Operators
and
Matrices
Volume 5, Number 2 (2011), 227–237

A RESULT ON POSITIVE MATRICES AND APPLICATIONS TO HANKEL OPERATORS

MARCUS CARLSSON

(*Communicated by L. Rodman*)

Abstract. Let S denote the shift operator on $l^2(\mathbb{N})$ and set $e_0 = (1,0,0,\ldots)$. A special case of the main result says that if W is a self-adjoint operator on $l^2(\mathbb{N})$ such that $W(e_0) = 0$ and $S^*WS \geqslant W$, then $W \geqslant 0$. We apply this result to AAK-type theorems on generalized Hankel operators, providing new insights related to previous work by S. Treil and A. Volberg [10].

1. Introduction

This paper contains two main results, Theorem A and Theorem B. The first is a positivity test for infinite matrices and the second is an application to generalized Hankel operators. We first introduce some notation. Let $\mathbb{N} = \{0,1,\ldots\}$ and let $\mathbb{M}_\infty$ denote the set of all functions from $\mathbb{N}\times\mathbb{N}$ into $\mathbb{C}$, i.e. $\mathbb{M}_\infty$ is the set of all "infinite matrices" $W = (w_{m,n})_{m,n\geqslant 0}$. Let c denote the set of sequences $a = (a_m)_{m=0}^\infty$ in $\mathbb{C}$ and let c_{00} denote the subset of sequences with finite support. We identify $\mathbb{M}_\infty$ with the set of all linear operators from c_{00} into c via $(Wa)_m = \sum_{n\geqslant 0} w_{m,n}a_n$, and we define W^* as the transpose of the conjugate of W, i.e. $W^* = (\overline{w_{n,m}})_{m,n\geqslant 0}$. Given $a,b \in c$ we write $\langle a,b\rangle = \sum_{n\geqslant 0} a_n\overline{b_n}$ whenever the sum is finite. We thus have

$$\langle Wa,b\rangle = \langle a,W^*b\rangle = \sum_{m,n} \overline{b_m}w_{m,n}a_n,$$

and we will also use the notation b^*Wa for the above expression. $W \in \mathbb{M}_\infty$ will be called Hermitian symmetric if $W^* = W$ and positive if $a^*Wa \geqslant 0$ for all $a \in c_{00}$. In this case we write $W \geqslant 0$. Moreover, if $W_1, W_2 \in \mathbb{M}_\infty$ we write $W_1 \geqslant W_2$ whenever $W_1 - W_2 \geqslant 0$. Finally, let S denote the shift operator on c_{00} defined by $Sa = (0,a_0,a_1,\ldots)$. Note that S^* is the backward shift.

To illustrate the first theorem, we present the following example. Let $w = (w_j)_{j=0}^\infty$ be a sequence in $\mathbb{R}$. If $w_0 = 0$ and $w_{j+1} \geqslant w_j$ for all $j \in \mathbb{N}$, then obviously $w_j \geqslant 0$ for all $j \in \mathbb{N}$. Another way to express this is to say that if $W \in \mathbb{M}_\infty$ is a diagonal matrix such that $W(e_0) = 0$ and $S^*WS \geqslant W$, then $W \geqslant 0$. Less trivial is the following result:

Mathematics subject classification (2010): Primary 47B35, Secondary 47B15, 47B37.
Keywords and phrases: Positive matrices, Hankel operators.

Paper OaM-05-16

THEOREM A. *Let* $W \in \mathbb{M}_\infty$ *be Hermitian symmetric such that* $W(e_0) = 0$. *If* $S^*WS \geqslant W$, *then*

$$W \geqslant 0.$$

Note that a matrix $T \in \mathbb{M}_\infty$ is Toeplitz if and only if $S^*TS = T$. An immediate consequence of Theorem A is therefore the following result.

COROLLARY 1.1. *Let* $W = (w_{m,n})$ *be Hermitian symmetric and set* $t_m = w_{0,m}$, $t_{-m} = w_{m,0}$ *for all* $m \in \mathbb{N}$. *Let* $T = (t_{n-m})_{m,n\geqslant 0}$ *be the corresponding Toeplitz matrix;*

$$T = \begin{pmatrix} w_{0,0} & w_{0,1} & w_{0,2} & \cdots \\ w_{1,0} & w_{0,0} & w_{0,1} & \ddots \\ w_{2,0} & w_{1,0} & w_{0,0} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

If $S^*WS \leqslant W$, *then* $W \leqslant T$. *If* $S^*WS \geqslant W$ *then* $W \geqslant T$.

We will now discuss applications of these results to generalized Hankel operators. By abuse of notation, let S denote the unilateral shift operator on $l^2 = l^2(\mathbb{N})$, i.e. S is the extension of the shift operator S defined earlier on the dense subset c_{00}. Recall that an operator $\Gamma : l^2 \to l^2$ is called a Hankel operator if it satisfies

$$\Gamma S = S^*\Gamma,$$

and that S^* is the backward shift. This definition is equivalent to demanding that the matrix representation of Γ in the standard basis $(e_m)_{m=0}^\infty$ (i.e. $e_0 = (1,0,0,\ldots)$, $e_1 = (0,1,0,\ldots)$, etc.) looks like a Hankel matrix

$$\Gamma \text{"="} \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots \\ \gamma_1 & \gamma_2 & \gamma_3 & \cdots \\ \gamma_2 & \gamma_3 & \gamma_4 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad \gamma_n \in \mathbb{C}. \tag{1.1}$$

Given a Hilbert space X we let $\mathscr{L}(X)$ denote the set of bounded operators on X. We define "generalized Hankel operators" as follows.

DEFINITION 1.2. Let X_1 and X_2 be Hilbert spaces and let $S \in \mathscr{L}(X_1)$ and $B \in \mathscr{L}(X_2)$ be given operators. A bounded operator $\Gamma : X_1 \to X_2$ will be called Hankel (with respect to S and B) if it satisfies

$$\Gamma S = B\Gamma. \tag{1.2}$$

It is easy to see that this definition is equivalent with the one in [7], (Vol 1, Part B, Sec 1.7), and that it is slightly more general than the one used by S. Treil and A. Volberg in [10].

EXAMPLE 1.3. As an example, let $w = (w_m)_{m=0}^{\infty}$ be any positive sequence and define l_w^2 as the completion of c_{00} with respect to the norm

$$\|a\|^2 = \sum_{m=0}^{\infty} |a_m|^2 w_m.$$

Let v be another positive sequence, let S be the shift operator on l_w^2 and let B be the backward shift operator on l_v^2. Then $\Gamma : l_w^2 \to l_v^2$ is Hankel if and only if its matrix representation (in the standard bases $(e_m)_{m=0}^{\infty}$ in l_w^2 and l_v^2 respectively) looks like (1.1). A concrete example would be to take $w_m = v_m = m+1$, in which case $l_w^2 = l_v^2$ correspond to the Dirichlet space.

We introduce more notation. Let $\Gamma : X_1, \to X_2$ be any bounded operator and recall that its singular values $\sigma_0, \sigma_1, \dots$ are defined as

$$\sigma_n = \inf\{\|\Gamma|_{\mathscr{M}}\| : \mathscr{M} \leqslant X_1 \text{ and } \mathsf{codim}\ \mathscr{M} = n\}, \tag{1.3}$$

where $\mathscr{M} \leqslant X_1$ means that $\mathscr{M}$ is a subspace and $\Gamma|_{\mathscr{M}}$ denotes the restriction of Γ to $\mathscr{M}$. Set $\sigma_\infty = \lim_{n\to\infty} \sigma_n$. Recall that for a compact operator Γ, the singular values appear as the eigenvalues of the operator $\sqrt{\Gamma^*\Gamma}$. A vector $u_n \in X_1$ will be called a σ_n-singular vector if $\|u_n\| = 1$ and

$$\sigma_n^2 u_n = \Gamma^* \Gamma u_n.$$

In the general case, the polar decomposition of Γ shows that such vectors always exist when $\sigma_n > \sigma_\infty$, whereas they may or may not exist if $\sigma_n = \sigma_\infty$. In the remainder, we assume that σ_n is such that a singular vector exists. (A very simple example of singular vectors is computed in Section 5). An operator S on some Hilbert space X is called expansive if $\|Sx\| \geqslant \|x\|$ for all $x \in X$ and contractive if $\|S\| \leqslant 1$. In the setting of Example 1.3, S is expansive if and only if w is increasing, and B is a contraction if and only if v is increasing. The subspace generated by taking the closure of the span of $\{S^m x : m \in \mathbb{N}\}$ will be denoted by $[x]_S$.

THEOREM B. *Let X_1 and X_2 be Hilbert spaces and let $\Gamma : X_1 \to X_2$ be a Hankel operator with respect to some operators $S \in \mathscr{L}(X_1)$ and $B \in \mathscr{L}(X_2)$, as in Definition* 1.2. *Moreover assume that S is expansive and that B is a contraction. Let $u_n \in X$ be a singular vector to Γ with singular value σ_n. Then*

$$\|\Gamma|_{[u_n]_S}\| = \sigma_n.$$

In particular, if $\sigma_{n-1} > \sigma_n$ then $\mathsf{codim}\ [u_n]_S \geqslant n$.

We note that in the classical case, i.e. when $X_1 = X_2$ is the (unweighted) l^2-space and S and B are as in Example 1.3, then the equality $\|\Gamma|_{[u_n]_S}\| = \sigma_n$ is a straightforward application of (1.2) and standard H^2-theory. In this case, we also have the famous theorem by Adamjan, Arov and Krein [1], known as the "AAK-theorem", which says that $\mathsf{codim}\ [u_n]_S = n$ if $\sigma_{n-1} > \sigma_n > \sigma_{n+1}$. (See [1] or [8] for a more accessible version.) Combined with the work of Butz [3], Theorem B can be used to give a new proof of the AAK-theorem that avoids using Nehari's theorem. Incidentally, this also provides a new proof of Nehari's theorem, although we will not pursue these matters here.

Extensions of Nehari's theorem in a similar setting as in Theorem B has been treated by Treil and Volberg in [10]. Although they prove that a form of the Nehari theorem does hold, we will in Section 5 show that a stronger extension, (of the type that one would need to imitate the classical proofs of the AAK-theorem), fails. Nevertheless, Treil and Volberg also give an extension of the AAK-theorem in [10]. An application of Theorem B leads to an improvement of this extension in certain important cases. We will discuss these matters further in Section 4.

The paper is outlined as follows. In Section 2 we prove Theorem A and in Section 3 we prove Theorem B. Section 4 contains a more detailed account of the AAK-theorem, its connections with Theorem B and the related work by Treil and Volberg. Finally, in Section 5 we give some counterexamples to natural questions arising in Section 4.

2. On positive matrices

In order to prove Theorem A we actually first need to prove a special case of Corollary 1.1. First some notation. Let X be a fixed Hilbert space and let $S \in \mathscr{L}(X)$. Given any $u \in X$, we associate with it the Toeplitz-matrix

$$T_{u,S} = \begin{pmatrix} \langle u,u\rangle & \langle Su,u\rangle & \langle S^2u,u\rangle & \cdots \\ \langle u,Su\rangle & \langle u,u\rangle & \langle Su,u\rangle & \cdots \\ \langle u,S^2u\rangle & \langle u,Su\rangle & \langle u,u\rangle & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \tag{2.1}$$

as well as the following matrix

$$N_{u,S} = \begin{pmatrix} \langle u,u\rangle & \langle Su,u\rangle & \langle S^2u,u\rangle & \cdots \\ \langle u,Su\rangle & \langle Su,Su\rangle & \langle S^2u,Su\rangle & \cdots \\ \langle u,S^2u\rangle & \langle Su,S^2u\rangle & \langle S^2u,S^2u\rangle & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \tag{2.2}$$

LEMMA 2.1. *Let $W = (w_{m,n})_{m,n\geqslant 0}$ be strictly positive Hermitian symmetric, set $t_m = w_{0,m}$ and $t_{-m} = w_{m,0}$ for all $m \in \mathbb{N}$. Let $T = (t_{n-m})_{m,n\geqslant 0}$ be the corresponding Toeplitz matrix. If $S^*WS \leqslant W$, then $W \leqslant T$.*

Proof. Let l_W denote the completion of c_{00} with respect to the norm

$$\|a\|_{l_W}^2 = a^*Wa.$$

Then S is a contraction on the space l_W. Thus S has a unitary dilation, i.e. there exists a Hilbert space Z that contains l_W as a subspace and a unitary operator $U \in \mathscr{L}(Z)$ such that $S^n x = P_{l_W} U^n x$ for all $x \in l_W$ and $n \in \mathbb{N}$, where P_{l_W} denotes the orthogonal projection in Z onto l_W. (See [6] for more details.) Let $a \in c_{00}$ be arbitrary and recall that $e_0 = (1,0,0,\dots)$. Clearly

$$\|a\|_{l_W}^2 = \|\sum a_n S^n e_0\|_{l_W}^2 = \|P_{l_W} \sum a_n U^n e_0\|_Z^2.$$

Moreover, it is easy to see that $T = T_{e_0,S} = T_{e_0,U} = N_{e_0,U}$, so

$$a^* T a = a^* N_{e_0,U} a = \|\sum a_n U^n e_0\|_Z^2.$$

Thus

$$a^*(T-W)a = \|\sum a_n U^n e_0\|_Z^2 - \|P_{l_W} \sum a_n U^n e_0\|_Z^2 \geqslant 0,$$

as desired. $\square$

THEOREM A. *If $W = (w_{m,n})_{m,n\geqslant 0} \in \mathbb{M}_\infty$ is Hermitian symmetric with $w_{0,m} = w_{m,0} = 0$ and $S^*WS \geqslant W$, then*

$$W \geqslant 0.$$

Proof. For each $k \in \mathbb{N}$ we define a new matrix $W_k = (w_{m,n}^k)_{m,n\geqslant 0}$ given by

$$\begin{cases} w_{m,n}^k = w_{m,n} & m,n \leqslant k \\ w_{m+l,k+l}^k = w_{m,k} & m \leqslant k, l > 0 \\ w_{k+l,m+l}^k = w_{k,m} & m \leqslant k, l > 0 \\ w_{m,n}^k = 0 & \text{elsewhere} \end{cases}$$

This cumbersome definition is easily visualized, here is W_3 :

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & w_{1,1} & w_{1,2} & w_{1,3} & 0 & 0 \\ 0 & w_{2,1} & w_{2,2} & w_{2,3} & w_{1,3} & \ddots \\ 0 & w_{3,1} & w_{3,2} & w_{3,3} & w_{2,3} & \ddots \\ 0 & 0 & w_{3,1} & w_{3,2} & w_{3,3} & \ddots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

For each fixed $a \in c_{00}$ we clearly have

$$\lim_{k\to\infty} a^* W_k a = a^* W a, \tag{2.3}$$

and hence it is enough to show that $W_k \geqslant 0$ for all $k \in \mathbb{N}$. Decompose an arbitrary a as $a = a_b + a_e$, where $a_b = (a_0, a_1, \dots, a_{k-1}, 0, 0, \dots)$. Note that

$$\begin{aligned} &(Sa_b)^* W_k (Sa_e) = a_b^* W_k a_e \\ &(Sa_e)^* W_k (Sa_e) = a_e^* W_k a_e \\ &a_b^* W_k a_b = a_b^* W a_b \leqslant (Sa_b)^* W (Sa_b) = (Sa_b)^* W_k (Sa_b) \end{aligned}$$

from which it follows that

$$W_k \leqslant S^* W_k S \tag{2.4}$$

Set $c_k = \max\{w_{m,n} : 0 \leqslant m,n \leqslant k\}$ and note that

$$\left| a^* W_k a \right| < 2kc_k \|a\|_{l^2}^2$$

whenever $a \neq 0$, which follows from the calculation

$$\begin{aligned} \left| a^* W_k a \right| &= \Big| \sum_n \sum_{|m-n| \leqslant k-1} w_{m,n}^k \overline{a_m} a_n \Big| \leqslant \sum_n \sum_{|m-n| \leqslant k-1} c_k \frac{(|a_m|^2 + |a_n|^2)}{2} \\ &\leqslant 2 \sum_n \sum_{|m-n| \leqslant k-1} c_k \frac{|a_n|^2}{2} \leqslant (2k-1) c_k \|a\|_{l^2}^2. \end{aligned}$$

Let $I \in \mathbb{M}_\infty$ denote the identity matrix and put $V_k = I - \frac{1}{2kc_k} W_k$. By the above calculation we have $V_k > 0$ and (2.4) implies that $S^* V_k S \leqslant V_k$. But then Lemma 2.1 applies which yields that $V_k \leqslant I$ or

$$0 \leqslant (I - V_k) = \frac{1}{2kc_k} W_k.$$

It follows that $W_k \geqslant 0$ for all k and hence $W \geqslant 0$ follows by (2.3). $\square$

The proof of Corollary 1.1 is immediate so we omit it. Note however the following reformulation.

COROLLARY 2.2. *Let X be a Hilbert space and let $u \in X$ be arbitrary. If $S \in \mathscr{L}(X)$ is a contraction, then*

$$N_{u,S} \leqslant T_{u,S}.$$

If S is expansive, then

$$T_{u,S} \leqslant N_{u,S}.$$

Proof. To simplify notation we will write $N(u,S) = N_{u,S}$. Given $a \in c_{00}$, it is easy to see that

$$\|a\|_{l_{N(u,S)}} = \|\sum a_n S^n u\|_X$$

and hence $a^*(S^* N_{u,S} S) a = \|Sa\|_{l_{N(u,S)}} = \|S \sum a_n S^n u\|_X$. By these identities, the corollary is easily seen to follow from Corollary 1.1. $\square$

3. Applications to Hankel operators

Given an operator $\Gamma : X_1 \to X_2$ and $x \in X_1$, recall that $\|\Gamma|_{[x]_S}\|$ denotes the operator norm of Γ restricted to the invariant subspace $[x]_S$ generated by x and $S \in \mathscr{L}(X_1)$. We can now prove Theorem B.

THEOREM B. *Let $\Gamma : X_1 \to X_2$ be a Hankel operator with respect to some operators $S \in \mathscr{L}(X_1)$ and $B \in \mathscr{L}(X_2)$, as in Definition* 1.2*, where S is expansive and B is a contraction. Let $\sigma_0 \geqslant \sigma_1 \geqslant \dots$ denote its singular values, let $n \in \mathbb{N}$ be fixed and let $u \in X$ be a singular vector with singular value σ_n. Then*

$$\|\Gamma|_{[u]_S}\| = \sigma_n.$$

Moreover, if $\sigma_{n-1} > \sigma_n$ then $\operatorname{codim} [u]_S \geqslant n$.

Proof. Put $v = \Gamma u$. By the polar decomposition of Γ it follows that $\|v\| = \sigma_n$. Obviously then $\|\Gamma|_{[u]_S}\| \geqslant \|\Gamma u\| = \|v\| = \sigma_n$, so we focus on proving the reverse inequality. It suffices to show that

$$\|\Gamma(\sum a_m S^m u)\| \leqslant \sigma_n \|\sum a_m S^m u\|$$

for all $a \in c_{00}$. Note that $\|\sum a_m S^m u\|^2 = a^* N_{u,S}\, a$ and similarly

$$\|\Gamma(\sum a_m S^m u)\|^2 = \|\sum a_m B^m \Gamma u\|^2 = a^* N_{v,B}\, a,$$

where we have used the Hankel commutation relation (1.2). This also yields

$$\langle B^m v, v\rangle = \langle B^m \Gamma u, \Gamma u\rangle = \langle \Gamma S^m u, \Gamma u\rangle = \langle S^m u, \Gamma^* \Gamma u\rangle = \sigma_n^2 \langle S^m u, u\rangle$$

which implies that $T_{v,B} = \sigma_n^2 T_{u,S}$. The desired inequality follows via Corollary 2.2 and the calculation

$$\begin{aligned}\sigma_n^2 \|\sum a_m S^m u\|^2 = \sigma_n^2 (a^* N_{u,S}\, a) \geqslant \sigma_n^2 (a^* T_{u,S}\, a) = \\ = a^* T_{v,B}\, a \geqslant a^* N_{v,B}\, a = \|\Gamma(\sum a_m S^m u)\|^2.\end{aligned}$$

The last part of the statement follows immediately by the first and the definition of singular numbers;

$$\inf\{\|\Gamma|_{\mathscr{M}}\| : \mathscr{M} \leqslant X \text{ and } \operatorname{codim} \mathscr{M} = n-1\} = \sigma_{n-1} > \sigma_n = \|\Gamma|_{[u]_S}\|. \qquad \square$$

REMARK. There are counterexamples to the conclusion of Theorem B if either of the restrictions on S and B are removed.

4. A review of generalized Hankel operators and the AAK-theorem

This section aims at putting Theorem B in its proper context. Recall the celebrated result by Adamyan, Arov and Krein [1], known as the AAK-theorem and usually stated as follows.

THEOREM. (AAK) *Let $\Gamma : l^2 \to l^2$ be a Hankel operator and let σ_n be its n'th singular value. Then there is a rank n Hankel operator K such that*

$$\sigma_n = \|\Gamma - K\|.$$

However, the theorem is much stronger because its proof provides a way of actually calculating the best rank n Hankel approximation. This in turn is related to the curious fact that the Fourier series defined by the n:th singular vector has precisely n zeroes in the unit disc, (assuming that $\sigma_{n+1} < \sigma_n < \sigma_{n-1}$). We outline this in more detail below.

It is easy to see that a rank 1 Hankel operator necessarily has the following form

$$\Gamma_{(z_0)} = \begin{pmatrix} 1 & z_0 & z_0^2 & \cdots \\ z_0 & z_0^2 & z_0^3 & \cdots \\ z_0^2 & z_0^3 & z_0^4 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad |z_0| < 1, \tag{4.1}$$

but it is not true that any finite rank Hankel operator is a sum of such. In fact, a "symbol" for the above rank one Hankel operator is easily seen to be $(1 - z_0\overline{z})^{-1}$ and in general, any rank n Hankel operator has a symbol of the form $r(\overline{z})$ where r is a rational function with degree n and all poles lie in $\{z \in \mathbb{C} : |z| > 1\}$, (see e.g. [8]). In terms of applications, the power of the AAK-theorem comes from the fact that the location of these poles can be easily calculated using the singular vectors. For simplicity, let us assume that σ_n is distinct and denote the corresponding singular vector by u_n. Let $\check{u}_n$ denote the analytic function in the unit disc defined by $\check{u}_n(z) = \sum_{m=0}^{\infty} u_n(m) z^m$. The proof of the AAK-theorem then shows that $\check{u}_n$ has precisely n roots $(z_j)_{j=1}^n$, counted with multiplicity, and that the poles of the rational symbol for K in the AAK-theorem are located at $(1/z_j)_{j=1}^n$, again counted with multiplicity. In particular, if $\check{u}_n$ has distinct zeroes, then the best rank n Hankel approximant of Γ is a sum of n matrices of the form (4.1) with z_0 replaced by $z_j,\ j = 1, \ldots, n$.

Using Beurling's and Nehari's theorem, a short argument shows that the AAK-theorem is equivalent with the following result

THEOREM. (AAK*) *Let $\Gamma : l^2 \to l^2$ be a Hankel operator and let σ_n be its n'th singular value. Then there is a singular vector u_n to σ_n such that* $\operatorname{codim} [u_n]_S = n$ *and* $\|\Gamma|_{[u_n]_S}\| = \sigma_n$.

We will now discuss S. Treil and A. Volberg's extension of the AAK-theorem in [10]. Throughout the remainder, X_1 and X_2 will denote Hilbert spaces and $\Gamma : X_1 \to X_2$ will denote a Hankel operator with respect to some operators $S \in \mathscr{L}(X_1)$ and $B \in \mathscr{L}(X_2)$.

THEOREM. (Treil, Volberg) *Assume that S is expansive and that B is a contraction and let $\Gamma : X_1 \to X_2$ be a Hankel operator. Let σ_n be a fixed singular value of Γ. Then there exists an S-invariant subspace $\mathscr{M}$ with* $\operatorname{codim} \mathscr{M} = n$ *such that* $\|\Gamma|_{\mathscr{M}}\| = \sigma_n$.

Their proof relies on a fixed point lemma by Ky Fan and does not imply anything concerning the singular vectors. In particular, it is not clear whether

$$\mathscr{M} = [u_n]_S \tag{4.2}$$

holds, which in terms of applications is important as it provides a way to actually calculate $\mathscr{M}$. Clearly (4.2) is not to be expected in the full generality of the above theorem. For instance, if $\operatorname{codim} S(X_1) > 1$ it is easy to see that $\operatorname{codim} [u]_S = \infty$ for all $u \in X_1$. However, for the concrete case considered in Example 1.3, this question is very natural. The conditions on S (the shift) and B (the backward shift) are then equivalent with the weights w and v being increasing. Take for instance a Hankel operator whose symbol is a polynomial from the Dirichlet space into itself, i.e. we set $w = (1,2,3,\dots)$ and consider $\Gamma : l^2_w \to l^2_w$. Fix n and let u_n be a σ_n-singular vector. Assume for simplicity that $\sigma_{n+1} < \sigma_n < \sigma_{n-1}$. Theorem B combined with (1.3) and well known characterizations of subspaces with finite codimension (see e.g. [2]) then show that $\check{u}_n(z) = \sum_{m=0}^{\infty} u_n(m) z^m$ has at least n zeroes in the unit disc, (counted with multiplicities). In the unweighted case we know that it has precisely n-zeroes, (counted with multiplicities). By studying Hankel operators $\Gamma : l^2_w \to l^2_v$ for a large variety of increasing weights, we have not been able to find one example where $\check{u}_n$ does not have n zeroes. Moreover, we are able to prove that we indeed have

$$\operatorname{codim} [u_n]_S = n \tag{4.3}$$

in the special case when $w = (1, R, R^2, \dots)$ for $R \geqslant 1$ and v is an arbitrary increasing sequence, as well as when $v = w = (j^\alpha)_{j=1}^\infty$ for $0 \leqslant \alpha \leqslant 1$. A full extension of the AAK-theorem in the second form (AAK*) is thus available, for instance, for Hankel operators from the Hardy to the Dirichlet space or from the Dirichlet space into itself. As the methods for the proof of (4.3) are quite lengthy and completely different from the ones presented here, and as we hope to improve the argument, we will publish this separately. The details can also be found in [4].

5. Counterexamples

In this section we provide a counterexample to two natural questions related to the material in the previous section. The first concerns the original version of the AAK-theorem in the weighted setting, the second concerns Nehari's theorem.

Let us consider a Hankel operator $\Gamma : l^2_w \to l^2_v$, where w, v are increasing sequences, and let n be such that $\sigma_{n+1} < \sigma_n < \sigma_{n-1}$. By the remarks at the end of the last section, $\check{u}_n$ has precisely n zeroes, (at least if $v = (1,1,1,\dots)$). We assume that these zeroes are distinct, (which generically is the case), and label them $z_1, \dots, z_n$. Theorem AAK naturally leads to the question of whether

$$\inf\left\{ \left\| \Gamma - \sum_{j=1}^{n} c_j \Gamma_{(z_j^k)_{k=0}^\infty} \right\|_{l^2_w \to l^2_v} : c_1, \dots, c_n \in \mathbb{C} \right\} = \sigma_n. \tag{5.1}$$

This is in general false, as the following example shows.

EXAMPLE 5.1. Assume that $w_0 = v_0 = 1$ and let Γ be defined by the sequence $\gamma = e_1$ as in (1.1). Since $\operatorname{Ran} S^2$ is a reducing subspace for Γ, it suffices to consider

$$\tilde{\Gamma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\tilde{\Gamma}^* = \begin{pmatrix} 1 & 0 \\ 0 & w_1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v_1 \end{pmatrix}$$

so $\tilde{\Gamma}^*\tilde{\Gamma} = \begin{pmatrix} v_1 & 0 \\ 0 & w_1^{-1} \end{pmatrix}$ and therefore $\sigma_0 = \sqrt{v_1}$, $u_0 = e_0$, $\sigma_1 = 1/\sqrt{w_1}$, $u_1 = e_1/\sqrt{w_1}$, since $w_1, v_1 \geqslant 1$. We thus get $\check{u}_0(z) = 1$ and $\check{u}_1 = z/\sqrt{w_1}$. Note that $\check{u}_0$ has no zeroes in $\mathbb{D}$ whereas $\check{u}_1$ has 1 zero at 0. The rank one Hankel operator corresponding to 0 is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. If (5.1) were to hold, then we should have

$$\inf\left\{ \left\| \begin{pmatrix} -c_1 & 1 \\ 1 & 0 \end{pmatrix} \right\|_{l^2_w \to l^2_v} : c \in \mathbb{C} \right\} = \sigma_1 = 1/\sqrt{w_1}. \tag{5.2}$$

(To be formally correct, we should introduce a new notation for the restriction of l^2_v and l^2_w to $\mathsf{Span}\,\{e_0, e_1\}$, but we omit this technicality.) However,

$$\left\| \begin{pmatrix} -c_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|_{l^2_v} \Big/ \left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|_{l^2_w} = \sqrt{c_1^2 + v_1},$$

which clearly is greater than $1/\sqrt{w_1}$ unless $c_1 = 0$ and $v_1 = w_1 = 1$.

Incidentally, this also provides a counterexample related to Nehari's theorem. Let us write $l^2(\mathbb{N})$ and $l^2(\mathbb{Z})$ for the standard unweighted l^2-spaces. Nehari's theorem says that an operator

$$\Gamma : l^2(\mathbb{N}) \to l^2(\mathbb{N})$$

that satisfies $B\Gamma = \Gamma S$ can be "lifted" to an operator $\tilde{\Gamma} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ such that $B\tilde{\Gamma} = \tilde{\Gamma}S$, $\|\tilde{\Gamma}\| = \|\Gamma\|$ and

$$\Gamma = P_{l^2(\mathbb{N})}\tilde{\Gamma}|_{l^2(\mathbb{N})}, \tag{5.3}$$

where $P_{l^2(\mathbb{N})} : l^2(\mathbb{Z}) \to l^2(\mathbb{N})$ denotes the orthogonal projection. Assume now that v, w are increasing positive sequences on $\mathbb{Z}$, and define $l^2_v(\mathbb{N})$, $l^2_v(\mathbb{Z})$, $l^2_w(\mathbb{N})$, $l^2_w(\mathbb{Z})$ and $P_{l^2_v(\mathbb{N})} : l^2_v(\mathbb{Z}) \to l^2_v(\mathbb{N})$ in the obvious way. The extension of Nehari's theorem by S. Treil and A. Volberg in [10] shows that any operator $\Gamma : l^2_w(\mathbb{N}) \to l^2_v(\mathbb{N})$ satisfying $B\Gamma = \Gamma S$ can be lifted to an operator $\tilde{\Gamma} : l^2_w(\mathbb{N}) \to l^2_v(\mathbb{Z})$ such that $B\tilde{\Gamma} = \tilde{\Gamma}S$, $\|\tilde{\Gamma}\| = \|\Gamma\|$ and

$$\Gamma = P_{l^2_v(\mathbb{N})}\tilde{\Gamma}.$$

However, Example 5.1 shows that it is not possible in general to find a $\tilde{\Gamma} : l^2_v(\mathbb{Z}) \to l^2_w(\mathbb{Z})$ with $B\tilde{\Gamma} = \tilde{\Gamma}S$, $\|\tilde{\Gamma}\| = \|\Gamma\|$ and

$$\Gamma = P_{l^2_v(\mathbb{N})}\tilde{\Gamma}|_{l^2_w(\mathbb{N})},$$

because this would imply that (5.2) holds.

Acknowledgements. This research was supported under NSF CMG grant DMS 0724644.

REFERENCES

[1] V. M. ADAMJAN, D. Z. AROV, M. G. KREIN, *Analytic properties of the Schmidt pairs of a Hankel operator and the generalized Schur-Takagi problem* (Russian), Mat. Sb. (N.S.), **86**, 128 (1971), 34–75.

[2] A. ALEMAN, *Finite codimensional invariant subspaces in Hilbert spaces of analytic functions*, J. Funct. Anal., **119**, 1 (1994), 1–18.

[3] J. R. BUTZ, *s-numbers of Hankel matrices*, J. Functional Analysis, **15** (1974), 297–305.

[4] M. CARLSSON, *AAK-theory on weighted spaces*, GMIG-project review report 2009, Purdue University.

[5] J.B. CONWAY, *A course in functional analysis*, Second edition, Graduate Texts in Mathematics, 96. Springer-Verlag, New York, 1990.

[6] C. FOIAS, S. SZ-NAGY, *Harmonic analysis of operators on Hilbert space.Harmonic analysis of operators on Hilbert space*, North-Holland Publishing Co., 1970.

[7] N. K. NIKOLSKI, *Operators, functions, and systems: an easy reading. Vol. 1. Hardy, Hankel, and Toeplitz*, Mathematical Surveys and Monographs, 92. American Mathematical Society, Providence, RI, 2002.

[8] V.V. PELLER, *Hankel operators and their applications*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.

[9] R. ROCHBERG, *A new characterization of Dirichlet type spaces and applications*, Illinois J. Math., **37**, 1 (1993), 101–122.

[10] S. TREIL, A. VOLBERG, *A fixed point approach to Nehari's problem and its applications*, 165–186, Oper. Theory Adv. Appl., 71, Birkhäuser, Basel, 1994.

(Received December 4, 2009)

Marcus Carlsson
Mathematics Department, Purdue University
150 N. University St. W. Lafayette
47906 IN, USA
e-mail: mcarlsso@math.purdue.edu