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FACTORING IN THE METAPLECTIC GROUP AND OPTICS

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Abstract. The metaplectic group is generated by the Fourier transform and multiplications by functions of particular exponential type. Based on the use of the metaplectic representation and a factorization of symplectic matrices, in this paper a bound on the number of terms needed to factor an arbitrary metaplectic operator is derived. The approach is constructive and numerically stable, leading to a reliable factorization algorithm in practice. The problem is partially motivated by the task of constructing lens systems in diffractive optics.

1. Introduction

Unitary operators on $L^2(\mathbb{R}^n)$ arising in diffractive optics and with a family of Schrödinger equations are constant multiples of integral operators of the form

$$f \longmapsto e^{-i(A^TCx,x)/2} \int_{\mathrm{R}(B^T)} e^{-i(B^TDy,y)/2-i(y,B^TCx)} f(By+Ax)dy \tag{1}$$

with $f \in L^2(\mathbb{R}^n)$ [2, 3, 7]. This operator formulation is taken from [7]; see Definition 1 and Theorem 2 therein. Here $A,B,C,D \in \mathbb{R}^{n\times n}$ are the blocks of a symplectic matrix $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $\mathrm{R}(B^T)$ denotes the row-space of the matrix B. (If $\mathrm{R}(B^T) = \{0\}$, then (1) is defined as $f \longmapsto e^{-i(A^TCx,x)/2} f(Ax)$.) These so-called metaplectic operators form the metaplectic group; see [2, 3] where also many historical remarks can be found. This paper is concerned with factoring such operators as a composition of Fourier transforms and multiplications by functions of the form $e^{\frac{i}{2}(Px,x)}$ for a symmetric matrix $P \in \mathbb{R}^{n\times n}$. These two families of operators are known to generate the metaplectic group; see, e.g., [2, Section 1.11].

We derive a bound on the length of generation by showing that the number of terms needed to factor an arbitrary metaplectic operator is at most 13. Based on a dimension count, we believe this to be optimal within a factor of $\frac{3}{2}$. We also show that generically 11 factors suffice. Our demonstration is constructive and numerically stable, allowing us to perform factorizations reliably in practice.

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Aside from being of theoretical interest, such factorizations are needed, for instance, in diffractive optics. (For diffractive optics, see [4] which has an engineering approach as opposed to [2].) There to Fourier transforms and multiplications by functions of the form $e^{\frac{i}{2}(Px,x)}$ correspond to simple lenses that can be used to build more complicated optical lens systems when applied in sequence. Since these operators generate the metaplectic group, any metaplectic operator can be regarded as an optical lens system. However, to construct an optical lens system given in terms of (a constant multiple of) an integral operator (1) in practice, a factorization of the metaplectic operator into its generators is required.

Our factorization is based on the so-called metaplectic representation linking metaplectic operators with symplectic matrices [2]. In this manner the task of factoring an integral operator (1) on $L^2(\mathbb{R}^n)$ is converted into the linear algebra problem of factoring symplectic matrices into factors corresponding to Fourier transforms and multiplications by functions of the form $e^{\frac{i}{2}(Px,x)}$. Once done, the metaplectic representation then gives a desired factorization of (1), modulo a constant which can be readily determined. Thereby the factorization is, in essence, contained in Theorem 3.1 and its proof (which is constructive).

The paper is organized as follows. In Section 2 basic properties of the metaplectic representation are briefly described. In Section 3, a factorization of symplectic matrices in view of the metaplectic representation is constructed. Section 4 is concerned with remarks on how to compute the factorization suggested numerically reliably in practice.

2. Metaplectic operators and their representation

The Wigner distribution of a function $f \in L^2(\mathbb{R}^n)$ is defined as

$$\mathbf{W}[f](x,\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i(\xi,y)} f(x+\frac{1}{2}y)\overline{f(x-\frac{1}{2}y)}dy \tag{2}$$

with $x, \xi \in \mathbb{R}^n$. Then a unitary operator $\mathscr{U}$ on $L^2(\mathbb{R}^n)$ is called metaplectic if it satisfies

$$(\mathbf{W} \circ \mathscr{U})[f] = \mathbf{W}[f] \circ S, \tag{3}$$

for a symplectic matrix $S \in \mathbb{R}^{2n\times 2n}$ [7, p. 246]. Recall that S is symplectic if

$$S^T JS = J, \quad \text{where } J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{4}$$

In particular, J is symplectic itself. For elementary properties of symplectic matrices, see, e.g., [6, Appendix 3]. It is also worth recalling that the set of symplectic matrices $\mathrm{Sp}(2n,\mathbb{R})$ is a subgroup of $\mathbb{R}^{2n\times 2n}$ as well as a smooth manifold of dimension $n(2n+1)$ [8, p. 198].

Forming a subgroup of the group of unitary operators on $L^2(\mathbb{R}^n)$ of the form (1), up to a constant multiple, the set of metaplectic operators $\mathrm{Mp}(2n,\mathbb{R})$ is called the metaplectic group. The metaplectic group is not a matrix group, i.e., it has no finite dimensional faithful representation. However, there exists a one-to-two mapping

$$\mu : \mathrm{Sp}(2n,\mathbb{R}) \to \mathrm{Mp}(2n,\mathbb{R})$$

satisfying

$$\mu(ST) = \pm\mu(S)\mu(T) \text{ for all } S,T \in \mathrm{Sp}(2n,\mathbb{R}). \tag{5}$$

This is defined in terms of a projective representation $\alpha : \mathrm{Sp}(2n,\mathbb{R}) \to \mathrm{Mp}(2n,\mathbb{R})$ for which there holds $\alpha(ST) = c_{ST}\alpha(S)\alpha(T)$ with a constant satisfying $|c_{ST}| = 1$. It is a fairly laborious task to show that the constant can actually be chosen to be ± 1 globally in $\mathrm{Sp}(2n,\mathbb{R})$; see [3, Chapter 4]. This one-to-two mapping (or sometimes its inverse) is called the metaplectic representation.

Consider the symplectic matrix J and symplectic matrices of the form

$$K = \begin{pmatrix} I & 0 \\ -P & I \end{pmatrix} \tag{6}$$

with P symmetric. Their images under the metaplectic representation can be found with the help of (1) as follows. We have

$$\mu(J) = c_J \mathscr{F}^{-1}, \tag{7}$$

where $\mathscr{F}$ denotes the Fourier transform on $L^2(\mathbb{R}^n)$. To $\mu(K)$ corresponds a multiplication by a smooth function as

$$(\mu(K)f)(x) = c_K e^{\frac{i}{2}(Px,x)} f(x) \text{ for } f \in L^2(\mathbb{R}^n). \tag{8}$$

The appearing constants satisfy $|c_J| = |c_K| = 1$. As described in [2, Section 1.11], these two type of operators generate the metaplectic group.

3. Factoring symplectic matrices for the metaplectic representation

Assume given a metaplectic operator (1). In what follows, we factor the appearing symplectic matrix S into a product of matrices of the form (6) and J. By using (5), the metaplectic representation then yields a factorization of (1) into the corresponding operators of type (7) and (8), modulo a constant. The constant can be easily recovered by operating once on a vector $f \in L^2(\mathbb{R}^n)$ with (1) and the factorization generated, and then by comparing the results.

In our factorization we improve on the techniques of [2, Chapter 1] to have an acceptable bound on the number of factors in view of practical construction. Whether optimal, we believe that different ideas are needed to improve these bounds. We give also reasons, based on a simple dimension count, why we believe that we are within a factor of $\frac{3}{2}$ from being optimal.

In our demonstration we need to employ the classical, although not so well-known fact that any square matrix is the product of two symmetric matrices [1]. All the proofs of this result that we are aware of are constructive but numerically unreliable. Therefore, to make this step also computationally viable, different techniques are needed. Here we rely on the algorithm recently proposed in [5] which gives a complete solution to this particular factorization problem.

THEOREM 3.1. *The group* $\mathrm{Sp}(2n,\mathbb{R})$ *is generated by its subgroup*

$$\left\{ \begin{pmatrix} I & 0 \\ -P & I \end{pmatrix} \in \mathbb{R}^{2n\times 2n} \, : \, P = P^T \right\} \tag{9}$$

and the element

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

i.e., every symplectic matrix

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

can be represented as the product of matrices of the above two types. There exists a representation requiring at most 13 generators.

Proof. We will proceed constructively. We are thus allowed to perform matrix products with $2n$-by-$2n$ matrices of the form

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I & 0 \\ -P & I \end{pmatrix},$$

where P is any real-entried symmetric matrix. These products remain symplectic since $\mathrm{Sp}(2n,\mathbb{R})$ is a group.

In what follows, P is symmetric. Observe first that

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}^3, \tag{10}$$

$$\begin{pmatrix} I & P \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -P & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{11}$$

Therefore we can reach any block-diagonal matrix of the form

$$\begin{pmatrix} P & 0 \\ 0 & P^{-1} \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & P^{-1} \\ -P & 0 \end{pmatrix} \tag{12}$$

$$= \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & P^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -P & I \end{pmatrix} \begin{pmatrix} I & P^{-1} \\ 0 & I \end{pmatrix}. \tag{13}$$

Clearly, if P is symmetric and nonsingular, then so is its inverse P^{-1}.

Any square matrix A can be written as the product $A = PQ$ with two symmetric matrices P and Q such that if A is real-entried, then so are P and Q [1, 5]. Therefore we can reach any block-diagonal matrix of the form $\begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}$ as

$$\begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & P^{-1} \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & Q^{-1} \end{pmatrix} \tag{14}$$

by using (13).

With these preparations, consider now an arbitrary symplectic matrix

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Suppose first that A is nonsingular. That both S and S^T being symplectic forces

$$I = A^T D - C^T B = A^T D - C^T A A^{-1} B = A^T D - A^T C A^{-1} B \tag{15}$$

$$A^T C = C^T A \text{ and } AB^T = BA^T \tag{16}$$

by (4). Then using (15) we have $D - CA^{-1}B = (A^T)^{-1}$ and

$$S = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & (A^T)^{-1} \end{pmatrix}.$$

By the fact that

$$\begin{pmatrix} A & B \\ 0 & (A^T)^{-1} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix},$$

we have

$$S = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}. \tag{17}$$

Each of the matrices on the right-hand side of this identity is reachable since CA^{-1} and $A^{-1}B$ are both symmetric by (16). Therefore S with an invertible $(1,1)$-block A is reachable.

Next we prove the claim concerning the number of factors, still assuming A is nonsingular. For notational simplicity, we denote by $K \equiv K(P)$ any generator matrix of the form

$$\begin{pmatrix} I & 0 \\ P & I \end{pmatrix}$$

with P symmetric. First, by (11) we have

$$\begin{pmatrix} I & P \\ 0 & I \end{pmatrix} = JKJ^T. \tag{18}$$

Moreover, transposing (13) and using (18) gives

$$\begin{pmatrix} P & 0 \\ 0 & P^{-1} \end{pmatrix} = KJKJ^T KJ. \tag{19}$$

Equipped with these, by (17) and (14) we can factor

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & P^{-1} \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & Q^{-1} \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}. \tag{20}$$

Using (18) and (19), S can be factored further as

$$\begin{aligned} S &= K(KJKJ^T KJ)(KJKJ^T KJ)JKJ^T \\ &= KJK(-I)JKJ\,KJK(-I)JK(-I)K(-J) \\ &= KJKJKJKJKJKJ \end{aligned} \tag{21}$$

by the fact that

$$J^T = -J \text{ and } J^2 = -I.$$

Consequently, when A is invertible, the number of generators needed is at most 12.

If B is invertible, then consider $S = -(SJ)J$ by factoring the symplectic matrix SJ with an invertible $(1,1)$-block. Then $J^2 = -I$ in the right-end cancels the minus sign in front and the number of factors is 11.

If C is invertible, then consider $S = J(-JS)$ by factoring the symplectic matrix $-JS$ with an invertible $(1,1)$-block. The number of factors is 13.

If D is invertible, then consider $S = -J(-JSJ)J$ by factoring the symplectic matrix $-JSJ$ with an invertible $(1,1)$-block. Then $J^2 = -I$ in the right-end cancels the minus in front and the number of factors is 12.

It remains to consider the case of all the blocks of S being singular. We show how to produce an invertible $(1,1)$-block. For this purpose, applying elementary row and column operations on A, there are invertible n-by-n matrices L and M such that

$$LAM = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \tag{22}$$

where I_r is the r-by-r identity matrix with $r = \operatorname{rank}(A)$. With this, let $\tilde{A} = LAM$ and $\tilde{C} = (L^T)^{-1}CM$. Write $\tilde{C}$ in blocks as $\tilde{C} = \begin{pmatrix} \underset{r\times r}{C_1} & \underset{r\times(n-r)}{C_2} \\ \underset{(n-r)\times r}{C_3} & \underset{(n-r)\times(n-r)}{C_4} \end{pmatrix}$. We have

$$\tilde{A}^T\tilde{C} = \begin{pmatrix} C_1 & C_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where C_1 is symmetric and $C_2 = 0$ by the condition that $\tilde{A}^T\tilde{C}$ be symmetric by (16). Denote $E = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix}$ which is symmetric. We have

$$\tilde{A} + E\tilde{C} = \begin{pmatrix} I_r & 0 \\ C_3 & C_4 \end{pmatrix}.$$

This matrix is invertible. For if it were not, then C_4 would have to be singular, i.e., there exists a nonzero vector $v_4 \in \mathbb{R}^{n-r}$ such that $C_4 v_4 = 0$. Then $\tilde{C}v = \tilde{A}v = 0$ with $v = (0, v_4) \in \mathbb{R}^n$. Therefore $(\tilde{D}^T\tilde{A} - \tilde{B}^T\tilde{C})v = 0$, contradicting $\tilde{D}^T\tilde{A} - \tilde{B}^T\tilde{C} = I$ in (15). Since

$$\tilde{A} + E\tilde{C} = L(A + L^{-1}E(L^T)^{-1}C)M,$$

we can conclude that $A + L^{-1}E(L^T)^{-1}C$ is invertible. Denote by $P = L^{-1}E(L^T)^{-1}$ which is symmetric. This yields us

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & -P \\ 0 & I \end{pmatrix} \begin{pmatrix} A + PC & B + PD \\ C & D \end{pmatrix} \tag{23}$$

Since $A + PC$ is invertible, the rightmost symplectic matrix can be reached by the arguments in the first part of the proof.

For the number of factors, consider the factorization (23). If we had reasoned similarly with S^T, the analogous factorization would read $S^T = K^T\tilde{S}$, i.e., $S = \tilde{S}^T K$. Hence using the factorization (21) with $\tilde{S}^T$ gives us

$$S = (KJKJKJKJKJKJ)K \tag{24}$$

requiring thus at most 13 factors. □

Observe that if we replace each J with $-J$ in (21), then the equality remains since there are 6 replacements in all. To $-J$ corresponds a Fourier transform instead of its inverse.

In particular, as shown in the proof, having an invertible B block is an ideal case by requiring at most 11 factors. This holds generically, i.e., in an open dense subset of $\mathrm{Sp}(2n,\mathbb{R})$.

THEOREM 3.2. *For a generic symplectic matrix* $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *the block* B *is nonsingular.*

Proof. The symplectic matrices are obtained as the image of the exponential map $M \mapsto e^M$ from the set of matrices $M \in \mathbb{R}^{2n\times 2n}$ satisfying $JM^TJ^T = M$. Equivalently,

$$M = \begin{pmatrix} N & S_1 \\ S_2 & -N \end{pmatrix}$$

with $N \in \mathbb{R}^{n\times n}$ and symmetric S_1 and S_2. Treating these matrices entrywise, the polynomial map $(N,S_1,S_2) \mapsto \det(I+M)$ is generically nonzero and therefore $I+M$ is generically invertible. Thus, the Cayley transform

$$M \mapsto (I-M)(I+M)^{-1} = S(M) = \begin{pmatrix} A(M) & B(M) \\ C(M) & D(M) \end{pmatrix}$$

is generically well-defined. Its range consists of symplectic matrices which do not have -1 as an eigenvalue. Since the Cayley transform is bijective, we can conclude that a generic symplectic matrix has this property. By Cramer's rule, the block $B(M)$ is a rational function of the entries of N, S_1 and S_2. Since $(N,S_1,S_2) \mapsto \det B(M)$ is generically nonzero (being nonzero, for example, at $S(M) = J$), we can conclude that generically B is invertible. □

We thus need 11 factors, generically. Based on a dimension count, the best possible number of factors is at least 7, generically. To see this, recall that the dimension of the set of real n-by-n symmetric matrices is $\frac{n(n+1)}{2}$. In the factorization (21) there are 6 factors from (9), hence consuming $6\frac{n(n+1)}{2} = 3(n+1)n$ parameters. As the dimension of $\mathrm{Sp}(2n,\mathbb{R})$ is $n(2n+1)$, we are likely to be within $\frac{3}{2}$ from having an optimal number of factors. (The best possible factorization, if existed, should be of the form $K_1JK_2JK_3JK_4$ to satisfy the dimension count.) This happens generically. In the nongeneric case of the block A being singular, there are 7 factors from (9) in the factorization (24).

For further remarks on the factors, consider the case of a symplectic matrix $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with an invertible A block factored as $A = PQ$ with $P, Q \in \mathbb{R}^{n\times n}$ symmetric. Then we have

$$S = K_1 J K_2 J K_3 J K_4 J K_5 J K_6 J, \tag{25}$$

where

$$K_1 = \begin{pmatrix} I & 0 \\ CA^{-1} + P^{-1} & I \end{pmatrix},\ K_2 = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix},\ K_3 = \begin{pmatrix} I & 0 \\ P^{-1} & I \end{pmatrix},$$

$$K_4 = \begin{pmatrix} I & 0 \\ Q^{-1} & I \end{pmatrix},\ K_5 = \begin{pmatrix} I & 0 \\ Q & I \end{pmatrix} \text{ and } K_6 = \begin{pmatrix} I & 0 \\ Q^{-1} + A^{-1}B & I \end{pmatrix}.$$

Observe that if B is symmetric and invertible, then we can take $P = -B$ and $Q = -B^{-1}A$ to make K_6 disappear. Then we have 9 factors in (25), followed by a multiplication by $-I = J^2$.

Let us emphasize that all the blocks of a symplectic matrix can be singular as the following example illustrates.

EXAMPLE 1. Consider the symplectic matrix $S \in \mathbb{R}^{4\times 4}$ from [7, p.257] with the blocks

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},\ D = A \text{ and } C = -B.$$

Each block is clearly singular. Now, as in the proof of Theorem 3.1, we need to perform an operation with an element from the subgroup (9) that yields an invertible $(1,1)$-block. Here we can proceed directly without resorting to (22). Namely, by taking, for instance, $G = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$ yields

$$SG = \begin{pmatrix} A+B & B \\ C+D & D \end{pmatrix}.$$

The block $A + B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is now invertible. Then $S = (SG)G^{-1}$ with $G^{-1} = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}$ also belonging to the subgroup (9). Thus the number of factors is at most $12 + 1 = 13$.

4. Factoring into symmetric matrices in practice

In what follows, we assume the $(1,1)$-block A of the appearing symplectic matrix $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ to be invertible. The approach is exactly analogous in the singular case, except that then one must employ the expansion (24) instead.

Assume hence given a metaplectic operator V as a constant multiple of (1). To factor V, we need to factor the appearing symplectic matrix S. (To have S, from V we immediately recover A, B and C. Then the condition (15) yields D.) The factors that need to be computed are listed blockwise in (25). We only need to describe how to compute the symmetric factors P and Q in a factorization $A = PQ$. Thereafter the blocks are readily computed by using the given formulae.

Denote by $\mathscr{X} \subset \mathbb{R}^{n\times n}$ the set of symmetric matrices. Let $\mathbf{P}$ be the orthogonal projection on $\mathbb{R}^{n\times n}$ onto $\mathscr{X}$ with respect to the standard inner product

$$(M,N) = \mathrm{trace}(N^T M) \quad \text{for } M,N \in \mathbb{R}^{n\times n}. \tag{26}$$

As is well-known, then $\mathbf{P}M = \frac{1}{2}(M + M^T)$ for any $M \in \mathbb{R}^{n\times n}$. Define a linear operator $\mathscr{L} : \mathscr{X} \to \mathbb{R}^{n\times n}$ as

$$X \longmapsto (I - \mathbf{P})AX = \frac{1}{2}(AX - XA^T). \tag{27}$$

If X is in the nullspace of $\mathscr{L}$, then $AX = P \in \mathscr{X}$ holds. In particular, if X is invertible, then we have $A = PQ$ with $Q = X^{-1}$, yielding a required factorization.

For this approach to be feasible, it is has been shown that the nullspace of $\mathscr{L}$ is at least n dimensional with invertible elements open and dense [5, Theorem 3.1.]. Consequently, with this amount of flexibility, we can conclude that the factorization (25) is far from being unique. How these degrees of freedom should be used depends on where the factorization is aimed at.

That this approach for computing some symmetric factors P and Q is really practical relies on the fact that there are several numerically stable ways for computing the nullspace of $\mathscr{L}$. For instance, one can use algorithms to compute the singular value decomposition of $\mathscr{L}$.

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