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STRONGLY APPROXIMATIVE SIMILARITY OF A DENSE CLASS OF OPERATORS

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Abstract. Two operators A,B on a complex separable Hilbert space $\mathscr{H}$ are said to be strongly approximatively similar, denoted by $A \sim_{sas} B$, if (i) given $\varepsilon > 0$, there exist compact operators K_i with $\|K_i\| < \varepsilon (i = 1,2)$ such that $A+K_1$ and $B+K_2$ are similar; and (ii) $\sigma_0(A) = \sigma_0(B)$ and $\dim \mathscr{H}(\lambda;A) = \dim \mathscr{H}(\lambda;B)$ for each $\lambda \in \sigma_0(A)$. In this paper, we characterize strongly approximative similarity for a class of operators which is dense in $\mathscr{B}(\mathscr{H})$ in the operator norm. As a result, we infer that the relation $\sim_{sas}$ is an equivalence relation for this class of operators. A corresponding classification is accordingly obtained.

1. Introduction

Throughout this paper, $\mathscr{H}$ ($\mathscr{H}_1, \mathscr{H}_2, \cdots, \mathscr{K}, \mathscr{K}_1, \mathscr{K}_2, \cdots$, etc.) will always denote a complex, separable, infinite dimensional Hilbert space. Denote by $\mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ the set of all bounded linear operators mapping $\mathscr{H}_1$ into $\mathscr{H}_2$. $\mathscr{K}(\mathscr{H}_1, \mathscr{H}_2)$ denotes the set of all compact operators in $\mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$. For $T \in \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$, denote the kernel and the range of T by $\ker T$ and $\operatorname{ran} T$ respectively. We simply write $\mathscr{B}(\mathscr{H})$ and $\mathscr{K}(\mathscr{H})$ instead of $\mathscr{B}(\mathscr{H}, \mathscr{H})$ and $\mathscr{K}(\mathscr{H}, \mathscr{H})$ respectively. $\mathscr{A}(\mathscr{H})$ denotes the quotient Calkin algebra $\mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$ and $\pi : \mathscr{B}(\mathscr{H}) \to \mathscr{A}(\mathscr{H})$ is the quotient map of $\mathscr{B}(\mathscr{H})$ onto $\mathscr{A}(\mathscr{H})$.

In operator theory, the classification of operators is in a key position. Perhaps similarity and unitary equivalence are the two most important equivalence relations on $\mathscr{B}(\mathscr{H})$. Two operators $A, B \in \mathscr{B}(\mathscr{H})$ are said to be *similar*, denoted by $A \sim B$, if there exists an invertible operator $X \in \mathscr{B}(\mathscr{H})$ such that $AX = XB$; if, in addition, X is unitary, we say that A and B are *unitarily equivalent*, denoted by $A \simeq B$. Given $T \in \mathscr{B}(\mathscr{H})$, the equivalence class containing T with respect to unitary equivalence (similarity), denoted by $\mathscr{U}(T)$(respectively, $\mathscr{S}(T)$), is called the *unitary orbit* (respectively, *similarity orbit*) of T. Of course, similarity and unitary equivalence can also be defined for operators acting on two different Hilbert spaces.

It is natural that people first restrict attention to special classes of operators. Hellinger's multiplicity theory characterizes the unitary orbits of normal operators. It follows

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from the well-known Putnam–Fuglede theorem that two normal operators are similar if and only if they are unitarily equivalent(see [18]). In 1978, Cowen and Douglas [4] introduced and studied a class of operators which possess an open set of eigenvalues. Today this class of operators is known as Cowen-Douglas operators. Cowen and Douglas gave a unitary classification of Cowen-Douglas operators in terms of complex geometry. In 2005, Jiang, Guo and Ji [15] gave a similarity classification of Cowen-Douglas operators using the ordered K_0-group of the commutant algebra as an invariant. However, it is very difficult to obtain a complete set of similarity invariants or unitary invariants for general operators acting on infinite dimensional Hilbert spaces. So people also investigate weakened notions of similarity and unitary equivalence.

Recall that two operators A and B on $\mathscr{H}$ are said to be *approximately unitarily equivalent* if $\overline{\mathscr{U}(A)} = \overline{\mathscr{U}(B)}$. Here and in what follows, given $\mathscr{E} \subset \mathscr{B}(\mathscr{H})$, let $\overline{\mathscr{E}}$ denote the norm-closure of $\mathscr{E}$. Using Voiculescu's non-commutative Weyl–von Neumann theorem [20], Hadwin [5] characterized the closure of $\mathscr{U}(T)$ for $T \in \mathscr{B}(\mathscr{H})$.

Two operators A and B on $\mathscr{H}$ are said to be *asymptotically similar* if $\overline{\mathscr{S}(A)} = \overline{\mathscr{S}(B)}$(see [9, Chapter 2]). The similarity orbit theorem of Apostol, Fialkow, Herrero and Voiculescu characterizes the closure of $\mathscr{S}(T)$ for $T \in \mathscr{B}(\mathscr{H})$ and hence provides a complete set of asymptotical similarity invariants(see [1, Theorem 9.1/9.2]).

Hadwin [6] introduced and studied the notion of approximate similarity of operators. Two operators A and B on $\mathscr{H}$ are said to be *approximately similar*, if there exists a sequence $\{X_n\}_{n=1}^{\infty}$ of invertible operators such that

$$\sup_n(\|X_n\| \cdot \|X_n^{-1}\|) < \infty \text{ and } X_n^{-1}AX_n \to B(n \to \infty).$$

In [7], Hadwin posed some new questions on approximate similarity. In order to continue our discussion, we first recall some notations and terminologies(see [9, Chapter 1]).

Let $T \in \mathscr{B}(\mathscr{H})$. We denote by $\sigma(T)$ the spectrum of T. If σ is a clopen subset of $\sigma(T)$, then we let $E(\sigma;T)$ denote the *Riesz idempotent* of T corresponding to σ. Denote by $\mathscr{H}(\sigma;T)$ the range of $E(\sigma;T)$. If λ is an isolated point in $\sigma(T)$, we simply write $\mathscr{H}(\lambda;T)$ instead of $\mathscr{H}(\{\lambda\};T)$; if, in addition, $\dim\mathscr{H}(\lambda;T) < \infty$, then λ is called a *normal eigenvalue* of T. The set of all normal eigenvalues of T will be denoted by $\sigma_0(T)$. For given $A \in \mathscr{B}(\mathscr{H}_1)$ and $B \in \mathscr{B}(\mathscr{H}_2)$, it is denoted by $A \stackrel{\mathscr{K}}{\sim} B$ that A is similar to a compact perturbation of B. It is easy to check that the relation $\stackrel{\mathscr{K}}{\sim}$ is an equivalence relation on Hilbert space operators.

In [12], the second author and Li introduced another weakened notion of similarity. Two operators $A, B \in \mathscr{B}(\mathscr{H})$ are said to be *strongly approximatively similar*, denoted by $A \sim_{sas} B$, if

(i) given $\varepsilon > 0$, there exist $K_i \in \mathscr{K}(\mathscr{H})$ with $\|K_i\| < \varepsilon (i = 1,2)$ such that $A + K_1 \sim B + K_2$; and

(ii) $\sigma_0(A) = \sigma_0(B)$ and $\dim\mathscr{H}(\lambda;A) = \dim\mathscr{H}(\lambda;B)$ for each $\lambda \in \sigma_0(A)$.

This is an interesting binary relation on $\mathscr{B}(\mathscr{H})$. Clearly, $\sim_{sas}$ implies $\stackrel{\mathscr{K}}{\sim}$. Also, it is easy to verify that the relation $\sim_{sas}$ is reflexive and symmetric. But it is not obvious whether or not $\sim_{sas}$ is transitive.

It is not difficult to prove that if $\mathscr{H}$ is a finite dimensional Hilbert space, then two operators A, B on $\mathscr{H}$ are strongly approximatively similar if and only if $\sigma(A) = \sigma(B)$ and $\dim \mathscr{H}(\lambda; A) = \dim \mathscr{H}(\lambda; B)$ for each $\lambda \in \sigma(A)$ (equivalently, they have the same characteristic polynomials).

The second author and Li [12] characterized $\sim_{sas}$ for certain classes of quasitriangular operators and hence proved that $\sim_{sas}$ is an equivalence relation for these classes of operators. They posed the following problem.

PROBLEM 1.1. ([12]) Is relation $\sim_{sas}$ always transitive?

In 2004, the second author and Li [13] gave a classification of essentially normal operators. As a consequence of this result, one can deduce that $\sim_{sas}$ is an equivalence relation for essentially normal operators, and a corresponding classification can be easily derived.

Recently the authors [14] have extended the results in [12] to a larger class of operators.

THEOREM 1.2. ([14]) *Let $S, T \in \mathscr{B}(\mathscr{H})$ be quasitriangular satisfying:*

(i) $\sigma(T) = \sigma(S) = \sigma_w(S)$ *is connected and* $\sigma_e(S) = \sigma_{lre}(S)$*;*

(ii) $\rho_{s-F}(S) \cap \sigma(S)$ *consists of at most finite components and each component* Ω *satisfies that* $\Omega = \operatorname{int}\overline{\Omega}$*, where* $\operatorname{int}\overline{\Omega}$ *is the interior of* $\overline{\Omega}$.

Then $S \sim_{sas} T$ *if and only if* $S \overset{\mathscr{K}}{\sim} T$.

In this paper, we develop new techniques to study strongly approximative similarity. We characterize $\sim_{sas}$ for a class $\mathrm{Nic}(\mathscr{H})$ of operators which is dense in $\mathscr{B}(\mathscr{H})$. As a result, we give a positive answer to Problem 1.1 for operators in $\mathrm{Nic}(\mathscr{H})$. To state our main result, we recall some notations and terminologies.

Throughout this paper, $\mathbb{C}$ and $\mathbb{N}$ denote the set of complex numbers and the set of positive integers respectively. For a subset Γ of $\mathbb{C}$, denote by $\operatorname{int}\Gamma$ the interior of Γ. Let $T \in \mathscr{B}(\mathscr{H})$; we shall denote by $\sigma_p(T)$ the point spectrum of T. T is called a *semi-Fredholm operator* if $\operatorname{ran} T$ is closed and either $\operatorname{nul} T$ or $\operatorname{nul} T^*$ is finite, where $\operatorname{nul} T := \dim\ker T$ and $\operatorname{nul} T^* := \dim\ker T^*$; in this case, $\operatorname{ind} T := \operatorname{nul} T - \operatorname{nul} T^*$ is called the *index* of T. Furthermore, if $-\infty < \operatorname{ind} T < \infty$, then T is called a *Fredholm operator*. We denote by $\sigma_e(T)$ the *essential spectrum* of T, that is, $\sigma_e(T) := \sigma(\pi(T))$. The set

$$\rho_{s-F}(T) := \{\lambda \in \mathbb{C} : \lambda - T \text{ is a semi-Fredholm operator }\}$$

is called the *semi-Fredholm domain* of T and $\sigma_{lre}(T) := \mathbb{C} \setminus \rho_{s-F}(T)$ is called the *Wolf spectrum* of T. The *Weyl spectrum* $\sigma_w(T)$ of T is defined by

$$\sigma_w(T) = \sigma_{lre}(T) \cup \{\lambda \in \rho_{s-F}(T) : \operatorname{ind}(\lambda - T) \neq 0\}.$$

Let $T \in \mathscr{B}(\mathscr{H})$. If σ is a clopen subset of $\sigma(T)$, then, by the classical Riesz decomposition theorem(see [19, Theorem 2.10]), $\mathscr{H}(\sigma;T)$ is an invariant space of T and $\sigma(T_\sigma) = \sigma$, where T_σ denotes the restriction of T to $\mathscr{H}(\sigma;T)$, that is, $T_\sigma := T|_{\mathscr{H}(\sigma;T)}$.

We define the following class of operators on $\mathscr{H}$.

DEFINITION 1.3. Let $\mathrm{Nic}(\mathscr{H})$ denote the collection of operators $T \in \mathscr{B}(\mathscr{H})$ which satisfy:

(i) $\sigma(T) = \sigma_w(T) \cup \sigma_0(T)$ and $\sigma(T)$ consists of finite connected components;

(ii) If σ is a connected component of $\sigma_w(T)$, then either $\sigma \subset \sigma_{lre}(T)$ or $\sigma = \overline{\Omega}$, where Ω is a nonempty bounded connected open subset of $\rho_{s-F}(T)$ and $\mathrm{int}\,\overline{\Omega} = \Omega$.

Using Apostol-Morrel simple models [3], one can observe that $\mathrm{Nic}(\mathscr{H})$ is dense in $\mathscr{B}(\mathscr{H})$(see also [9, Theorem 6.1]). The main result of this paper is the following theorem.

THEOREM 1.4. (Main Theorem) *Let* $S,T \in \mathrm{Nic}(\mathscr{H})$. *Then* $S \sim_{sas} T$ *if and only if* $\sigma(S) = \sigma(T)$ *and* $S_\sigma \overset{\mathscr{K}}{\sim} T_\sigma$ *for each clopen subset* σ *of* $\sigma(T)$.

It follows immediately from Theorem 1.4 that $\sim_{sas}$ is an equivalence relation on $\mathrm{Nic}(\mathscr{H})$. Hence the above result gives a classification of operators in $\mathrm{Nic}(\mathscr{H})$ with respect to strongly approximatively similarity.

In Section 2, we shall make some preparations for the proof of Main Theorem. Section 3 is devoted to the proof of Main Theorem.

2. Preparation

It is convenient to introduce some notations and terminologies.

Given $\delta > 0$ and a subset σ of $\mathbb{C}$, denote $\sigma_\delta := \{z \in \mathbb{C} : \mathrm{dist}(z,\sigma) < \delta\}$ and $\sigma^* := \{z \in \mathbb{C} : \bar{z} \in \sigma\}$. Recall that an operator T on $\mathscr{H}$ is said to be *quasitriangular*(see [8]), if there is a sequence $\{P_n\}_{n=1}^{\infty}$ of finite-rank orthogonal projections increasing to the unit operator I with respect to the strong operator topology such that $\lim_{n\to\infty} \|(I - P_n)TP_n\| = 0$. It is well-known that T is quasitriangular if and only if $\mathrm{ind}(\lambda - T) \geqslant 0$ for all $\lambda \in \rho_{s-F}(T)$(see [2]). If both T and T^* are quasitriangular, then we say that T is *biquasitriangular*.

An operator T on $\mathscr{H}$ is said to be an $\mathscr{I}+\mathscr{K}$ operator, denoted by $T \in (\mathscr{I}+\mathscr{K})$, if T is invertible and T is a compact perturbation of the unit operator on $\mathscr{H}$. Clearly, $T \in (\mathscr{I}+\mathscr{K})$ implies $T^{-1} \in (\mathscr{I}+\mathscr{K})$. Denote by $(\mathscr{I}+\mathscr{K})(\mathscr{H})$ the set of all $\mathscr{I}+\mathscr{K}$ operators on $\mathscr{H}$. Two operators $A,B \in \mathscr{B}(\mathscr{H})$ are said to be $(\mathscr{I}+\mathscr{K})$-*similar*, denoted by $A \sim_{\mathscr{I}+\mathscr{K}} B$, if there exists $X \in (\mathscr{I}+\mathscr{K})(\mathscr{H})$ such that $AX = XB$. Note that $A \sim_{\mathscr{I}+\mathscr{K}} B$ implies that A is a compact perturbation of B. If $T \in \mathscr{B}(\mathscr{H})$, $\mathscr{M}$ is a subspace of $\mathscr{H}$ and $1 \leqslant n \leqslant \infty$, we let $P_{\mathscr{M}}$ denote the orthogonal projection of $\mathscr{H}$ onto $\mathscr{M}$, and let $T^{(n)}$ denote the operator $\oplus_{i=1}^n T$ acting on $\oplus_{i=1}^n \mathscr{H}$ (orthogonal direct sum of n copies of $\mathscr{H}$). Given $A \in \mathscr{B}(\mathscr{H}_1)$ and $B \in \mathscr{B}(\mathscr{H}_2)$, the operator $\tau_{A,B} : X \mapsto AX - XB$ is called the Rosenblum operator induced by A and B. Obviously $\tau_{A,B}$ is a bounded linear operator on $\mathscr{B}(\mathscr{H}_2, \mathscr{H}_1)$.

An operator T on a Hilbert space $\mathscr{H}$ is said to be a Cowen-Douglas operator if there exist Ω, a connected open subset of $\mathbb{C}$, and n, a positive integer, such that

(i) $\Omega \subset \sigma(T)$;

(ii) $\mathrm{nul}(\lambda - T) = n$ for $\lambda \in \Omega$;

(iii) $\mathrm{ran}(\lambda - T) = \mathscr{H}$ for $\lambda \in \Omega$; and

(iv) $\vee\{\ker(\lambda - T) : \lambda \in \Omega\} = \mathscr{H}$.

The collection of such operators T is denoted by $\mathscr{B}_n(\Omega)$. Under the assumption of (i), (ii) and (iii), the condition (iv) in the above definition is equivalent to any one of the following conditions(see [4] or [16, Proposition 1.41]):

(v) $\vee\{\ker(\lambda - T)^n : n \geqslant 1\} = \mathscr{H}$ for each $\lambda \in \Omega$;

(vi) $\vee\{\ker(\lambda_n - T) : n \geqslant 1\} = \mathscr{H}$ for all sequences $\{\lambda_n\}_{n=1}^{\infty} \subset \Omega$ with $\lambda_n \to \lambda_1 (n \to \infty)$.

The reader is referred to [4] or [16, Chapter 1] for more results on Cowen-Douglas operators.

LEMMA 2.1. ([12], Lemma 2.2) *Let $A, B \in \mathscr{B}(\mathscr{H})$ satisfy the following conditions:*

(i) $B = A + K_0, K_0 \in \mathscr{K}(\mathscr{H})$;

(ii) $\sigma(A)$ is a connected infinite set and $\sigma(A) = \sigma(B)$;

(iii) There exists a denumerable dense subset $\Gamma = \{\lambda_i : i \geqslant 1\}$ of $\sigma(A)$ such that $\vee\{\ker(\lambda_i - A) : i \geqslant 1\} = \mathscr{H}$ and $\mathrm{nul}(\lambda_i - A) = 1$ for $i \geqslant 1$.

Then, given $\varepsilon > 0$, there exists $K \in \mathscr{K}(\mathscr{H})$ with $\|K\| < \varepsilon$ such that $A \sim_{\mathscr{I}+\mathscr{K}} B + K$.

LEMMA 2.2. ([12], Proposition 3.1) *Suppose that Ω is a nonempty bounded connected open subset of $\mathbb{C}$ and $k \in \mathbb{N}$. Let $B \in \mathscr{B}_1(\Omega)$ satisfy $\sigma(B) = \overline{\Omega}$ and $S = B^{(k)}$. If T is a compact perturbation of S and $\sigma(T) = \sigma(S)$, then, given $\varepsilon > 0$, there exists K compact with $\|K\| < \varepsilon$ such that $T + K \sim_{\mathscr{I}+\mathscr{K}} S$.*

LEMMA 2.3. ([16], Lemma 1.10) *Let $A, B \in \mathscr{B}(\mathscr{H})$ and denote $\tau = \tau_{A,B}|_{\mathscr{K}(\mathscr{H})}$. Then $\tau^* = -\tau_{B,A}|_{C^1(\mathscr{H})}$ and $(\tau^*)^* = \tau_{A,B} \in \mathscr{B}(\mathscr{B}(\mathscr{H}))$, where $C^1(\mathscr{H})$ is the Schatten 1-class on $\mathscr{H}$ and τ^* is the dual operator of τ.*

LEMMA 2.4. *Let $A_i \in \mathscr{B}(\mathscr{H}_i)(i=1,2)$ and $C, C_0 \in \mathscr{B}(\mathscr{H}_2, \mathscr{H}_1)$ satisfying that $C - C_0$ is compact. Suppose that*

$$S = \begin{bmatrix} A_1 & C \\ 0 & A_2 \end{bmatrix} \begin{matrix} \mathscr{H}_1 \\ \mathscr{H}_2 \end{matrix}, \quad T = \begin{bmatrix} A_1 & C_0 \\ 0 & A_2 \end{bmatrix} \begin{matrix} \mathscr{H}_1 \\ \mathscr{H}_2 \end{matrix}.$$

If $\ker \tau_{A_2,A_1} \cap \mathscr{K}(\mathscr{H}_1, \mathscr{H}_2) = \{0\}$, then, given $\varepsilon > 0$, there exists $K \in \mathscr{K}(\mathscr{H}_1 \oplus \mathscr{H}_2)$ with $\|K\| < \varepsilon$ such that $T + K \sim_{\mathscr{I}+\mathscr{K}} S$.

Proof. Since $\ker \tau_{A_2,A_1}|_{\mathscr{K}(\mathscr{H}_1,\mathscr{H}_2)} = \{0\}$, it follows from Lemma 2.3 that $\tau_{A_1,A_2}|_{\mathscr{K}(\mathscr{H}_2,\mathscr{H}_1)}$ has dense range. Then there exist $E, \overline{K} \in \mathscr{K}(\mathscr{H}_2, \mathscr{H}_1)$ with $\|\overline{K}\| < \varepsilon$ such that $A_1 E - E A_2 = C_0 - C + \overline{K}$. Set

$$K = \begin{bmatrix} 0 & \overline{K} \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathscr{H}_1 \\ \mathscr{H}_2 \end{matrix}, \quad X = \begin{bmatrix} I_1 & E \\ 0 & I_2 \end{bmatrix} \begin{matrix} \mathscr{H}_1 \\ \mathscr{H}_2 \end{matrix},$$

where I_i is the unit operator on $\mathscr{H}_i(i=1,2)$. Then $X \in (\mathscr{I} + \mathscr{K})$ and K is compact satisfying $\|K\| < \varepsilon$. A direct computation shows that $X(T+K)X^{-1} = S$. This completes the proof. $\square$

LEMMA 2.5. ([16], Proposition 1.14) *Let $A, B \in \mathscr{B}(\mathscr{H})$. Assume that*

$$\mathscr{H} = \vee\{\ker(\lambda - A)^k : \lambda \in \Gamma, k \geqslant 1\}$$

for a certain subset Γ of $\sigma_p(A)$, and $\sigma_p(B) \cap \Gamma = \emptyset$. Then $\ker \tau_{B,A} = \{0\}$.

LEMMA 2.6. *Let $A, B, C \in \mathscr{B}(\mathscr{H})$ and assume that there exists a countable subset Γ_1 of $\sigma(A)$ and a countable subset Γ_2 of $\sigma(B)$ satisfying the following conditions:*

(i) $\sigma_p(A) \cap \Gamma_2 = \emptyset = \sigma_p(B) \cap \Gamma_1$ *and* $\mathrm{nul}(A - \lambda) = 1 = \mathrm{nul}(B - \mu)$ *for all* $\lambda \in \Gamma_1, \mu \in \Gamma_2$*;*

(ii) $\vee\{\ker(A - \lambda) : \lambda \in \Gamma_1\} = \mathscr{H} = \vee\{\ker(B - \lambda) : \lambda \in \Gamma_2\}$.

Then, given $\varepsilon > 0$, there exists $K \in \mathscr{K}(\mathscr{H})$ with $\|K\| < \varepsilon$ such that $\mathrm{nul}(T - \lambda) = 1$ *for all $\lambda \in \Gamma_1 \cup \Gamma_2$ and $\vee\{\ker(T - \lambda) : \lambda \in \Gamma_1 \cup \Gamma_2\} = \mathscr{H} \oplus \mathscr{H}$, where*

$$T = \begin{bmatrix} A & C+K \\ 0 & B \end{bmatrix} \begin{matrix} \mathscr{H} \\ \mathscr{H} \end{matrix}.$$

Proof. Assume that $\Gamma_1 = \{a_i : i \geqslant 1\}$ and $\Gamma_2 = \{b_i : i \geqslant 1\}$, where $a_i \neq a_j$ and $b_i \neq b_j$ for $i \neq j$. By hypothesis, we can choose two orthonormal bases $\{e_i\}_{i=1}^{\infty}, \{f_i\}_{i=1}^{\infty}$

of $\mathscr{H}$ such that

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \left[\begin{array}{cccc|cccc} a_1 & a_{1,2} & a_{1,3} & \cdots & c_{1,1} & c_{1,2} & c_{1,3} & \cdots \\ & a_2 & a_{2,3} & \cdots & c_{2,1} & c_{2,2} & c_{2,3} & \cdots \\ & & a_3 & \cdots & c_{3,1} & c_{3,2} & c_{3,3} & \cdots \\ & & & \ddots & \vdots & \vdots & \vdots & \ddots \\ \hline & & & & b_1 & b_{1,2} & b_{1,3} & \cdots \\ & & & & & b_2 & b_{2,3} & \cdots \\ & & & & & & b_3 & \cdots \\ & & & & & & & \ddots \end{array}\right] \begin{array}{c} e_1 \\ e_2 \\ e_3 \\ \vdots \\ f_1 \\ f_2 \\ f_3 \\ \vdots \end{array}.$$

For each $j \in \mathbb{N}$, since $\sum_{i=1}^{\infty} |c_{i,j}|^2 < \infty$, there exists $k_j \in \mathbb{N}$ such that $\sum_{i=k_j+1}^{\infty} |c_{i,j}|^2 < (\frac{\varepsilon}{2^j})^2$. We may also assume that $k_1 < k_2 < k_3 < \cdots$. Then it is easy to see that there exists $K \in \mathscr{K}(\mathscr{H})$ with $\|K\| < \varepsilon$ such that

$$\overline{C} := C + K = \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ c_{k_1,1} & \vdots & \vdots & \cdots \\ 0 & \vdots & \vdots & \cdots \\ 0 & c_{k_2,2} & \vdots & \cdots \\ \vdots & 0 & \vdots & \cdots \\ & 0 & c_{k_3,3} & \cdots \\ & \vdots & 0 & \cdots \\ & & \vdots & \ddots \end{bmatrix}.$$

Set $T = \begin{bmatrix} A & \overline{C} \\ 0 & B \end{bmatrix}$. It follows from $\sigma_p(B) \cap \Gamma_1 = \emptyset$ that $\mathrm{nul}(T - a_i) = 1$ for all $i \geqslant 1$. On the other hand, $\vee\{\ker(\lambda - A) : \lambda \in \Gamma_1\} = \mathscr{H}$ implies $\vee\{\ker(\lambda - T) : \lambda \in \Gamma_1\} \supset \mathscr{H} \oplus \{0\}$. Then, to complete the proof, it suffices to prove that $\mathrm{nul}(T - b_i) = 1$ for all $i \geqslant 1$ and $(0, f_i) \in \vee\{\ker(\lambda - T) : \lambda \in \Gamma_1 \cup \Gamma_2\}$ for all $i \geqslant 1$.

Since $f_1 \in \ker(B - b_1)$, $\overline{C} f_1 \in \vee\{e_1, e_2, \cdots, e_{k_1}\} \subset \mathrm{ran}(A - b_1)$, there exists $x_1 \in \mathscr{H}$ such that $(A - b_1)x_1 + \overline{C} f_1 = 0$. Then $(x_1, f_1) \in \ker(T - b_1) \subset \vee\{\ker(\lambda - T) : \lambda \in \Gamma_1 \cup \Gamma_2\}$ and hence $(0, f_1) \in \vee\{\ker(\lambda - T) : \lambda \in \Gamma_1 \cup \Gamma_2\}$. Since $b_1 \notin \sigma_p(A)$, we obtain $\mathrm{nul}(T - b_1) = 1$.

Note that there exists $\lambda \in \mathbb{C}$ such that $\lambda f_1 + f_2 \in \ker(B - b_2)$ and

$$\overline{C}(\lambda f_1 + f_2) \in \vee\{e_1, e_2, \cdots, e_{k_2}\} \subset \mathrm{ran}(A - b_2),$$

using a similar argument as above, one can check that $\mathrm{nul}(T - b_2) = 1$ and $(0, \lambda f_1 + f_2) \in \vee\{\ker(T - \lambda) : \lambda \in \Gamma_1 \cup \Gamma_2\}$. Moreover, $(0, f_2) \in \vee\{\ker(T - \lambda) : \lambda \in \Gamma_1 \cup \Gamma_2\}$.

By using a similar method as above, we can prove that $\mathrm{nul}(T-b_i)=1$ and $(0,f_i)\in\vee\{\ker(T-\lambda):\lambda\in\Gamma_1\cup\Gamma_2\}$ for all $i\in\mathbb{N}$. This completes the proof. $\square$

PROPOSITION 2.7. *Let* $S=\begin{bmatrix}A & R\\ 0 & B^{(n)}\end{bmatrix}\begin{matrix}\mathscr{H}\\ \mathscr{K}\end{matrix}$ *satisfy the following conditions:*

(i) $n\in\mathbb{N}, B\in\mathscr{B}_1(\Omega), \sigma_p(B)=\Omega$ *and* $\sigma(B)=\overline{\Omega}=\sigma(A)=\sigma_{lre}(A)$*, where* Ω *is a nonempty bounded connected open subset of* $\mathbb{C}$*;*

(ii) $\Gamma:=\sigma_p(A)$ *is a denumerable dense subset of* $\partial\Omega$ *such that* $\bigvee_{\lambda\in\Gamma}\ker(\lambda-A)=\mathscr{H}$ *and* $\mathrm{nul}(\lambda-A)=1$ *for all* $\lambda\in\Gamma$.

If T *is a compact perturbation of* S *with* $\sigma(T)=\sigma(S)$*, then, given* $\varepsilon>0$*, there exists* $K\in\mathscr{K}(\mathscr{H}\oplus\mathscr{K})$ *with* $\|K\|<\varepsilon$ *such that* $T+K\sim_{\mathscr{I}+\mathscr{K}}S$*.*

Proof. Without loss of generality, we assume that $0\in\Omega$. For each $j\in\mathbb{N}$, set

$$P_j=I_{\mathscr{H}}\oplus P^{(n)}_{\ker B^j}=\begin{bmatrix}I_{\mathscr{H}} & & & \\ & P_{\ker B^j} & & \\ & & \ddots & \\ & & & P_{\ker B^j}\end{bmatrix},$$

where $I_{\mathscr{H}}$ is the unit operator on $\mathscr{H}$. Then $\{P_j\}_{i=1}^{\infty}$ is a sequence of orthogonal projections and it follows from the definition of Cowen-Douglas operators that $P_j\overset{\mathrm{SOT}}{\to}I$ (the unit operator on $\mathscr{H}\oplus\mathscr{K}$). Here SOT denotes the strong operator topology. Assume that $T=S+K_0$, where K_0 is compact. Thus, for given $\varepsilon>0$, there exists j_0 such that $\|P_{j_0}K_0P_{j_0}-K_0\|<\frac{\varepsilon}{6}$ and $\sigma(T+P_{j_0}K_0P_{j_0}-K_0)\subset\sigma(T)_{\frac{\varepsilon}{6}}$. Set $K_1=P_{j_0}K_0P_{j_0}-K_0$. Then $T+K_1=S+P_{j_0}K_0P_{j_0}$. Assume that

$$B=\begin{bmatrix}B_0 & *\\ 0 & B_1\end{bmatrix}\begin{matrix}\ker B^{j_0}\\ \mathrm{ran}\,(B^*)^{j_0}\end{matrix},$$

thus

$$S=\begin{bmatrix}A & R_1 & R_2\\ 0 & B_0^{(n)} & *\\ 0 & 0 & B_1^{(n)}\end{bmatrix}=\begin{bmatrix}A & R_1 & R_{2,1} & R_{2,2}\\ 0 & B_0^{(n)} & H & *\\ 0 & 0 & B_1 & 0\\ 0 & 0 & 0 & B_1^{(n-1)}\end{bmatrix},$$

where $\begin{bmatrix}R_1, R_2\end{bmatrix}=R$ and $\begin{bmatrix}R_{2,1}, R_{2,2}\end{bmatrix}=R_2$. We note that both H and R_1 are compact. Denote $C=\begin{bmatrix}A & R_1\\ 0 & B_0^{(n)}\end{bmatrix}$. It is easily seen that C is an operator acting on $\mathrm{ran}\,P_{j_0}$. Set

$$\overline{C}=C+P_{j_0}K_0|_{\mathrm{ran}\,P_{j_0}}, E=\begin{bmatrix}R_{2,1}\\ H\end{bmatrix}\text{ and }D=\begin{bmatrix}\overline{C} & E\\ 0 & B_1\end{bmatrix}.$$

Then

$$T+K_1=S+P_{j_0}K_0P_{j_0}=\begin{bmatrix}\begin{bmatrix}\overline{C} & E\\ 0 & B_1\end{bmatrix} & \begin{matrix}*\\0\end{matrix}\\ 0 & B_1^{(n-1)}\end{bmatrix}\triangleq\begin{bmatrix}D & *\\ 0 & B_1^{(n-1)}\end{bmatrix}.$$

Obviously, $\overline{C}$ is biquasitriangular and

$$\sigma_{lre}(\overline{C})=\sigma_{lre}(C)=\overline{\Omega}=\sigma_{lre}(A)=\sigma(A).$$

It is easy to prove that if $\lambda\in\sigma_0(\overline{C})$, then $\lambda\notin\overline{\Omega}$ and hence $\lambda-B_1$ is invertible. This implies that $\lambda\in\sigma_0(T+K_1)$. Then $\sigma_0(\overline{C})\subset\sigma_0(T+K_1)$. Likewise, one can verify that $\sigma_0(T+K_1)\subset\sigma_0(\overline{C})$. Hence $\sigma_0(\overline{C})=\sigma_0(T+K_1)$. Since $\sigma_0(T+K_1)\subset\sigma(T+K_1)\subset\sigma(T)_{\frac{\varepsilon}{6}}=\sigma_{lre}(\overline{C})_{\frac{\varepsilon}{6}}$, we obtain $\sigma_0(\overline{C})\subset\sigma_{lre}(\overline{C})_{\frac{\varepsilon}{6}}$ and hence

$$\max\{\mathrm{dist}(\lambda,\partial\rho_{s-F}(\overline{C})):\lambda\in\sigma_0(\overline{C})\}=\max\{\mathrm{dist}(\lambda,\sigma_{lre}(\overline{C})):\lambda\in\sigma_0(\overline{C})\}<\frac{\varepsilon}{6}.$$

By [9, Theorem 3.48], we may directly assume that $\sigma(\overline{C})=\sigma_{lre}(\overline{C})=\overline{\Omega}$. Hence it is easy to check that $\sigma(D)=\sigma_{lre}(D)=\overline{\Omega}$. Set

$$G_0=\begin{bmatrix}A & R_1 & R_{2,1}\\ 0 & J & f\otimes e\\ 0 & 0 & B_1\end{bmatrix},$$

where J is a nj_0-order Jordan block acting on the underlying space of $B_0^{(n)}$ with $\sigma(J)=\{0\}$, $e\in\ker B_1$ with $\|e\|=1$ and $f\in\ker J^*$ with $\|f\|=1$. Then G_0 is a compact perturbation of D. Note that $\begin{bmatrix}J & f\otimes e\\ 0 & B_1\end{bmatrix}$ is similar to B(see [13, Lemma 3.1]), then, by Lemma 2.6, there exist a compact perturbation $\overline{R_1}$ of R_1 and a compact perturbation $\overline{R_{2,1}}$ of $R_{2,1}$ such that

$$G:=\begin{bmatrix}A & \overline{R_1} & \overline{R_{2,1}}\\ 0 & J & f\otimes e\\ 0 & 0 & B_1\end{bmatrix}$$

satisfies the condition (iii) of Lemma 2.1. Obviously G is a compact perturbation of D and $\sigma(G)=\overline{\Omega}=\sigma(D)$. By Lemma 2.1, there exist $X_1\in(\mathscr{I}+\mathscr{K})$ and a compact K_2 with $\|K_2\|<\frac{\varepsilon}{6}$ such that

$$X_1(T+K_1+K_2)X_1^{-1}=\begin{bmatrix}G & *\\ 0 & B_1^{(n-1)}\end{bmatrix}=\begin{bmatrix}A & \overline{R_1} & \overline{R_{2,1}} & *\\ 0 & J & f\otimes e & F_1\\ 0 & 0 & B_1 & F_2\\ 0 & 0 & 0 & B_1^{(n-1)}\end{bmatrix}.$$

It is easy to see that F_1,F_2 are compact, then $L:=\begin{bmatrix}J & f\otimes e & F_1\\ 0 & B_1 & F_2\\ 0 & 0 & B_1^{(n-1)}\end{bmatrix}$ is a compact perturbation of $B^{(n)}$ and $\sigma(L)=\sigma(B^{(n)})$. By Lemma 2.2, there exist $X_2\in(\mathscr{I}+\mathscr{K})$

and K_3 compact with $\|K_3\| < \frac{\varepsilon}{6}$ such that

$$X_2X_1(T + \sum_{i=1}^{3} K_i)X_1^{-1}X_2^{-1} = \begin{bmatrix} A & R_0 \\ 0 & B^{(n)} \end{bmatrix}.$$

Observe that $R_0 - R$ is compact and, by Lemma 2.5, $\ker \tau_{B^{(n)},A} = \{0\}$. Using Lemma 2.4, there exist $X_3 \in (\mathscr{I} + \mathscr{K})$ and K_4 compact with $\|K_4\| < \frac{\varepsilon}{6}$ such that

$$X_3X_2X_1(T + \sum_{i=1}^{4} K_i)X_1^{-1}X_2^{-1}X_3^{-1} = \begin{bmatrix} A & R \\ 0 & B^{(n)} \end{bmatrix} = S.$$

Since $X_3X_2X_1 \in (\mathscr{I} + \mathscr{K})$ and $\sum_{i=1}^{4} K_i$ is compact with $\|\sum_{i=1}^{4} K_i\| < \varepsilon$, we conclude the proof. $\square$

PROPOSITION 2.8. *Let S be an operator on $\mathscr{H} \oplus \mathscr{K}^{(\infty)}$ which can be written as*

$$S = \begin{bmatrix} A & R \\ 0 & B^{(\infty)} \end{bmatrix} = \begin{array}{c} \begin{bmatrix} A & R_1 & R_2 & R_3 & \cdots \\ 0 & B & 0 & 0 & \cdots \\ 0 & 0 & B & 0 & \cdots \\ 0 & 0 & 0 & B & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array} \begin{array}{c} \mathscr{H} \\ \mathscr{K} \\ \mathscr{K} \\ \mathscr{K} \\ \vdots \end{array},$$

where

(i) $B \in \mathscr{B}_1(\Omega), \sigma_p(B) = \Omega$ and $\sigma(B) = \overline{\Omega} = \sigma(A) = \sigma_{lre}(A)$, where Ω is a nonempty bounded connected open subset of $\mathbb{C}$;

(ii) $\Gamma := \sigma_p(A)$ is a denumerable dense subset of $\partial\Omega$ such that $\vee\{\ker(\lambda - A) : \lambda \in \Gamma\} = \mathscr{H}$ and $\mathrm{nul}(\lambda - A) = 1$ for all $\lambda \in \Gamma$.

If T is a compact perturbation of S with $\sigma(T) = \sigma(S)$, then, given $\varepsilon > 0$, there exists a compact operator K with $\|K\| < \varepsilon$ such that $T + K \sim_{\mathscr{I}+\mathscr{K}} S$.

Proof. Assume that $S = T + K_0$, where K_0 is compact. For each $j \in \mathbb{N}$, set

$$P_j = I_{\mathscr{H}} \oplus I_{\mathscr{K}}^{(j)} \oplus 0^{(\infty)} = \begin{bmatrix} I_{\mathscr{H}} & & & & & \\ & I_{\mathscr{K}} & & & & \\ & & \ddots & & & \\ & & & I_{\mathscr{K}} & & \\ & & & & 0 & \\ & & & & & \ddots \end{bmatrix} \begin{array}{c} \mathscr{H} \\ \mathscr{K} \\ \vdots \\ \mathscr{K} \\ \mathscr{K} \\ \vdots \end{array},$$

where $I_{\mathscr{H}}$ is the unit operator on $\mathscr{H}$ and $I_{\mathscr{K}}$ is the unit operator on $\mathscr{K}$. Then it is obvious that $P_j \xrightarrow{SOT} I$ (the unit operator on $\mathscr{H} \oplus \mathscr{K}^{(\infty)}$). Thus, for given $\varepsilon > 0$, there exists j_0 such that $\|P_{j_0}K_0P_{j_0}-K_0\|<\frac{\varepsilon}{6}$ and $\sigma(T+P_{j_0}K_0P_{j_0}-K_0)\subset\sigma(T)_{\frac{\varepsilon}{6}}$. Denote

$$P_{j_0}K_0P_{j_0}=\begin{bmatrix}\overline{K_0} & 0\\ 0 & 0\end{bmatrix}\begin{matrix}\operatorname{ran}P_{j_0}\\ \operatorname{ran}(I-P_{j_0})\end{matrix}, S_1=\begin{bmatrix}A & R_1 & R_2 & \cdots & R_{j_0}\\ & B & & & \\ & & B & & \\ & & & \ddots & \\ & & & & B\end{bmatrix}\begin{matrix}\mathscr{H}\\ \mathscr{K}\\ \mathscr{K}\\ \vdots\\ \mathscr{K}\end{matrix}.$$

Set $K_1=P_{j_0}K_0P_{j_0}-K_0$ and $\overline{S_1}=S_1+\overline{K_0}$. Then $\overline{S_1}$ is biquasitriangular, $\sigma_{lre}(\overline{S_1})=\overline{\Omega}$ and

$$\begin{aligned}T+K_1=S+P_{j_0}K_0P_{j_0}&=\begin{bmatrix}S_1 & *\\ 0 & B^{(\infty)}\end{bmatrix}\begin{matrix}\operatorname{ran}P_{j_0}\\ \operatorname{ran}(I-P_{j_0})\end{matrix}+\begin{bmatrix}\overline{K_0} & 0\\ 0 & 0\end{bmatrix}\begin{matrix}\operatorname{ran}P_{j_0}\\ \operatorname{ran}(I-P_{j_0})\end{matrix}\\ &=\begin{bmatrix}\overline{S_1} & *\\ 0 & B^{(\infty)}\end{bmatrix}\begin{matrix}\operatorname{ran}P_{j_0}\\ \operatorname{ran}(I-P_{j_0})\end{matrix}.\end{aligned}$$

It is easy to check that $\sigma_0(\overline{S_1})\subset\sigma_0(T+K_1)\subset\sigma(T+K_1)\subset\sigma(T)_{\frac{\varepsilon}{6}}=\sigma_{lre}(\overline{S_1})_{\frac{\varepsilon}{6}}$. By [9, Theorem 3.48], we may directly assume that $\sigma(\overline{S_1})=\sigma_{lre}(\overline{S_1})=\overline{\Omega}=\sigma(S_1)$. Note that $\overline{S_1}$ is a compact perturbation of S_1 and S_1 satisfies the hypothesis of Proposition 2.7, then, by Proposition 2.7, there exist $X_1\in(\mathscr{I}+\mathscr{K})$ and $K_2\in\mathscr{K}(\mathscr{H}\oplus\mathscr{K}^{(\infty)})$ with $\|K_2\|<\frac{\varepsilon}{6}$ such that

$$X_1(T+K_1+K_2)X_1^{-1}=\begin{bmatrix}S_1 & *\\ 0 & B^{(\infty)}\end{bmatrix}\begin{matrix}\operatorname{ran}P_{j_0}\\ \operatorname{ran}(I-P_{j_0})\end{matrix}=\begin{bmatrix}A & * & *\\ 0 & B^{(n)} & E\\ 0 & 0 & B^{(\infty)}\end{bmatrix}.$$

Obviously E is compact. By [16, Lemma 3.10], a straightforward computation shows that if X is compact and $B^{(\infty)}X-XB^{(n)}=0$, then $X=0$. By Lemma 2.4, there exist $X_2\in(\mathscr{I}+\mathscr{K})$ and $K_3\in\mathscr{K}(\mathscr{H}\oplus\mathscr{K}^{(\infty)})$ with $\|K_3\|<\frac{\varepsilon}{6}$ such that

$$X_2X_1(T+\sum_{i=1}^{3}K_i)X_1^{-1}X_2^{-1}=\begin{bmatrix}A & R_0\\ 0 & B^{(\infty)}\end{bmatrix}\begin{matrix}\mathscr{H}\\ \mathscr{K}^{(\infty)}\end{matrix}.$$

Note that $R-R_0$ is compact. Since $\Gamma\cap\sigma_p(B^{(\infty)})=\emptyset$ and $\vee\{\ker(\lambda-A):\lambda\in\Gamma\}=\mathscr{H}$, it follows from Lemma 2.5 that $\ker\tau_{B^{(\infty)},A}=\{0\}$. Using Lemma 2.4, we can choose $X_3\in(\mathscr{I}+\mathscr{K})$ and $K_4\in\mathscr{K}(\mathscr{H}\oplus\mathscr{K}^{(\infty)})$ with $\|K_4\|<\frac{\varepsilon}{6}$ such that

$$X_3X_2X_1(T+\sum_{i=1}^{4}K_i)X_1^{-1}X_2^{-1}X_3^{-1}=\begin{bmatrix}A & R\\ 0 & B^{(\infty)}\end{bmatrix}\begin{matrix}\mathscr{H}\\ \mathscr{K}^{(\infty)}\end{matrix}=S.$$

Note that $K:=\sum_{i=1}^{4}K_i$ is compact with $\|K\|<\varepsilon$ and $X_3X_2X_1\in(\mathscr{I}+\mathscr{K})$. So we complete the proof. $\square$

3. Proof of Main Theorem

First, we give some useful lemmas.

LEMMA 3.1. ([2] or [17] Lemma 3.2.6) *For $T \in \mathscr{B}(\mathscr{H})$, a nonempty set $\Gamma \subset \sigma_{lre}(T)$ and $\varepsilon > 0$, there exists a compact operator K with $\|K\| < \varepsilon$ such that $T + K = \begin{bmatrix} A & * \\ 0 & N \end{bmatrix}$, where N is a diagonal normal operator of uniform infinite multiplicity, $\sigma(N) = \overline{\Gamma}, \sigma(A) = \sigma(T), \sigma_{lre}(A) = \sigma_{lre}(T)$ and $\operatorname{ind}(\lambda - A) = \operatorname{ind}(\lambda - T)$ for each $\lambda \in \rho_{s-F}(T)$.*

Let Ω be a nonempty bounded connected open subset of $\mathbb{C}$ such that $\operatorname{int} \overline{\Omega} = \Omega$. Then, for given $\lambda_0 \in \Omega$, there exists a probability measure μ supported by $\Gamma := \partial\Omega$ such that $f(\lambda_0) = \int_\Gamma f d\mu$ for every function f analytic on some neighborhood of $\overline{\Omega}$([10, page 123]). Let $M(\Gamma)$ be the operator "multiplication by λ" on $L^2(\Gamma, \mu)$, then the subspace $H^2(\Gamma)$ spanned by the functions analytic on some neighborhood of $\overline{\Omega}$ is an invariant subspace of $M(\Gamma)$. Hence $M(\Gamma)$ can be written as

$$M(\Gamma) = \begin{bmatrix} M_+(\Gamma) & Z \\ 0 & M_-(\Gamma) \end{bmatrix} \begin{matrix} H^2(\Gamma) \\ L^2(\Gamma,\mu) \ominus H^2(\Gamma) \end{matrix}.$$

LEMMA 3.2. ([10] or [17] Lemma 3.2.4) *Let $M(\Gamma), M_+(\Gamma)$ and $M_-(\Gamma)$ be as above. Then*

(i) $M(\Gamma)$ is normal and both $M_+(\Gamma)$ and $M_-(\Gamma)$ are essentially normal;

(ii) $\sigma(M(\Gamma)) = \sigma_e(M(\Gamma)) = \sigma_e(M_+(\Gamma)) = \sigma_e(M_-(\Gamma)) = \Gamma$ and $\sigma(M_+(\Gamma)) = \sigma(M_-(\Gamma)) = \overline{\Omega}$;

(iii) $\sigma_p(M_-(\Gamma)) = \Omega = \sigma_p(M_+(\Gamma)^*)^*$ and $\operatorname{ind}(\lambda - M_-(\Gamma)) = \operatorname{nul}(\lambda - M_-(\Gamma)) = -\operatorname{ind}(\lambda - M_+(\Gamma)) = \operatorname{nul}(\lambda - M_+(\Gamma))^* = 1, \forall \lambda \in \Omega$;

(iv) $M_+(\Gamma)^* \in \mathscr{B}_1(\Omega^*)$ and $M_-(\Gamma) \in \mathscr{B}_1(\Omega)$.

LEMMA 3.3. ([11], Lemma 2.12) *Let $T \in \mathscr{B}(\mathscr{H})$ be biquasitriangular. Suppose that $\sigma(T) = \sigma_{lre}(T)$ is a perfect set. If Γ is a perfect subset of $\sigma(T)$ and Γ intersects each clopen subset of $\sigma(T)$, then, given $\varepsilon > 0$, there exists $K \in \mathscr{K}(\mathscr{H})$ with $\|K\| < \varepsilon$ satisfying the following conditions:*

(i) $\sigma(T+K) = \sigma(T)$ and $\sigma_p(T+K)$ is a denumerable dense subset of Γ;

(ii) $\vee\{\ker(T+K-\lambda) : \lambda \in \sigma_p(T+K)\} = \mathscr{H}$ and $\operatorname{nul}(T+K-\lambda) = 1$ for all $\lambda \in \sigma_p(T+K)$.

PROPOSITION 3.4. *Let $S \in \mathscr{B}(\mathscr{H})$. Assume that $\sigma(S) = \sigma_e(S) = \overline{\Omega}$ and $\sigma_{lre}(S) = \partial\Omega$, where Ω is a nonempty bounded connected open subset of $\mathbb{C}$ and $\operatorname{int}\overline{\Omega} = \Omega$. If $T \in \mathscr{B}(\mathscr{H})$ and $\sigma(T) = \sigma(S)$, then $T \sim_{sas} S$ if and only if $T \overset{\mathscr{K}}{\sim} S$.*

Proof. By definition, it is obvious that $T \sim_{sas} S$ implies $T \overset{\mathscr{K}}{\sim} S$. Now, we assume that $T \overset{\mathscr{K}}{\sim} S$. We are going to prove that $T \sim_{sas} S$. It follows from $\sigma_e(S) = \overline{\Omega}$ and $\sigma_{lre}(S) = \partial\Omega$ that $\mathrm{ind}(\lambda - S) = \infty$ for all $\lambda \in \Omega$, or $\mathrm{ind}(\lambda - S) = -\infty$ for all $\lambda \in \Omega$. Without loss of generality, we may assume that $\mathrm{ind}(\lambda - S) = \infty$ for all $\lambda \in \Omega$ (otherwise we deal with S^*). Denote $\Gamma = \partial\Omega$.

For given $\varepsilon > 0$, it follows from Lemma 3.1 that there exists $K_1 \in \mathscr{K}(\mathscr{H})$ with $\|K_1\| < \frac{\varepsilon}{4}$ such that

$$S + K_1 = \begin{bmatrix} A & * \\ 0 & N \end{bmatrix},$$

where $\sigma(A) = \sigma(S), \sigma_{lre}(A) = \sigma_{lre}(S)$, $\mathrm{ind}(\lambda - A) = \mathrm{ind}(\lambda - S)$ for each λ in $\rho_{s-F}(S)$ and N is normal with $\sigma(N) = \Gamma$. Let $M(\Gamma)$ be given as in Lemma 3.2. By the non-commutative Weyl-von Neumann theorem(see [20]), there exists $K_2 \in \mathscr{K}(\mathscr{H})$ with $\|K_2\| < \frac{\varepsilon}{4}$ such that

$$S + K_1 + K_2 \simeq \begin{bmatrix} A & * \\ 0 & M(\Gamma)^{(\infty)} \end{bmatrix}.$$

Assume that $M(\Gamma) = \begin{bmatrix} M_+(\Gamma) & Z \\ 0 & M_-(\Gamma) \end{bmatrix}$. Denote $B = M_-(\Gamma)$. Thus

$$\begin{aligned} S + K_1 + K_2 &\simeq \begin{bmatrix} A & * & * \\ 0 & M_+(\Gamma)^{(\infty)} & Z^{(\infty)} \\ 0 & 0 & B^{(\infty)} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A & * \\ 0 & M_+(\Gamma)^{(\infty)} \end{bmatrix} & * \\ 0 & B^{(\infty)} \end{bmatrix} \\ &\triangleq \begin{bmatrix} A_1 & * \\ 0 & B^{(\infty)} \end{bmatrix}. \end{aligned}$$

It is obvious that A_1 is biquasitriangular and $\sigma_{lre}(A_1) = \sigma(A_1) = \overline{\Omega} = \sigma(S)$. It follows from $\mathrm{int}\,\overline{\Omega} = \Omega$ that $\partial\Omega$ is a perfect set. Then, by Lemma 3.3, there exists a compact operator $\overline{K_3}$ on the underlying space of A_1 with $\|\overline{K_3}\| < \frac{\varepsilon}{4}$ such that $A_2 := A_1 + \overline{K_3}$ satisfies

(i) $\sigma(A_2) = \sigma(A_1)$ and $\Gamma_1 := \sigma_p(A_2)$ is a denumerable dense subset of $\partial\Omega$, and

(ii) $\vee\{\ker(A_2 - \lambda) : \lambda \in \Gamma_1\}$ is precisely the underlying space of A_2 and $\mathrm{nul}(A_2 - \lambda) = 1$ for each $\lambda \in \Gamma_1$.

Then there exists $K_3 \in \mathscr{K}(\mathscr{H})$ with $\|K_3\| < \frac{\varepsilon}{4}$ such that

$$S + K_1 + K_2 + K_3 \simeq \begin{bmatrix} A_2 & * \\ 0 & B^{(\infty)} \end{bmatrix} = \begin{bmatrix} A_2 & R_1 & R_2 & \cdots \\ & B & & \\ & & B & \\ & & & \ddots \end{bmatrix},$$

where

(iii) $B \in \mathscr{B}_1(\Omega), \sigma_p(B) = \Omega$ and $\sigma(B) = \overline{\Omega} = \sigma(A_2) = \sigma_{lre}(A_2)$, and

(iv) $\Gamma_1 = \sigma_p(A_2)$ is a denumerable dense subset of $\partial\Omega$ such that $\vee\{\ker(\lambda - A_2) : \lambda \in \Gamma_1\}$ is the underlying space of A_2 and $\mathrm{nul}(\lambda - A_2) = 1$ for all $\lambda \in \Gamma_1$.

By our former hypothesis, T is similar to a compact perturbation of $S + \sum_{i=1}^3 K_i$ and $\sigma(T) = \sigma(S) = \sigma(S + \sum_{i=1}^3 K_i)$. By Proposition 2.8, there exists $K_4 \in \mathscr{K}(\mathscr{H})$ with $\|K_4\| < \varepsilon$ such that $S + \sum_{i=1}^3 K_i \sim T + K_4$. Note that $\|\sum_{i=1}^3 K_i\| < \varepsilon$, thus we conclude the proof. $\square$

LEMMA 3.5. ([9], Proposition 1.7) *Let $\mathscr{B}$ be a Banach algebra with identity and $a \in \mathscr{B}$. If f is analytic on a neighborhood of $\sigma(a)$, then, given $\varepsilon > 0$, there exists $\delta > 0$ such that $f(b)$ is well defined and $\|f(a) - f(b)\| < \varepsilon$ for all $b \in \mathscr{B}$ with $\|a - b\| < \delta$.*

LEMMA 3.6. ([19], Theorem 2.10) *Let $T \in \mathscr{B}(\mathscr{H})$ and suppose that $\sigma(T) = \sigma_1 \cup \sigma_2$, where $\sigma_i (i = 1,2)$ are clopen subsets of $\sigma(T)$ and $\sigma_1 \cap \sigma_2 = \emptyset$. Then $\mathscr{H}(\sigma_1;T) + \mathscr{H}(\sigma_2;T) = \mathscr{H}$, $\mathscr{H}(\sigma_1;T) \cap \mathscr{H}(\sigma_2;T) = \{0\}$ and $\sigma(T_{\sigma_i}) = \sigma_i (i = 1,2)$. In particular, T admits the following matrix representation*

$$T = \begin{bmatrix} T_{\sigma_1} & 0 \\ 0 & T_{\sigma_2} \end{bmatrix} \begin{matrix} \mathscr{H}(\sigma_1;T) \\ \mathscr{H}(\sigma_2;T) \end{matrix}.$$

PROPOSITION 3.7. *Let $T \in \mathscr{B}(\mathscr{H})$ and suppose that $\sigma(T) = \sigma_1 \cup \sigma_2$, where σ_1, σ_2 are clopen subsets of $\sigma(T)$ and $\sigma_1 \cap \sigma_2 = \emptyset$. Let Ω_1, Ω_2 be two Cauchy domains such that $\sigma_i \subset \Omega_i (i = 1,2)$ and $\Omega_1 \cap \Omega_2 = \emptyset$. Then there exists $\delta > 0$ such that the following conditions hold for any $K \in \mathscr{K}(\mathscr{H})$ with $\|K\| < \delta$:*

(i) $\sigma(T + K)$ is the disjoint union of two clopen subsets Γ_1 and Γ_2, where $\Gamma_i \subset \Omega_i (i = 1,2)$; and

(ii) $(T + K)_{\Gamma_i} \overset{\mathscr{K}}{\sim} T_{\sigma_i} (i = 1,2)$.

Proof. By Lemma 3.6, T can be written as

$$T = \begin{bmatrix} T_{\sigma_1} & 0 \\ 0 & T_{\sigma_2} \end{bmatrix} \begin{matrix} \mathscr{H}(\sigma_1;T) \\ \mathscr{H}(\sigma_2;T) \end{matrix}.$$

Set

$$f(\lambda) = \begin{cases} 1, & \lambda \in \Omega_1, \\ 0, & \lambda \in \Omega_2. \end{cases}$$

Then f is analytic on $\Omega_1 \cup \Omega_2$ and $f(T) = E(\sigma_1;T)$. By the upper semi-continuity of spectrum and Lemma 3.5, there exists $\delta > 0$ such that $\sigma(S) \subset \Omega_1 \cup \Omega_2$ (then $f(S)$ is well defined) and, moreover, $\|f(T) - f(S)\| < \frac{1}{2\|f(T)\|+2}$ for all $S \in \mathscr{B}(\mathscr{H})$ with $\|S - T\| < \delta$. We shall prove that δ satisfies all requirements.

Arbitrarily choose a $K \in \mathscr{K}(\mathscr{H})$ with $\|K\| < \delta$. Denote $S = T + K$ and $\Gamma_i = \sigma(S) \cap \Omega_i (i = 1,2)$. Then Γ_1, Γ_2 are two disjoint clopen subsets of $\sigma(S)$ and $\sigma(S) =$

$\Gamma_1 \cup \Gamma_2$. Then $f(S)$ is well defined, $f(S) = E(\Gamma_1; S)$ and $\|f(T) - f(S)\| < \frac{1}{2\|f(T)\|+2}$. Denote $P = f(T)$ and $Q = f(S)$. Thus, by Lemma 3.6, S can be written as

$$S = \begin{bmatrix} S_{\Gamma_1} & 0 \\ 0 & S_{\Gamma_2} \end{bmatrix} \begin{matrix} \operatorname{ran} Q \\ \operatorname{ran}(I-Q) \end{matrix},$$

where $\sigma(S_{\Gamma_i}) = \Gamma_i (i = 1,2)$. It suffices to prove that $T_{\sigma_i} \stackrel{\mathscr{K}}{\sim} S_{\Gamma_i} (i = 1,2)$. Since S is a compact perturbation of T, it is easy to see that Q is a compact perturbation of P. Set $W = PQ + (I-P)(I-Q)$. Then

$$\begin{aligned} W &= I + PQ - P + PQ - Q \\ &= I + P(Q-P) + (P-Q)Q. \end{aligned}$$

Note that $\|P(Q-P) + (P-Q)Q\| \leqslant \|P(Q-P)\| + \|(P-Q)Q\| \leqslant \|P-Q\| \cdot (\|P\| + \|Q\|) < 1$, we obtain $W \in (\mathscr{I} + \mathscr{K})$. Then $TW - WS \in \mathscr{K}(\mathscr{H})$. Denote $W_1 = PW|_{\operatorname{ran} Q}, W_2 = (I-P)W|_{\operatorname{ran}(I-Q)}$. It is easy to see that both $W_1 \in \mathscr{B}(\operatorname{ran} Q, \operatorname{ran} P)$ and $W_2 \in \mathscr{B}(\operatorname{ran}(I-Q), \operatorname{ran}(I-P))$ are invertible. Hence it follows from

$$P(TW - WS)Q = (PTP)(PQ) - (PQ)(QSQ) = P(T_{\sigma_1} W_1 - W_1 S_{\Gamma_1})Q$$

that $T_{\sigma_1} W_1 - W_1 S_{\Gamma_1}$ is compact. Similarly, one can check that $T_{\sigma_2} W_2 - W_2 S_{\Gamma_2}$ is also compact. So we have proved that $T_{\sigma_i} \stackrel{\mathscr{K}}{\sim} (T+K)_{\Gamma_i} (i = 1,2)$. $\square$

COROLLARY 3.8. *Let $A, B \in \mathscr{B}(\mathscr{H})$. If $A \sim_{sas} B$ and $\sigma(A) = \sigma(B)$, then $A_\sigma \stackrel{\mathscr{K}}{\sim} B_\sigma$ for each clopen subset σ of $\sigma(A)$.*

Proof. Arbitrarily choose a clopen subset σ of $\sigma(T)$. Without loss of generality, we assume that $\sigma \neq \emptyset$ and $\sigma \neq \sigma(T)$. Set $\sigma_1 = \sigma(T) \setminus \sigma$. We can choose two disjoint Cauchy domains Ω and Ω_1 such that $\sigma \subset \Omega, \sigma_1 \subset \Omega_1$. Then we can choose a common positive number δ such that δ satisfies the conditions (i) and (ii) in Proposition 3.7 for both A and B. Since $A \sim_{sas} B$, there exist K and K_1 in $\mathscr{K}(\mathscr{H})$ with $\|K\| + \|K_1\| < \delta$ such that $A + K \sim B + K_1$. Set $\Gamma = \sigma(A+K) \cap \Omega$. By our assumption, Γ is a clopen subset of $\sigma(A+K)$. Since $A + K \sim B + K_1$, it follows easily that $(A+K)_\Gamma \sim (B+K_1)_\Gamma$. By Proposition 3.7, we obtain $A_\sigma \stackrel{\mathscr{K}}{\sim} (A+K)_\Gamma$ and $B_\sigma \stackrel{\mathscr{K}}{\sim} (B+K_1)_\Gamma$. Therefore we can conclude that $A_\sigma \stackrel{\mathscr{K}}{\sim} B_\sigma$. $\square$

Now, we are going to give the proof of Main Theorem.

Proof of Main Theorem. "$\Longrightarrow$". By definition, $T \sim_{sas} S$ implies that $\sigma_0(T) = \sigma_0(S)$, $T \stackrel{\mathscr{K}}{\sim} S$ and hence $\sigma_w(T) = \sigma_w(S)$. Since $T, S \in \operatorname{Nic}(\mathscr{H})$, $\sigma(T) = \sigma_w(T) \cup \sigma_0(T)$ and $\sigma(S) = \sigma_w(S) \cup \sigma_0(S)$, we have $\sigma(T) = \sigma(S)$. Now the proof of the necessity follows immediately from Corollary 3.8.

"$\Longleftarrow$". Now we assume that $\sigma(T) = \sigma(S)$ and $T_\sigma \stackrel{\mathscr{K}}{\sim} S_\sigma$ for each clopen subset σ of $\sigma(T)$. We shall prove that $T \sim_{sas} S$.

Without loss of generality, we assume that $\{\sigma_i\}_{i=1}^n$ is an enumeration of the connected components of $\sigma(T)$ (or, equivalently, $\sigma(S)$). $T \in \mathrm{Nic}(\mathscr{H})$ implies that each σ_i is a clopen subset of $\sigma(T)$. Then, by Lemma 3.6, T and S can be represented as

$$T = \begin{array}{c} \begin{bmatrix} T_1 & & & \\ & T_2 & & \\ & & \ddots & \\ & & & T_n \end{bmatrix} \end{array} \begin{array}{l} \mathscr{H}(\sigma_1;T) \\ \mathscr{H}(\sigma_2;T) \\ \vdots \\ \mathscr{H}(\sigma_n;T) \end{array}, S = \begin{array}{c} \begin{bmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_n \end{bmatrix} \end{array} \begin{array}{l} \mathscr{H}(\sigma_1;S) \\ \mathscr{H}(\sigma_2;S) \\ \vdots \\ \mathscr{H}(\sigma_n;S) \end{array},$$

where $T_i = T_{\sigma_i}$ and $S_i = S_{\sigma_i} (1 \leqslant i \leqslant n)$. Then $\sigma(T_i) = \sigma_i = \sigma(S_i)$ and $T_i \overset{\mathscr{K}}{\sim} S_i (1 \leqslant i \leqslant n)$. Obviously, to complete the proof, it suffices to prove that $T_i \sim_{sas} S_i$ for all $1 \leqslant i \leqslant n$.

Arbitrarily fix an i with $1 \leqslant i \leqslant n$.

Case 1. $\dim \mathscr{H}(\sigma_i;T) < \infty$. It follows from $T_i \overset{\mathscr{K}}{\sim} S_i$ that $\dim \mathscr{H}(\sigma_i;T) = \dim \mathscr{H}(\sigma_i;S)$. Since σ_i is a connected component of $\sigma(T)$, we infer that $\sigma_i (= \sigma(T_i) = \sigma(S_i))$ is a singleton. Then, by the theorem of Jordan canonical form, we obtain $T_i \sim_{sas} S_i$.

Case 2. $\dim \mathscr{H}(\sigma_i;T) = \infty$ and $\sigma_i \subset \sigma_{lre}(T)$. In this case, it is easy to see that $\sigma(T_i) = \sigma_{lre}(T_i) = \sigma_i$. Then T_i is a biquasitriangular operator and $\sigma(T_i) (= \sigma_{lre}(T_i) = \sigma_i)$ is connected. Since $T_i \overset{\mathscr{K}}{\sim} S_i$ and $\sigma(T_i) = \sigma(S_i)$, it follows from Theorem 1.2 that $T_i \sim_{sas} S_i$.

Case 3. $\dim \mathscr{H}(\sigma_i;T) = \infty$ and $\sigma = \overline{\Omega}$, where Ω is a nonempty bounded connected open subset of $\rho_{s-F}(T)$ and $\mathrm{int}\,\overline{\Omega} = \Omega$. Since $\mathrm{ind}(\lambda - T)$ is continuous on $\rho_{s-F}(T)$, there exists $k, -\infty \leqslant k \leqslant \infty$, such that $k = \mathrm{ind}(T - \lambda) = \mathrm{ind}(T_i - \lambda)$ for all $\lambda \in \Omega$.

Since $\sigma(T) = \sigma_w(T) \cup \sigma_0(T)$, we can infer that $k \neq 0$. Without loss of generality, we may assume that $k > 0$ (otherwise we deal with T_i^* and S_i^*). In view of Theorem 1.2 (in the case that $0 < n < \infty$) and Proposition 3.4 (in the case that $n = \infty$), we can conclude that $T_i \sim_{sas} S_i$. This completes the proof. $\square$

Let $A, B \in \mathscr{B}(\mathscr{H})$ and assume that $\sigma(A) = \sigma(B)$. It is trivial to see that if $A_\sigma \overset{\mathscr{K}}{\sim} B_\sigma$ for each clopen subset σ of $\sigma(A)$, then $A \overset{\mathscr{K}}{\sim} B$. But the converse is not necessarily true.

EXAMPLE 3.9. Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{array}{l} \mathbb{C} \\ \mathscr{H} \\ \mathscr{H} \\ \mathscr{H} \end{array}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{array}{l} \mathbb{C} \\ \mathscr{H} \\ \mathscr{H} \\ \mathscr{H} \end{array}$$

where I is the unit operator on $\mathscr{H}$. It is obvious that A is a compact perturbation of B and $\sigma(A) = \sigma(B) = \{0,1\}$. So $A \overset{\mathscr{K}}{\sim} B$. Set $\sigma = \{0\}$. Then σ is a clopen subset of

$\sigma(A)(=\sigma(B))$ and

$$A_\sigma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \mathbb{C} \\ \mathscr{H}, \\ \mathscr{H} \end{matrix} \quad B_\sigma = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathscr{H} \\ \mathscr{H} \end{matrix}.$$

We claim that A_σ is not similar to any compact perturbation of B_σ.

In fact, if not, then there exists an invertible operator $X \in \mathscr{B}(\mathbb{C} \oplus \mathscr{H}^{(2)}, \mathscr{H}^{(2)})$ such that $B_\sigma X - XA_\sigma$ is compact. Define $X_1 \in \mathscr{B}(\mathscr{H}^{(2)})$ by $X_1 y = Xy$, where $y \in \mathscr{H}^{(2)}$. It is obvious that X_1 is a Fredholm operator on $\mathscr{H}^{(2)}$ and $\operatorname{ind} X_1 = -1$. Since $\{B_\sigma X - XA_\sigma\}|_{\mathscr{H}^{(2)}} = B_\sigma X_1 - X_1 B_\sigma$, we deduce that $B_\sigma X_1 - X_1 B_\sigma$ is compact. Without loss of generality, we assume that

$$X_1 = \begin{bmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{bmatrix} \begin{matrix} \mathscr{H} \\ \mathscr{H} \end{matrix}.$$

Since $B_\sigma X_1 - X_1 B_\sigma$ is compact, a straightforward computation shows that $X_{2,1}, X_{1,1} - X_{2,2} \in \mathscr{K}(\mathscr{H})$. Note that X_1 is a Fredholm operator, then both $X_{1,1}$ and $X_{2,2}$ are Fredholm, and $\operatorname{ind} X_1 = 2 \cdot \operatorname{ind} X_{1,1} \neq -1$, a contradiction. Thus we can conclude that A_σ is not similar to any compact perturbation of B_σ.

We conclude this paper with the following question.

QUESTION 3.10. Let $A, B \in \mathscr{B}(\mathscr{H})$. Assume that $\sigma(A) = \sigma(B)$ and $A_\sigma \overset{\mathscr{K}}{\sim} B_\sigma$ for each clopen subset σ of $\sigma(A)$. In this case, is it necessary that $A \sim_{sas} B$?

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