X_i $(i\geqslant 0)$, let us define the generalized Duhamel product $\underset{i}{\circledast}$ by the formula (see, for instance [10, p. 189] and [4]):

$$x\underset{i}{\circledast}y:=\sum_{n,m\geqslant i}\frac{w_{n+m-i}}{w_nw_m}x_ny_me_{n+m-i}. \tag{2}$$

(Since $X_0=X$, instead $\underset{0}{\circledast}$ we will write simple $\circledast$). By virtue of conditions of theorem, the formula (2) is correctly defined. For every fixed $n\geqslant i$, let us denote

$$R_n(x):=\sum_{k\geqslant n}x_ke_k.$$

Then we have:

$$\begin{aligned}\|R_n(x)\| &= \left\|\sum_{k\geqslant n}x_ke_k\right\| = \Big\|x_0e_0+x_1e_1+\ldots+x_{n-1}e_{n-1}\\ &\quad + \sum_{k\geqslant n}x_ke_k-(x_0e_0+x_1e_1+\ldots+x_{n-1}e_{n-1})\Big\|\\ &\leqslant \|x\|+\|x_0e_0\|+\|x_1e_1\|+\ldots+\|x_{n-1}e_{n-1}\|.\end{aligned}$$

By considering that every basis is uniformly minimal, we have that

$$\|x_je_j\|\leqslant\frac{1}{d}\|x\|,$$

for all $j\geqslant 0$, where $d:=\inf\limits_{i\geqslant 0}dist\left(\frac{x_i}{\|x_i\|},Lin(x_j:j\neq i)\right)$ is the constant of uniform minimality of $\{e_n\}_{n\geqslant 0}$, which implies that

$$\|R_n(x)\|\leqslant\|x\|+\frac{1}{d}n\|x\|=\left(\frac{1}{d}n+1\right)\|x\| \tag{3}$$

On the other hand, it is easy to verify that

$$\left\|T^k\right\|\leqslant\sup_{n\geqslant 1}|\lambda_n\lambda_{n+1}\ldots\lambda_{n+k-1}|<+\infty \tag{4}$$

Then, by using inequalities (3), (4) and conditions of theorem, we have:

$$\begin{aligned}\left\|x\underset{i}{\circledast}y\right\| &= \left\|\sum_{n,m\geqslant i}\frac{w_{n+m-i}}{w_nw_m}x_ny_me_{n+m-i}\right\| = \left\|\sum_{n\geqslant i}\frac{x_n}{w_n}\sum_{m\geqslant i}\frac{w_{n+m-i}}{w_m}y_me_{n+m-i}\right\|\\ &\leqslant \left\|\frac{x_i}{w_i}\sum_{m\geqslant i}y_me_m\right\|+\left\|\frac{x_{i+1}}{w_{i+1}}\sum_{m\geqslant i}\frac{w_{m+1}}{w_m}y_me_{m+1}\right\|\\ &\quad +\ldots+\left\|\frac{x_{i+N-1}}{w_{i+N-1}}\sum_{m\geqslant i}\frac{w_{m+N-i}}{w_m}y_me_{n+N-1}\right\|+\left\|\sum_{n\geqslant N}\frac{x_n}{w_n}\sum_{m\geqslant i}\frac{w_{n+m-i}}{w_m}y_me_{n+m-i}\right\|\end{aligned}$$

$$
\begin{aligned}
&= \left|\frac{x_i}{w_i}\right| \|y\| + \left|\frac{x_{i+1}}{w_{i+1}}\right| \|Ty\| + \ldots + \left|\frac{x_{i+N-1}}{w_{i+N-1}}\right| \left\|T^{N-1}y\right\| \\
&\quad + \left|\frac{y_i}{w_i}\right| \|R_N(x)\| + \left|\frac{y_{i+1}}{w_{i+1}}\right| \|R_N(Tx)\| + \ldots + \left|\frac{y_{i+N-1}}{w_{i+N-1}}\right| \left\|R_N\left(T^{N-1}x\right)\right\| \\
&\quad + \sum_{n\geqslant N}\sum_{m\geqslant N} \left|\frac{w_{n+m-i}}{w_n w_m}\right| . |x_n|\,|y_m|\,\|e_{n+m-i}\| \leqslant C_{i,N,d}\,\|x\|\,\|y\|.
\end{aligned}
$$

Thus

$$\left\|x \underset{i}{\circledast} y\right\|_{X_i} \leqslant C_{i,N,d}\,\|x\|_{X_i}\,\|y\|_{X_i} \tag{5}$$

for every $x,y \in X_i$ $(i \geqslant 0)$. It is clear from (2) and (5) that $\left(X_i, \underset{i}{\circledast}\right)$ is a Banach algebra with the property that $w_i e_i \underset{i}{\circledast} f = f \underset{i}{\circledast} w_i e_i = f$ for all $f \in X_i$. Every $x \in X_i$ defines the following "generalized Duhamel operator":

$$\mathscr{D}_{x,i} y := x \underset{i}{\circledast} y \quad (y \in X_i).$$

It follows from this and (2) that

$$T_i^n y = w_{i+n} e_{i+n} \underset{i}{\circledast} y = \mathscr{D}_{y,i}(w_{i+n} e_{i+n}) \tag{6}$$

for all $y \in X_i$ and $n \geqslant 0$. Indeed, for every $y \in X_i$ and $n \geqslant 0$ we have

$$
\begin{aligned}
T_i^n y &= T^n y = T^n \left(\sum_{m\geqslant i} y_m e_m\right) = \sum_{m\geqslant i} y_m T^n e_m \\
&= \sum_{m\geqslant i} y_m \lambda_m \lambda_{m+1} \ldots \lambda_{m+n-1} e_{m+n} \\
&= \sum_{m\geqslant i} y_m \frac{w_{m+n}}{w_m} e_{m+n} = \sum_{m\geqslant i} w_{i+n} y_m \frac{w_{i+n+m-i}}{w_{i+n} w_m} e_{i+m+n-i} \\
&= \sum_{m\geqslant i} w_{i+n} y_m \left(e_{i+n} \underset{i}{\circledast} e_m\right) = w_{i+n} e_{i+n} \underset{i}{\circledast} \sum_{m\geqslant i} y_m e_m \\
&= w_{i+n} e_{i+n} \underset{i}{\circledast} y = \mathscr{D}_{y,i}(w_{i+n} e_{i+n}),
\end{aligned}
$$

which implies (6). Hence, $T|X_i$ is a well splitting operator.

To prove that $T|X_i$ is a strong splitting operator, it suffices to show that the operator $\mathscr{D}_{x,i}$ is invertible in X_i if and only if $x_i \neq 0$.

Indeed, if $\mathscr{D}_{x,i}$ is invertible in X_i, then there exists $y \in X_i$ such that $x \underset{i}{\circledast} y = w_i e_i$. By considering (2), from this we obtain that

$$\left(x \underset{i}{\circledast} y\right)_i = \frac{x_i}{w_i} y_i = w_i,$$

that is $x_i y_i = w_i^2 \neq 0$, and therefore $x_i \neq 0$.

Now we prove the inverse assertion. Since

$$\mathscr{D}_{x,i} = \mathscr{D}_{x_i e_i + (x - x_i e_i), i}$$

and $w_i e_i \underset{i}{\circledast} y = y$ for all $y \in X_i,$ we have

$$\mathscr{D}_{x,i} = \frac{x_i}{w_i} I_{X_i} + \mathscr{D}_{x - x_i e_i, i}.$$

Therefore, to prove invertibility of $\mathscr{D}_{x,i}$ it suffices to show that some power of $\mathscr{D}_{x-x_i e_i,i}$ is compact and $\ker \mathscr{D}_{x,i} = \{0\}$; from which by a classical theorem of S.M. Nikolski (see [11]) we will obtain invertibility of $\mathscr{D}_{x,i}$. For this purpose we set $x' := x - x_i e_i$ and $\widetilde{x} := [x']^{\underset{i}{\circledast} N+1} = x' \underset{i}{\circledast} \ldots \underset{i}{\circledast} x'$ $(N+1 \text{ time})$. Simple calculus show that

$$\widetilde{x}_i = \widetilde{x}_{i+1} = \ldots = \widetilde{x}_{i+N} = 0.$$

Therefore for each $y \in X_i$ we have:

$$\begin{aligned}
\mathscr{D}_{x',i}^{N+1} y &= \left[x'\right]^{\underset{i}{\circledast} N+1} \underset{i}{\circledast} y = \widetilde{x} \underset{i}{\circledast} y \\
&= \sum_{n \geqslant i} \frac{\widetilde{x}_n}{w_n} \sum_{m \geqslant i} \frac{w_{n+m-i}}{w_m} y_m e_{n+m-i} \\
&= \sum_{n \geqslant i+N+1} \frac{\widetilde{x}_n}{w_n} \sum_{m \geqslant i} \frac{w_{n+m-i}}{w_m} y_m e_{n+m-i} \\
&= \frac{y_i}{w_i} \sum_{n \geqslant i+N+1} \widetilde{x}_n e_n + \frac{y_{i+1}}{w_{i+1}} R_{i+N+1}\left(T\widetilde{x}\right) + \ldots \\
&\quad + \frac{y_{i+N}}{w_{i+N}} R_{i+N+1}\left(T^N \widetilde{x}\right) + \sum_{n \geqslant i+N+1} \sum_{m \geqslant i+N+1} \frac{w_{n+m-i}}{w_n w_m} \widetilde{x}_n y_m e_{n+m-i} \\
&= \sum_{j=i}^{i+N} \frac{y_j}{w_j} R_{i+N+1}\left(T^{j-i} \widetilde{x}\right) + \sum_{n=i+N+1}^{M} \sum_{m=i+N+1}^{M} \frac{w_{n+m-i}}{w_n w_m} \widetilde{x}_n y_m e_{n+m-i} \\
&\quad + \sum_{n=i+N+1}^{M} \sum_{m=M+1}^{\infty} \frac{w_{n+m-i}}{w_n w_m} \widetilde{x}_n y_m e_{n+m-i} \\
&\quad + \sum_{n=M+1}^{\infty} \sum_{m=i+N+1}^{M} \frac{w_{n+m-i}}{w_n w_m} \widetilde{x}_n y_m e_{n+m-i}
\end{aligned}$$

Let us define the following finite-rank operator:

$$\mathscr{K}_M^{(i)} y := \sum_{j=i}^{i+N} \frac{y_j}{w_j} R_{i+N+1}\left(T^{j-i} \widetilde{x}\right) + \sum_{n=i+N+1}^{M} \sum_{m=i+N+1}^{M} \frac{w_{n+m-i}}{w_n w_m} \widetilde{x}_n y_m e_{n+m-i}.$$

Considering the conditions (a), (b) of the theorem, we obtain

$$\begin{aligned}\left\|\mathscr{D}_{x',i}^{N+1}-\mathscr{K}_M^{(i)}\right\|_{L(X_i)} &= \sup_{\|y\|_{X_i}\leqslant 1}\left\|\mathscr{D}_{x',i}^{N+1}y-\mathscr{K}_M^{(i)}y\right\|_{X_i}\\ &= \sup_{\|y\|_{X_i}\leqslant 1}\left\|\sum_{n=i+N+1}^{M}\sum_{m=i+N+1}^{\infty}\frac{w_{n+m-i}}{w_nw_m}\widetilde{x}_ny_me_{n+m-i}\right.\\ &\quad\left.+\sum_{n=M+1}^{\infty}\sum_{m=i+N+1}^{\infty}\frac{w_{n+m-i}}{w_nw_m}\widetilde{x}_ny_me_{n+m-i}\right\|_{X_i}\\ &\leqslant C_i\left(\sum_{n=i+N+1}^{M}\sum_{m=M+1}^{\infty}\left|\frac{w_{n+m-i}}{w_nw_m}\right|+\sum_{n=M+1}^{\infty}\sum_{m=i+N+1}^{\infty}\left|\frac{w_{n+m-i}}{w_nw_m}\right|\right)\\ &\to 0 \ (\text{as } M\to+\infty),\end{aligned}$$

which means that $\mathscr{D}_{x',i}^{N+1}$ is a compact operator on X_i.

We now prove that $\ker\mathscr{D}_{x,i}=\{0\}$. In fact, if $y\in X_i\cap\ker\mathscr{D}_{x,i}$, that is $x\underset{i}{\circledast}y=0$, then simple calculations show that

$$\begin{aligned}\frac{x_i}{w_i}y_i &= 0\\ \frac{x_i}{w_i}y_{i+1}+\frac{x_{i+1}}{w_i}y_i &= 0\\ \frac{x_i}{w_i}y_{i+2}+\frac{w_{i+2}}{w_{i+1}^2}x_{i+1}y_{i+1}+\frac{x_{i+2}}{w_i}y_i &= 0\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots&\end{aligned}$$

Since $x_i\neq 0$, from this infinite system we obtain that

$$y_i=y_{i+1}=y_{i+2}=\ldots=0,$$

that is $y=0$. Thus, we deduce that $\mathscr{D}_{x,i}$ is invertible operator on X_i, and consequently $T|X_i$ $(i\geqslant 0)$ is a strong splitting operator on X_i, as desired.

(ii) Obviously, all subspaces X_i $(i=1,2,\ldots)$ are nontrival T-invariant subspaces and

$$X\supset X_1\supset X_2\supset\ldots\supset\{0\}.$$

Therefore, since Span $\{T^nx:n\geqslant 0\}$, $x\in X$, is T-invariant subspace, the operator T is unicellular in X if and only if for all $x\in X$, $x\neq 0$,

$$\operatorname{Span}\{T^nx:n\geqslant 0\}=X_i$$

for some $i=i(x)$, $i=0,1,2,\ldots$. On the other hand, it follows easily from the splitting property of operators $T|X_i$ $(i\geqslant 0)$ that

$$\operatorname{Span}\{T^nx:n\geqslant 0\}=X_i\Leftrightarrow x\in X_i \text{ and } x_i\neq 0.$$

In fact, by considering formula (6), we have

$$\begin{aligned}\operatorname{Span}\{T^n x : n \geqslant 0\} &= \operatorname{Span}\{\mathscr{D}_{x,i}(w_{i+n}e_{i+n}) : n \geqslant 0\} \\ &= \operatorname{clos}\mathscr{D}_{x,i}\operatorname{Span}\{w_{i+n}e_{i+n} : n \geqslant 0\} \\ &= \operatorname{clos}\mathscr{D}_{x,i}X_i,\end{aligned}$$

and therefore

$$\operatorname{Span}\{T^n x : n \geqslant 0\} = X_i \Leftrightarrow \operatorname{clos}\mathscr{D}_{x,i}X_i = X_i.$$

It remains only to show that

$$\operatorname{clos}\mathscr{D}_{x,i}X_i = X_i \Leftrightarrow x_i \neq 0.$$

Indeed, if $\operatorname{clos}\mathscr{D}_{x,i}X_i = X_i$, then there exists a sequence $\left\{x^{(n)}\right\} \subset X_i$ such that $x \underset{i}{\circledast} x^{(n)} \to w_i e_i$ as $n \to \infty$, or $\frac{1}{w_i} x_i x_i^{(n)} \to w_i \neq 0$ as $n \to \infty$, from which we deduce that $x_i \neq 0$.

Conversely, if $x \in X_i$ and $x_i \neq 0$, then as we proved already in item (i), an operator $\mathscr{D}_{x,i}$ is invertible in X_i, in particular, $\operatorname{clos}\mathscr{D}_{x,i}X_i = X_i$, as desired. The theorem is proved. $\square$

Let $\mathscr{K} \in L(Y)$ be an operator on the Banach space Y such that for every $x \in X$ there exists $x(\mathscr{K}) := \sum\limits_{n\geqslant 0} x_n \mathscr{K}^n$ and $\|x(\mathscr{K})\| \asymp \|x\|$, i.e., there exist the constants $c_1, c_2 > 0$ satisfying

$$c_1\|x\| \leqslant \|x(\mathscr{K})\| \leqslant c_2\|x\| \tag{7}$$

for all $x \in X$. Then, the map $\Gamma x := x(\mathscr{K})$ defines the continuous homomorphism from the algebra $(X, \circledast)$ to the algebra $L(Y)$, where $\circledast$ is the usual Duhamel product. Indeed, let us define $X(\mathscr{K}) := \{x(\mathscr{K}) : x \in X\}$, that is $X(\mathscr{K}) = \Gamma X$. In $X(\mathscr{K})$ we define the following product:

$$x(\mathscr{K})\,\widehat{\circledast}\,y(\mathscr{K}) := (x \circledast y)(\mathscr{K}).$$

Then

$$\Gamma(x \circledast y) = (x \circledast y)(\mathscr{K}) = x(\mathscr{K})\,\widehat{\circledast}\,y(\mathscr{K}) = \Gamma x \widehat{\circledast} \Gamma y.$$

Clearly, $\Gamma e_0 = I$ and $\Gamma(w_1 e_1) = \mathscr{K}$.

On the other hand, it follows from (7) that Γ is continuous. Thus, Γ is a continuous homomorphism (i.e., Γ is a representation of algebra $(X, \circledast)$ in $L(Y)$).

In the following theorem we describe all closed ideals of the algebra $\left(X(\mathscr{K}), \widehat{\circledast}\right)$.

THEOREM 3. *Every closed nontrivial ideal E of the algebra $\left(X(\mathscr{K}), \widehat{\circledast}\right)$ has the form $E = X_i(\mathscr{K})$ for some $i \geqslant 1$.*

Proof. Let us define in $X(\mathscr{K})$ the following operator:

$$Ax(\mathscr{K}) := (Tx)(\mathscr{K}),$$

where T is the weighted shift operator, as in Theorem 2. It is clear from (7) that A is a linear bounded operator in $X(\mathscr{K})$. It is also clear that

$$Ax(\mathscr{K}) = (Tx)(\mathscr{K}) = (w_1e_1 \circledast x)(\mathscr{K}) = \mathscr{K}\widehat{\circledast}x(\mathscr{K})$$

for all $x(\mathscr{K}) \in X(\mathscr{K})$. If $x(\mathscr{K}) \in X_i(\mathscr{K})$, where $X_i \in Lat(T)$, $X_i = \text{Span}\{e_k : k \geqslant i\}$ $= \left\{x \in X : x = \sum\limits_{n=i}^{\infty} x_n e_n\right\}$, then $Ax(\mathscr{K}) = \mathscr{K}^{i+1}\underset{i}{\widehat{\circledast}}x(\mathscr{K})$, where $\underset{i}{\circledast}$ is the Duhamel product in X_i (see formula (2)). Now, it is clear that in order to describe the closed nontrivial ideals of the algebra $\left(X(\mathscr{K}), \widehat{\circledast}\right)$, it suffices to describe the closed nontrivial A-invariant subspaces in $\left(X(\mathscr{K}), \widehat{\circledast}\right)$. For this purpose, note that $X_i(\mathscr{K}) \in Lat(A)$, $i \geqslant 1$ (this follows, for example, from the relation $A\Gamma = \Gamma T$ and inclusion $X_i \in Lat(T)$, $i \geqslant 1$). Therefore, it suffices to show that

$$\text{Span}\{A^n x(\mathscr{K}) : n \geqslant 0\} = X_i(\mathscr{K}) \Leftrightarrow x(\mathscr{K}) \in X_i \text{ and } x_i \neq 0.$$

The implication $\Longrightarrow$ is obvious. Let us prove the reverse implication $\Longleftarrow$. In fact,

$$\begin{aligned}\text{Span}\{A^n x(\mathscr{K}) : n \geqslant 0\} &= \text{Span}\left\{\mathscr{K}^{i+n}\underset{i}{\widehat{\circledast}}x(\mathscr{K}) : n \geqslant 0\right\} \\ &= \operatorname{clos}\mathscr{D}_{x(\mathscr{K}),\underset{i}{\widehat{\circledast}}}\text{Span}\left\{\mathscr{K}^{i+n} : n \geqslant 0\right\} \\ &= \operatorname{clos}\mathscr{D}_{x(\mathscr{K}),\underset{i}{\widehat{\circledast}}}X_i(\mathscr{K}) = X_i(\mathscr{K}),\end{aligned}$$

because the operator $\mathscr{D}_{x(\mathscr{K}),\underset{i}{\widehat{\circledast}}}$ with $x_i \neq 0$ is invertible in $X_i(\mathscr{K})$, which completes the proof of theorem. $\square$

Now we give some applications of formula (6).

THEOREM 4. *Suppose that all conditions of Theorem 2 are satisfied. Then we have:*

(i) $\{T|X_i\}' = \{\mathscr{D}_{x,i} : x \in X_i\}$, $i = 0, 1, 2, ...$, *i.e., the commutant of operator* $T|X_i$ *consists from the Duhamel operators* $\mathscr{D}_{x,i}$, $x \in X_i$.

(ii) If $\|e_i\| = \frac{1}{|w_i|}$, *then* $\|p(T|X_i)\|_{L(X_i)} = C_{i,N,d}\|q\|_{X_i}$ *for all polynomials* $p = \sum\limits_{n \geqslant i} p_n e_n$, *where* $q := \sum\limits_{n \geqslant i} w_n p_n e_n$ *and* $C_{i,N,d} > 0$ *is the constant, as in the inequality* (5).

Proof. It follows from the formula (6) that

$$Tx = x \underset{i}{\circledast} w_{i+1}e_{i+1},\ x \in X_i.$$

Then, by virtue of commutativity and associativity of the Duhamel product $\underset{i}{\circledast}$ $(i \geqslant 0)$, we have

$$\{\mathscr{D}_{x,i} : x \in X_i\} \subset \{T|X_i\}'.$$

Conversely, let $A \in \{T|X_i\}'$ be an arbitrary operator, that is

$$(T|X_i)A = A(T|X_i),$$

or

$$T_iA = AT_i,$$

where $T_i = T|X_i$. Then

$$T_1^k A = AT_1^k \quad (\forall k \geqslant 0),$$

which implies that

$$T_1^k Aw_ie_i = AT_1^k w_ie_i.$$

From this, by considering formula (6), we obtain

$$\begin{aligned} w_{i+k}e_{i+k} \underset{i}{\circledast} Aw_ie_i &= A\left(w_{i+k}e_{i+k} \underset{i}{\circledast} w_ie_i\right) \\ &= Aw_{i+k}e_{i+k}, \end{aligned}$$

and hence

$$Ae_{i+k} = Aw_ie_i \circledast e_{i+k} \quad (\forall k \geqslant 0),$$

therefore

$$Ap = Aw_ie_i \underset{i}{\circledast} p$$

for all polynomials $p = \sum_{n \geqslant i} p_ne_n \in X_i$. Since $\left(X_i, \underset{i}{\circledast}\right)$ is a Banach algebra, we have

$$Ax = Aw_ie_i \underset{i}{\circledast} x$$

for all $x \in X_i$. By setting $y := Aw_ie_i$, we have $A = \mathscr{D}_{y,i}$, where $y \in X_i$, which completes the proof of (i).

(ii) It is easy to see from the formula (6) that $p(T_i)x = q \underset{i}{\circledast} x$ for all $x \in X_i$ and polynomials $p = \sum_{n \geqslant i} p_ne_n \in X_i$, where q is the vector polynomial of the form $q := \sum_{n \geqslant i} w_np_ne_n$. Then we have that (see inequality (5))

$$\|p(T_i)x\|_{X_i} = \left\|q \underset{i}{\circledast} x\right\|_{X_i} \leqslant C_{i,N,d}\|q\|_{X_i}\|x\|_{X_i} \quad (\forall x \in X_i),$$

that is

$$\|p(T_i)\|_{L(X_i)} \leqslant C_{i,N,d}\|q\|_{X_i},$$

and since $q \underset{i}{\circledast} w_ie_i = q$, we have

$$p(T_i)C_{i,N,d}w_ie_i = q \underset{i}{\circledast} C_{i,N,d}w_ie_i = C_{i,N,d}\left(q \underset{i}{\circledast} w_ie_i\right) = C_{i,N,d}q.$$

From this, by considering that $\|w_ie_i\| = 1$, we obtain that

$$\|p(T_i)\|_{L(X_i)} = C_{i,N,d}\|q\|_{X_i},$$

which completes the proof of theorem. $\square$

4. Extended eigenvalues and extended eigenvectors for $T|X_i$

Following Biswas, Lambert and Petrovic [12], we say that a complex number λ is an extended eigenvalue of A if there exists a nonzero operator $B \in L(X)$ such that

$$AB = \lambda BA;$$

such an operator B is called extended eigenvector corresponding to λ. The set of all extended eigenvalues of A will be called the extended point spectrum, and will be denoted as $\sigma_p^{ext}(A)$. It is easy to see that if $\lambda_n \neq 0,\ n = 0,1,2,...,$ then for the corresponding weighted shift operator $T,\ Te_n = \lambda_n e_{n+1},\ n \geqslant 0,$ we have that $\ker T = \{0\}$. Therefore, $\lambda = 0$ is not an extended eigenvalue of T, and hence $\sigma_p^{ext}(T) \subset \mathbb{C}\setminus\{0\}$. Thus, $\sigma_p^{ext}(T|X_i) \subset \mathbb{C}\setminus\{0\}$ for all $i \geqslant 0$. The basic facts about the extended eigenvalues and extended eigenvectors of operators can be found in [12]–[18].

The following result describes the set of all extended eigenvectors of all operators $T|X_i\ (i \geqslant 0)$, which essentially improves Theorem 3 in [2] and Theorem 1 in [5].

THEOREM 5. *Let X, T and T_i be the same as in Theorem 2. Suppose that $\lambda \in \mathbb{C}\setminus\{0\}$ is an extended eigenvalue for T_i and $A \in L(X_i)$ is a nonzero operator. Then:*

(i) if $|\lambda| \leqslant 1$, then $\lambda A T_i = T_i A$ if and only if $AD_\lambda = \lambda^i \mathscr{D}_{Aw_ie_i,i}$, where D_λ, $D_\lambda e_n = \lambda^n e_n\ (n \geqslant 0)$, is a diagonal operator.

(ii) if $|\lambda| > 1$, then $\lambda A T_i = T_i A$ if and only if $A = \lambda^i \mathscr{D}_{Aw_ie_i,i} D_{\frac{1}{\lambda}}$.

Proof. (i) If $\lambda A T_i = T_i A$, then $\lambda^n A T_i^n = T_i^n A,\ n \geqslant 0$. In particular,

$$A\lambda^n T_i^n w_i e_i = T_i^n A w_i e_i,\ n \geqslant 0.$$

Using formula (6), from this we obtain that

$$A\left(\lambda^n w_{i+n} e_{i+n} \underset{i}{\circledast} w_i e_i\right) = Aw_i e_i \underset{i}{\circledast} w_{i+n} e_{i+n},$$

or

$$A\left(\lambda^{i+n} e_{i+n}\right) = \lambda^i A w_i e_i \underset{i}{\circledast} e_{i+n},$$

that is,

$$AD_\lambda e_{i+n} = \lambda^i A w_i e_i \underset{i}{\circledast} e_{i+n} \quad (n \geqslant 0).$$

From this

$$AD_\lambda P = \lambda^i \mathscr{D}_{Aw_ie_i,i} P$$

for all polynomials $P = \sum_{n \geqslant i} P_n e_n \in X_i$. Since $\left(X_i, \underset{i}{\circledast}\right)$ is a Banach algebra, from this we have that

$$AD_\lambda x = \lambda^i \mathscr{D}_{Aw_ie_i,i} x \quad (\forall x \in X_i),$$

that is

$$AD_\lambda = \lambda^i \mathscr{D}_{Aw_ie_i,i} \tag{8}$$

where $\mathscr{D}_{Aw_ie_i,i}$ is the Duhamel operator on X_i.

Conversely, let us prove that every nonzero operator A satisfying (8), is an extended eigenvector for the operator $T_i = T|X_i$. In fact, for all polynomials $P = \sum\limits_{m=i}^{\deg P} P_m e_m$, let us denote $P_{\frac{1}{\lambda}} := \sum\limits_{m \geqslant i} \frac{1}{\lambda^m} P_m e_m$. Then we have

$$\begin{aligned}
T_iAP &= T_iAD_\lambda D_{\frac{1}{\lambda}} p = T_iAD_\lambda P_{\frac{1}{\lambda}} = \lambda^i T_i \mathscr{D}_{Aw_ie_i,i} P_{\frac{1}{\lambda}} \\
&= \lambda^i \left(w_{i+1}e_{i+1} \underset{i}{\circledast} \mathscr{D}_{Aw_ie_i,i} P_{\frac{1}{\lambda}} \right) \\
&= \lambda^i \mathscr{D}_{Aw_ie_i,i} \mathscr{D}_{w_{i+1}e_{i+1,i}} P_{\frac{1}{\lambda}} \\
&= \lambda^i \mathscr{D}_{Aw_ie_i,i} \left(w_{i+1}e_{i+1} \underset{i}{\circledast} P_{\frac{1}{\lambda}} \right) \\
&= \lambda^i \mathscr{D}_{Aw_ie_i,i} \lambda^{i+1} \left(\frac{w_{i+1}e_{i+1}}{\lambda^{i+1}} \underset{i}{\circledast} P_{\frac{1}{\lambda}} \right) \\
&= \lambda\lambda^i \left(\lambda^i \mathscr{D}_{Aw_ie_i,i} \left(\frac{w_{i+1}e_{i+1}}{\lambda^{i+1}} \underset{i}{\circledast} P_{\frac{1}{\lambda}} \right) \right) \\
&= \lambda\lambda^i \left(AD_\lambda \left(\frac{w_{i+1}e_{i+1}}{\lambda^{i+1}} \underset{i}{\circledast} \sum_{m=i}^{\deg P} P_m \frac{1}{\lambda^m} e_m \right) \right) \\
&= \lambda AD_\lambda \left[\frac{w_{i+1}}{\lambda} \sum_{m=i}^{\deg P} P_m \frac{1}{\lambda^m} \left(e_{i+1} \underset{i}{\circledast} e_m \right) \right] \\
&= \lambda AD_\lambda \left(\frac{w_{i+1}}{\lambda} \sum_{m=i}^{\deg P} P_m \frac{1}{\lambda^m} \frac{w_{i+1+m-i}}{w_{i+1}w_m} e_{i+1+m-i} \right) \\
&= \lambda AD_\lambda \left(\frac{1}{\lambda} \sum_{m=i}^{\deg P} P_m \frac{1}{\lambda^m} \frac{w_{m+1}}{w_m} e_{m+1} \right) \\
&= \lambda AD_\lambda \sum_{m=i}^{\deg P} P_m \frac{1}{\lambda^{m+1}} \lambda_m e_{m+1} \\
&= \lambda AD_\lambda D_{\frac{1}{\lambda}} T \sum_{m=i}^{\deg P} P_m e_m = \lambda AT_iP.
\end{aligned}$$

Thus

$$T_iAx = \lambda AT_ix$$

for all $x \in X_i$, that is, $T_iA = \lambda AT_i$, as desired.

(ii) If $|\lambda| > 1$ and $\lambda AT_i = T_iA$, then $AT_i = \frac{1}{\lambda} T_iA$. From this

$$AT_i^n = \frac{1}{\lambda^n} T_i^n A,$$

which implies that

$$AT_i^n w_ie_i = \frac{1}{\lambda^n} T_i^n Aw_ie_i,$$

that is (see formula (6)),

$$A\left(w_{i+n}e_{i+n}\underset{i}{\circledast}w_ie_i\right)=\frac{1}{\lambda^n}\left(w_{i+n}e_{i+n}\underset{i}{\circledast}Aw_ie_i\right)$$
$$=\left(w_{i+n}\frac{1}{\lambda^n}e_{i+n}\underset{i}{\circledast}Aw_ie_i\right),$$

or

$$Ae_{i+n}=Aw_ie_i\underset{i}{\circledast}\frac{\lambda^i}{\lambda^{i+n}}e_{i+n}=\lambda^iAw_ie_i\underset{i}{\circledast}\frac{1}{\lambda^{i+n}}e_{i+n}\ (n\geqslant 0),$$

which implies that

$$AP=\lambda^i\mathscr{D}_{Aw_ie_i,i}P_{\frac{1}{\lambda}}=\lambda^i\mathscr{D}_{Aw_ie_i,i}D_{\frac{1}{\lambda}}P$$

for all polynomials $p\in X_i$, and hence $Ax=\lambda^i\mathscr{D}_{Aw_ie_i,i}D_{\frac{1}{\lambda}}x$ for all $x\in X_i$, which means that

$$A=\lambda^i\mathscr{D}_{Aw_ie_i,i}D_{\frac{1}{\lambda}}. \tag{9}$$

Conversely, let us prove that every nonzero operator $A\in L(X_i)$, with representation (9), is the extended eigenvector for the operator T_i. Indeed, for all polynomials $P=\sum\limits_{m=i}^{\deg P}P_me_m$ we have

$$\begin{aligned}
T_iAP&=T_i\left(\lambda^i\mathscr{D}_{Aw_ie_i,i}D_{\frac{1}{\lambda}}P\right)\\
&=\left(w_{i+1}e_{i+1}\underset{i}{\circledast}\lambda^i\left(Aw_ie_i\underset{i}{\circledast}D_{\frac{1}{\lambda}}p\right)\right)\\
&=\lambda^iAw_ie_i\underset{i}{\circledast}\left(w_{i+1}e_{i+1}\underset{i}{\circledast}D_{\frac{1}{\lambda}}p\right)\\
&=\lambda^iAw_ie_i\underset{i}{\circledast}\left(w_{i+1}e_{i+1}\underset{i}{\circledast}\sum_{m=i}^{\deg P}\frac{P_m}{\lambda^m}e_m\right)\\
&=\lambda^iAw_ie_i\underset{i}{\circledast}\left(w_{i+1}\sum_{m=i}^{\deg P}\frac{P_m}{\lambda^m}\left(e_{i+1}\circledast e_m\right)\right)\\
&=\lambda^iAw_ie_i\underset{i}{\circledast}\left(w_{i+1}\sum_{m=i}^{\deg P}\frac{P_m}{\lambda^m}\frac{w_{i+1+m-i}}{w_{i+1}w_m}e_{i+1+m-i}\right)\\
&=\lambda^iAw_ie_i\underset{i}{\circledast}\left(\sum_{m=i}^{\deg P}\frac{P_m}{\lambda^m}\frac{w_{m+1}}{w_m}e_{m+1}\right)\\
&=\lambda^iAw_ie_i\underset{i}{\circledast}\sum_{m=i}^{\deg P}\frac{P_m}{\lambda^m}\lambda_me_{m+1}\\
&=\lambda^iAw_ie_i\underset{i}{\circledast}\lambda\sum_{m=i}^{\deg P}\frac{P_m}{\lambda^{m+1}}\lambda_me_{m+1}
\end{aligned}$$

$$
\begin{aligned}
&= \lambda\left(\lambda^i A w_i e_i \underset{i}{\circledast} D_{\frac{1}{\lambda}} \sum_{m=i}^{\deg P} P_m \lambda_m e_{m+1}\right)\\
&= \lambda\left(\lambda^i A w_i e_i \underset{i}{\circledast} D_{\frac{1}{\lambda}} T \sum_{m=i}^{\deg P} P_m e_m\right)\\
&= \lambda\left(\lambda^i A w_i e_i \underset{i}{\circledast} D_{\frac{1}{\lambda}} T_i \sum_{m=i}^{\deg P} P_m e_m\right)\\
&= \lambda\left(\lambda^i A w_i e_i \underset{i}{\circledast} D_{\frac{1}{\lambda}} T_i P\right)\\
&= \lambda\left(\lambda^i \mathscr{D}_{Aw_i e_i, i} D_{\frac{1}{\lambda}} T_i P\right)\\
&= \lambda A T_i P,
\end{aligned}
$$

thus

$$T_i A P = \lambda A T_i P$$

for all polynomials $p \in X_i$, and therefore $T_i A = \lambda A T_i$, which completes the proof. Theorem 5 is proved. $\square$

Acknowledgements. The authors are grateful to the referee for his very useful remarks, which improved the pesentation of the paper.

This work is supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK) with Project 107T649.

REFERENCES

[1] N. K. NIKOLSKI, *Treatise on the shift operator*, vol. 273, Springer, Berlin, 1986.

[2] M. T. KARAEV, *On extended eigenvalues and extended eigenvectors of some operator classes*, Proc. Amer. Math. Soc., **134** (2006), 2383–2392.

[3] N. K. NIKOLSKI, *Basisness and unicellularity of weighted shift operators*, Izvestiya Akad. Nauk SSSR, Ser. Mat., **32** (1968), 1123–1137 (in Russian).

[4] M. T. KARAEV, *On some applications of the ordinary and extended Duhamel products*, Sibirskii Matem. Zhurnal, **46**, 3(2005), 553–566 (in Russian).

[5] M. GÜRDAL, *Description of extended eigenvalues and extended eigenvectors of integration operator on the Wiener algebra*, Expo. Math., **27**, 2 (2009), 153–160.

[6] M. T. KARAEV, *Addition of spectral multiplicities and invariant subspaces*, Ph. D. Thesis, Baku, 1991 (in Russian).

[7] M. T. KARAEV, *An addition theorem on multiplicities of spectrum*, Trudy IMM AN Azerb., **8**, 16 (1998), 123–128 (in Russian).

[8] A. SHIELDS, *Weighted shift operators and analytic function theory. Topics in operator theory*, 128, Math. Surveys, No:13, Amer. Math. Soc., Providence, R.I., 1974, pp. 49.

[9] V. M. KADETS, *Bases with individual brackets and bases with individual permutations* (Russian), Teor. Funkt. i Funktsional Anal. i Prilozhen, **49** (1988) , 43–51; Translation in J. Soviet. Math., **49** (1990), 1064–1069.

[10] N. K. NIKOLSKI, *Selected problems of weighted approximation and spectral analysis*, Translated from Proceedings of the Steklov Institute of Mathematics, **120** (1974), 1–270; American Math. Soc., Providence, R.I., 1976, 276pp.

[11] L. V. KANTOROVICH AND G. P. AKILOV, *Functional Analysis*, Nauka, Moscow, 1977.

[12] A. BISWAS, A. LAMBERT AND S. PETROVIC, *Extended eigenvalues and the Volterra operator*, Glasgow Math. J., **44** (2002), 521–534.
[13] A. BISWAS, A. LAMBERT AND S. PETROVIC, *On extended eigenvectors for operators*, Preprint.
[14] A. LAMBERT, *Hyperinvariant subspaces and extended eigenvalues*, New York J. Math., **10** (2004), 83–88.
[15] A. LAMBERT AND S. PETROVIC, *Beyond hyperinvariance for compact operators*, J. Funct. Anal., **219** (2005), 93–108.
[16] A. BISWAS, S. PETROVIC, *On extended eigenvalues of operators*, Integral Equations and Operator Theory, **55** (2006), 233–248.
[17] S. PETROVIC, *On the extended eigenvalues of some Volterra operators*, Integral Equations and Operator Theory, **57** (2007), 593–598.
[18] S. SHKARIN, *Compact operators without extended eigenvalues*, J. Math. Anal. Appl., **332** (2007), 455–462.

(Received October 15, 2009)

M.T. Karaev
Suleyman Demirel University
Isparta Vocational School
32260, Isparta
Turkey
e-mail: garayev@fef.sdu.edu.tr

M. Gürdal
Suleyman Demirel University
Department of Mathematics
32260, Isparta
Turkey
e-mail: gurdal@fef.sdu.edu.tr