Operators
and
Matrices
Volume 5, Number 2 (2011), 239–260

GENERALIZED BICIRCULAR PROJECTIONS VIA RANK PRESERVING MAPS ON THE SPACES OF SYMMETRIC AND ANTISYMMETRIC OPERATORS

AJDA FOŠNER AND DIJANA ILIŠEVIĆ

(*Communicated by N.-C. Wong*)

Abstract. We study several related maps from the space of symmetric (respectively, antisymmetric) operators, acting on a complex Hilbert space, to itself: rank preserving linear maps (more precisely, maps preserving rank one operators in the symmetric case, and maps preserving rank two operators in the antisymmetric case), surjective linear isometries and generalized bicircular projections.

1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space and let $\mathscr{B}(\mathscr{H})$ be the algebra of all bounded linear operators on $\mathscr{H}$. Throughout this paper we will fix an orthonormal basis $\{e_\lambda : \lambda \in \Lambda\}$ of $\mathscr{H}$. For $x \in \mathscr{H}$ we have $x = \sum_{\lambda\in\Lambda} \langle x, e_\lambda \rangle e_\lambda$ and define $\overline{x} = \sum_{\lambda\in\Lambda} \langle e_\lambda, x \rangle e_\lambda$. Notice that $\langle \overline{x}, \overline{y} \rangle = \langle y, x \rangle$ for all $x, y \in \mathscr{H}$.

Let $T \in \mathscr{B}(\mathscr{H})$. If $S \in \mathscr{B}(\mathscr{H})$ is such that $\langle Te_\lambda, e_\mu \rangle = \langle Se_\mu, e_\lambda \rangle$ for all $\lambda, \mu \in \Lambda$, then S is called the transpose of T associated to the basis $\{e_\lambda : \lambda \in \Lambda\}$ and it is denoted by T^t. An easy computation shows $T^t x = \overline{T^* \overline{x}}$ for all $x \in \mathscr{H}$. This implies $\langle T^t x, y \rangle = \langle T\overline{y}, \overline{x} \rangle$ for all $x, y \in \mathscr{H}$. If $T^t = T$, then T is called a symmetric operator, and if $T^t = -T$, then T is called an antisymmetric operator.

For $x, y \in \mathscr{H}$ we write $x \otimes y$ for the operator defined by $(x \otimes y)(\xi) = \langle \xi, y \rangle x$ for all $\xi \in \mathscr{H}$. Clearly, $x \otimes y \in \mathscr{B}(\mathscr{H})$ and $\|x \otimes y\| = \|x\| \cdot \|y\|$. If x and y are nonzero, then $x \otimes y$ is rank one. It can be easily verified that $(x \otimes y)^t = \overline{y} \otimes \overline{x}$ for all $x, y \in \mathscr{H}$.

By $\mathscr{S}(\mathscr{H})$ and $\mathscr{A}(\mathscr{H})$ we will denote the linear subspaces of all symmetric and antisymmetric operators in $\mathscr{B}(\mathscr{H})$, respectively. The matrix space $\mathscr{S}_n(\mathbb{C})$ of all $n \times n$ complex symmetric matrices, and the matrix space $\mathscr{K}_n(\mathbb{C})$ of all $n \times n$ complex skew-symmetric matrices, equipped with the spectral norm, can be considered as a special (finite dimensional) case of $\mathscr{S}(\mathscr{H})$ and $\mathscr{A}(\mathscr{H})$, respectively.

Let X be a complex Banach space. A linear map $P : X \to X$ satisfying $P^2 = P$ is called a linear projection on X. By $\overline{P}$ is denoted its complementary projection $I - P$, where I is the identity operator on X.

Mathematics subject classification (2010): Primary: 47A65, Secondary: 47B49.

Keywords and phrases: Generalized bicircular projections, rank preserving maps, symmetric operators, antisymmetric operators.

The second author was supported by the Ministry of Science, Education and Sports of the Republic of Croatia (project No. 037-0372784-2757).

Paper OaM-05-17

DEFINITION 1.1. A linear projection P on X is called bicircular if the map $P+\lambda\overline{P}$ is an isometry for all modulus one $\lambda\in\mathbb{C}$, and it is called a generalized bicircular projection if the map $P+\lambda\overline{P}$ is an isometry for some modulus one $\lambda\in\mathbb{C},\ \lambda\neq 1$.

The structure of bicircular projections on $\mathscr{S}(\mathscr{H})$ and $\mathscr{A}(\mathscr{H})$ is determined in [9, Theorems 2.3 and 2.5], and the structure of generalized bicircular projections on $\mathscr{S}_n(\mathbb{C})$ and $\mathscr{K}_n(\mathbb{C})$, with respect to the spectral norm, in [4, Theorems 5.1 and 5.2]. The aim of this paper is to fill a gap between these results by deducing the form of generalized bicircular projections on $\mathscr{S}(\mathscr{H})$ and $\mathscr{A}(\mathscr{H})$, where $\mathscr{H}$ is infinite dimensional. Although most of our results hold for finite dimensional $\mathscr{H}$ as well, we will assume throughout the paper that $\mathscr{H}$ is infinite dimensional.

By [9, Theorem 2.3] there are only trivial bicircular projections on $\mathscr{S}(\mathscr{H})$, that is, zero and the identity operator. However, if P is a bicircular projection on $\mathscr{A}(\mathscr{H})$, then either P or $\overline{P}$ is of the form $X\mapsto RX+XR^t$ with $R=x\otimes x$ for some norm one $x\in\mathscr{H}$, see [9, Theorem 2.5].

According to the recent general result [5, Theorem 2.1] in the setting of JB*-triples, a generalized bicircular projection P acting on a JB*-triple A is either bicircular, or $\lambda=-1$ and $P=\frac{1}{2}(I+\varphi)$ for some surjective linear isometry $\varphi:A\to A$ satisfying $\varphi^2=I$. Since $\mathscr{S}(\mathscr{H})$ and $\mathscr{A}(\mathscr{H})$ belong to the class of JB*-triples and the structure of bicircular projections on these spaces has been already known, it remains to determine the structure of surjective linear isometries on these spaces. It turns out that this problem can be reduced to the problem of linear maps preserving rank, more precisely, preserving rank one operators in $\mathscr{S}(\mathscr{H})$ and preserving rank two operators in $\mathscr{A}(\mathscr{H})$. Recall that every surjective linear isometry $\varphi:A\to A$, where $A=\mathscr{S}(\mathscr{H})$ or $A=\mathscr{A}(\mathscr{H})$, satisfies $\varphi(XY^*X)=\varphi(X)\varphi(Y)^*\varphi(X)$ for all $X,Y\in A$ (the corresponding result in a more general setting of JB*-triples can be found in [6] and [3, Theorem D]). Our purpose is to find an explicit formula for φ.

The linear preserver problem which concerns characterization of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant has been one of the most active and fertile subjects in matrix theory during the past one hundred years. In the last few decades a lot of results on linear preservers, not only on matrix algebras, but also on more general rings and operator algebras, have been obtained. It has turned out that one of the most important classes of linear preserver problems in matrix and operator theory is the one concerning rank. More information on linear (as well as nonlinear) preserver problems can be found e.g. in a book [7], where an extensive list of references on this subject is given.

2. Generalized bicircular projections on $\mathscr{S}(\mathscr{H})$

If $X\in\mathscr{B}(\mathscr{H})$ is a rank one operator and u is a nonzero element of its range, it is an easy exercise to prove the existence of $v\in\mathscr{H}$ such that $X=u\otimes\overline{v}$ (e.g. [8, p. 56]). If $X\in\mathscr{S}(\mathscr{H})$, then $u\otimes\overline{v}=v\otimes\overline{u}$. Thus there exists $\alpha\in\mathbb{C}$ such that $v=\alpha u$. If we define $x=\alpha^{1/2}u$, then

$$\begin{aligned}\langle (u\otimes\overline{v})e_\lambda, e_\mu\rangle &= \langle v, e_\lambda\rangle\langle u, e_\mu\rangle = \alpha\langle u, e_\lambda\rangle\langle u, e_\mu\rangle \\ &= \langle x, e_\lambda\rangle\langle x, e_\mu\rangle = \langle (x\otimes\overline{x})e_\lambda, e_\mu\rangle\end{aligned}$$

for all $\lambda,\mu\in\Lambda$. Hence, $X = x\otimes\overline{x}$. If $x\otimes\overline{x} = y\otimes\overline{y}$, then $y = x$ or $y = -x$. Therefore, every symmetric rank one operator has the form $x\otimes\overline{x}$ for some unique (up to a sign) $x\in\mathscr{H}$.

LEMMA 2.1. *A nonzero operator $X\in\mathscr{S}(\mathscr{H})$ is rank one if and only if $X\mathscr{S}(\mathscr{H})X = \mathbb{C}X$.*

Proof. If $X\in\mathscr{S}(\mathscr{H})$ is rank one, then there exists $x\in\mathscr{H}$ such that $X = x\otimes\overline{x}$. For all $Y\in\mathscr{S}(\mathscr{H})$ and $y\in\mathscr{H}$ we have

$$\begin{aligned}XYXy &= (x\otimes\overline{x})Y(x\otimes\overline{x})y = (x\otimes\overline{x})Y(\langle y,\overline{x}\rangle x) \\ &= \langle y,\overline{x}\rangle(x\otimes\overline{x})Yx = \langle y,\overline{x}\rangle\langle Yx,\overline{x}\rangle x = \langle Yx,\overline{x}\rangle(x\otimes\overline{x})y = \langle Yx,\overline{x}\rangle Xy.\end{aligned}$$

Therefore, $X\mathscr{S}(\mathscr{H})X\subseteq\mathbb{C}X$. Fix $\lambda\in\mathbb{C}$. If $\langle Yx,\overline{x}\rangle = 0$ for all $Y\in\mathscr{S}(\mathscr{H})$, then $X\mathscr{S}(\mathscr{H})X = 0$. In particular, $XX^*X = 0$ which yields $X = 0$. Hence, there exists $Y\in\mathscr{S}(\mathscr{H})$ such that $\langle Yx,\overline{x}\rangle\neq 0$. If we define $Z = \frac{\lambda}{\langle Yx,\overline{x}\rangle}Y$, then $Z\in\mathscr{S}(\mathscr{H})$ and $XZX = \lambda X$. Hence, $\mathbb{C}X\subseteq X\mathscr{S}(\mathscr{H})X$.

Assume that a nonzero $X\in\mathscr{S}(\mathscr{H})$ is such that $X\mathscr{S}(\mathscr{H})X = \mathbb{C}X$. There exists $\lambda\in\Lambda$ such that $Xe_\lambda\neq 0$. Furthermore, there exists $\alpha\in\mathbb{C}$ with the property $X(e_\lambda\otimes e_\lambda)X = \alpha X$. Thus,

$$\langle Xx, e_\lambda\rangle Xe_\lambda = \alpha Xx$$

for all $x\in\mathscr{H}$. If $\alpha = 0$, then $\langle Xx, e_\lambda\rangle Xe_\lambda = 0$ for all $x\in\mathscr{H}$. This implies $\langle Xx, e_\lambda\rangle = 0$ for all $x\in\mathscr{H}$. In particular, for all $\mu\in\Lambda$,

$$\langle Xe_\lambda, e_\mu\rangle = \langle Xe_\mu, e_\lambda\rangle = 0,$$

which yields $Xe_\lambda = 0$; a contradiction. Hence, $\alpha\neq 0$. Thus,

$$Xx = \frac{1}{\alpha}\langle Xx, e_\lambda\rangle Xe_\lambda\in\mathbb{C}(Xe_\lambda),$$

so X is rank one. $\square$

LEMMA 2.2. *If $A\in\mathscr{S}(\mathscr{H})$ is such that $XAX = 0$ for all rank one operators $X\in\mathscr{S}(\mathscr{H})$, then $A = 0$.*

Proof. Let $x\in\mathscr{H}$ be nonzero and let $X = x\otimes\overline{x}$. Then

$$\begin{aligned}0 = XAX\overline{x} &= (x\otimes\overline{x})A(x\otimes\overline{x})\overline{x} \\ &= \langle x,x\rangle(x\otimes\overline{x})Ax = \langle x,x\rangle\langle Ax,\overline{x}\rangle x.\end{aligned}$$

Thus $\langle Ax,\overline{x}\rangle = 0$ for all $x \in \mathscr{H}$. Inserting $x+\overline{y}$ instead of x, and using the assumption that A is symmetric, we get

$$0 = \langle Ax, y\rangle + \langle A\overline{y},\overline{x}\rangle = 2\langle Ax, y\rangle$$

for all $x, y \in \mathscr{H}$. Hence, $A = 0$. $\square$

LEMMA 2.3. *If $A, B \in \mathscr{B}(\mathscr{H})$ are such that $AXA^t = BXB^t$ for all rank one operators $X \in \mathscr{S}(\mathscr{H})$, then $A = B$ or $A = -B$.*

Proof. For all $x, y \in \mathscr{H}$ we have

$$A(x \otimes \overline{x})A^t\overline{y} = \langle A^t\overline{y},\overline{x}\rangle Ax = \langle Ax, y\rangle Ax$$

and analogously

$$B(x \otimes \overline{x})B^t\overline{y} = \langle Bx, y\rangle Bx.$$

Our assumption yields

$$\langle Ax, y\rangle Ax = \langle Bx, y\rangle Bx. \tag{1}$$

This implies

$$\langle Ax, y\rangle^2 = \langle Bx, y\rangle^2. \tag{2}$$

Assume $A \neq B$ and $A \neq -B$. Let $x_1 \in \mathscr{H}$ be such that $Ax_1 \neq Bx_1$. Then there exists $y_1 \in \mathscr{H}$ such that $\langle Ax_1, y_1\rangle \neq \langle Bx_1, y_1\rangle$. By (2), we have $\langle Ax_1, y_1\rangle = -\langle Bx_1, y_1\rangle$. Then (1) implies

$$\langle Bx_1, y_1\rangle(Ax_1 + Bx_1) = 0.$$

Since $\langle Bx_1, y_1\rangle = -\langle Ax_1, y_1\rangle \neq -\langle Bx_1, y_1\rangle$, we have $\langle Bx_1, y_1\rangle \neq 0$. Hence, $Ax_1 = -Bx_1$. Let $x_2 \in \mathscr{H}$ be such that $Ax_2 \neq -Bx_2$. In the same manner we conclude $Ax_2 = Bx_2$. Inserting $x_1 + x_2$ instead of x and y_1 instead of y in (1), we get

$$\langle Bx_1, y_1\rangle Bx_2 + \langle Bx_2, y_1\rangle Bx_1 = 0, \tag{3}$$

which implies

$$\langle Bx_1, y_1\rangle\langle Bx_2, y_1\rangle = 0.$$

Since $\langle Bx_1, y_1\rangle \neq 0$, we have $\langle Bx_2, y_1\rangle = 0$. Notice that $Bx_2 = Ax_2 \neq -Bx_2$ implies $Bx_2 \neq 0$. Then the first summand in (3) is nonzero, and the second one is zero; a contradiction. Hence, $A = B$ or $A = -B$. $\square$

PROPOSITION 2.4. *Let $\varphi : \mathscr{S}(\mathscr{H}) \to \mathscr{S}(\mathscr{H})$ be a bounded injective linear map preserving rank one operators. Then there exists a unique (up to a sign) $T \in \mathscr{B}(\mathscr{H})$ such that*

$$\varphi(X) = TXT^t$$

for all rank one $X \in \mathscr{S}(\mathscr{H})$. Furthermore, the map T is injective.

Proof. Let us fix a nonzero $x_0 \in \mathscr{H}$ and write $\varphi(x_0 \otimes \overline{x_0}) = u_0 \otimes \overline{u_0}$ with $u_0 \in \mathscr{H}$. Let $x \in \mathscr{H}$ be such that x and x_0 are linearly independent and write $\varphi(x \otimes \overline{x}) = u \otimes \overline{u}$ for some $u \in \mathscr{H}$. Then u and u_0 are also linearly independent.

Notice that

$$\begin{aligned}\varphi(x_0 \otimes \overline{x} + x \otimes \overline{x_0}) &= \frac{1}{2}\varphi\big((x_0+x) \otimes (\overline{x_0+x})\big) - \frac{1}{2}\varphi\big((x_0-x) \otimes (\overline{x_0-x})\big)\\ &= \frac{1}{2} v \otimes \overline{v} - \frac{1}{2} w \otimes \overline{w}\end{aligned}$$

for some linearly independent $v, w \in \mathscr{H}$. On the other hand,

$$\begin{aligned}\varphi(x_0 \otimes \overline{x} + x \otimes \overline{x_0}) &= \varphi\big((x_0+x) \otimes (\overline{x_0+x})\big) - \varphi(x_0 \otimes \overline{x_0}) - \varphi(x \otimes \overline{x})\\ &= v \otimes \overline{v} - u_0 \otimes \overline{u_0} - u \otimes \overline{u}.\end{aligned}$$

Hence,

$$\frac{1}{2} v \otimes \overline{v} + \frac{1}{2} w \otimes \overline{w} = u_0 \otimes \overline{u_0} + u \otimes \overline{u}.$$

If $\xi \in \mathscr{H}$ is such that $\langle w, \xi \rangle = 0$ and $\langle v, \xi \rangle = 1$, then

$$v = 2\langle u_0, \xi \rangle u_0 + 2\langle u, \xi \rangle u,$$

that is, $v = \alpha u_0 + \beta u$ for some $\alpha, \beta \in \mathbb{C}$. This implies

$$\begin{aligned}\varphi(x_0 \otimes \overline{x} + x \otimes \overline{x_0}) &= (\alpha u_0 + \beta u) \otimes (\overline{\alpha u_0 + \beta u}) - u_0 \otimes \overline{u_0} - u \otimes \overline{u}\\ &= (\alpha^2 - 1) u_0 \otimes \overline{u_0} + \alpha\beta(u_0 \otimes \overline{u} + u \otimes \overline{u_0}) + (\beta^2 - 1) u \otimes \overline{u}. \qquad (4)\end{aligned}$$

Assume that $\beta^2 - 1 \neq 0$. Let $\lambda = \frac{(\alpha\beta)^2}{\beta^2-1} - \alpha^2 + 1$. Then

$$\begin{aligned}&\varphi(x_0 \otimes \overline{x} + x \otimes \overline{x_0} + \lambda x_0 \otimes \overline{x_0})\\ &= \varphi(x_0 \otimes \overline{x} + x \otimes \overline{x_0}) + \lambda \varphi(x_0 \otimes \overline{x_0})\\ &= (\alpha^2 - 1 + \lambda) u_0 \otimes \overline{u_0} + \alpha\beta(u_0 \otimes \overline{u} + u \otimes \overline{u_0}) + (\beta^2 - 1) u \otimes \overline{u}\\ &= u_0 \otimes \overline{\left(\frac{(\alpha\beta)^2}{\beta^2 - 1} u_0 + \alpha\beta u\right)} + u \otimes \overline{(\alpha\beta u_0 + (\beta^2 - 1) u)}\\ &= \left(\frac{\alpha\beta}{\beta^2 - 1} u_0 + u\right) \otimes \overline{(\alpha\beta u_0 + (\beta^2 - 1) u)}\\ &= \frac{1}{\beta^2 - 1}\big(\alpha\beta u_0 + (\beta^2 - 1) u\big) \otimes \overline{\big(\alpha\beta u_0 + (\beta^2 - 1) u\big)}\end{aligned}$$

is rank one. Thus, $\lambda \neq 0$. However,

$$\begin{aligned}&\varphi(x_0 \otimes \overline{x} + x \otimes \overline{x_0} + \lambda x_0 \otimes \overline{x_0})\\ &= \varphi\big((\lambda^{-1/2} x + \lambda^{1/2} x_0) \otimes (\overline{\lambda^{-1/2} x + \lambda^{1/2} x_0})\big) - \lambda^{-1} \varphi(x \otimes \overline{x})\end{aligned}$$

is rank two; a contradiction. Therefore, $\beta^2 - 1 = 0$. Analogously, $\alpha^2 - 1 = 0$. Hence, $\alpha\beta = 1$ or $\alpha\beta = -1$. Let us define $Tx = \alpha\beta u$. Then (4) can be written as

$$\varphi(x_0 \otimes \overline{x} + x \otimes \overline{x_0}) = u_0 \otimes \overline{Tx} + Tx \otimes \overline{u_0}.$$

We also have

$$\varphi(x\otimes\overline{x}) = u\otimes\overline{u} = \alpha\beta u\otimes\overline{\alpha\beta u} = Tx\otimes\overline{Tx}.$$

If $x=\mu x_0$ for some $\mu\in\mathbb{C}$, define $Tx=\mu u_0$. Then

$$\begin{aligned}
&\varphi(x_0\otimes\overline{x}+x\otimes\overline{x_0}) = 2\mu\varphi(x_0\otimes\overline{x_0}) = 2\mu u_0\otimes\overline{u_0}\\
&= u_0\otimes\overline{\mu u_0}+\mu u_0\otimes\overline{u_0} = u_0\otimes\overline{Tx}+Tx\otimes\overline{u_0},
\end{aligned}$$

and also

$$\varphi(x\otimes\overline{x}) = \mu^2\varphi(x_0\otimes\overline{x_0}) = \mu u_0\otimes\overline{\mu u_0} = Tx\otimes\overline{Tx}.$$

Hence, for all $x\in\mathscr{H}$,

$$\varphi(x\otimes\overline{x}) = Tx\otimes\overline{Tx},$$

$$\varphi(x_0\otimes\overline{x}+x\otimes\overline{x_0}) = Tx_0\otimes\overline{Tx}+Tx\otimes\overline{Tx_0}.$$

Since φ is linear, for all $\lambda\in\mathbb{C},\ x,y\in\mathscr{H}$,

$$\begin{aligned}
&Tx_0\otimes\overline{T(\lambda x+y)}+T(\lambda x+y)\otimes\overline{Tx_0}\\
&=\varphi(x_0\otimes(\overline{\lambda x+y})+(\lambda x+y)\otimes\overline{x_0})\\
&=\lambda\varphi(x_0\otimes\overline{x}+x\otimes\overline{x_0})+\varphi(x_0\otimes\overline{y}+y\otimes\overline{x_0})\\
&=\lambda(Tx_0\otimes\overline{Tx}+Tx\otimes\overline{Tx_0})+Tx_0\otimes\overline{Ty}+Ty\otimes\overline{Tx_0}\\
&=Tx_0\otimes(\overline{\lambda Tx+Ty})+(\lambda Tx+Ty)\otimes\overline{Tx_0},
\end{aligned}$$

which implies

$$Tx_0\otimes\left(\overline{T(\lambda x+y)-(\lambda Tx+Ty)}\right)+\left(T(\lambda x+y)-(\lambda Tx+Ty)\right)\otimes\overline{Tx_0}=0$$

and we finally conclude that T is linear as well. Furthermore, for all $x\in\mathscr{H}$,

$$\|Tx\|^2 = \|Tx\otimes\overline{Tx}\| = \|\varphi(x\otimes\overline{x})\| \leqslant \|\varphi\|\cdot\|x\otimes\overline{x}\| = \|\varphi\|\cdot\|x\|^2,$$

hence $T\in\mathscr{B}(\mathscr{H})$.

For all $x,y,z\in\mathscr{H}$,

$$\begin{aligned}
&\langle T(x\otimes\overline{x})T^t y,z\rangle = \langle T^t y,\overline{x}\rangle\langle Tx,z\rangle = \langle Tx,\overline{y}\rangle\langle Tx,z\rangle\\
&=\langle y,\overline{Tx}\rangle\langle Tx,z\rangle = \langle (Tx\otimes\overline{Tx})y,z\rangle = \langle\varphi(x\otimes\overline{x})y,z\rangle.
\end{aligned}$$

Hence, $\varphi(X)=TXT^t$ for all rank one $X\in\mathscr{S}(\mathscr{H})$. By Lemma 2.3, we conclude that such T is unique, up to a sign.

It remains to prove that T is injective. Let $x\in\mathscr{H}$ be such that $Tx=0$. Then $\varphi(x\otimes\overline{x})=0$, thus injectivity of φ yields $x\otimes\overline{x}=0$ and finally $x=0$. $\square$

THEOREM 2.5. *Let $\varphi:\mathscr{S}(\mathscr{H})\to\mathscr{S}(\mathscr{H})$ be a surjective linear isometry. Then there exists a unitary $U\in\mathscr{B}(\mathscr{H})$ such that*

$$\varphi(X)=UXU^t$$

for all $X\in\mathscr{S}(\mathscr{H})$.

Proof. If $X \in \mathscr{S}(\mathscr{H})$ is a rank one operator, then $X\mathscr{S}(\mathscr{H})X = \mathbb{C}X$ according to Lemma 2.1. This implies

$$\mathbb{C}\varphi(X) = \varphi(\mathbb{C}X) = \varphi(X\mathscr{S}(\mathscr{H})X) = \varphi(X)\varphi(\mathscr{S}(\mathscr{H}))^*\varphi(X) = \varphi(X)\mathscr{S}(\mathscr{H})\varphi(X).$$

Lemma 2.1 implies that $\varphi(X)$ is rank one as well.

According to Proposition 2.4 there exists $U \in \mathscr{B}(\mathscr{H})$ such that

$$\varphi(X) = UXU^t$$

for all rank one $X \in \mathscr{S}(\mathscr{H})$.

Let $X, Y \in \mathscr{S}(\mathscr{H})$ and let Y be a rank one operator. Then

$$\varphi(YX^*Y) = \varphi(Y)\varphi(X)^*\varphi(Y)$$

implies

$$\varphi(YX^*Y) = UYU^t\varphi(X)^*UYU^t,$$

that is

$$\varphi(YX^*Y) = \varphi(YU^t\varphi(X)^*UY)$$

since $YU^t\varphi(X)^*UY \in \mathscr{S}(\mathscr{H})$ is either zero or a rank one operator. Injectivity of φ yields

$$YX^*Y = YU^t\varphi(X)^*UY.$$

Lemma 2.2 implies

$$X^* = U^t\varphi(X)^*U,$$

hence

$$X = U^*\varphi(X)(U^t)^* \tag{5}$$

for all $X \in \mathscr{S}(\mathscr{H})$. Since φ is bijective, replacing X with $\varphi^{-1}(X)$ we get

$$\varphi^{-1}(X) = U^*X(U^t)^*,$$

that is

$$X = \varphi(U^*X(U^t)^*)$$

for all $X \in \mathscr{S}(\mathscr{H})$. In particular, if $X \in \mathscr{S}(\mathscr{H})$ is rank one, then $U^*X(U^t)^* \in \mathscr{S}(\mathscr{H})$ is either zero or rank one, so we have

$$X = UU^*X(U^t)^*U^t.$$

Lemma 2.3 yields $UU^* = I$ or $UU^* = -I$. Since UU^* is positive, $UU^* = I$. Then (5) implies

$$\varphi(X) = UXU^t \tag{6}$$

for all $X \in \mathscr{S}(\mathscr{H})$. Inserting (6) in (5) we get

$$X = U^*UXU^t(U^t)^*,$$

thus $U^*U = I$ or $U^*U = -I$ by Lemma 2.3. Since U^*U is positive, $U^*U = I$. Hence, U is unitary. $\square$

REMARK 2.1. The proof of Theorem 2.5 can be obtained using the results from [1]. However, we decided to prove it via Proposition 2.4 for the sake of completeness, since we deal with the maps having more properties than those in [1].

COROLLARY 2.6. *Let* $P : \mathscr{S}(\mathscr{H}) \to \mathscr{S}(\mathscr{H})$ *be a nontrivial linear projection and* $\lambda \neq 1$ *a modulus one complex number. Then* $P + \lambda\overline{P}$ *is an isometry if and only if* $\lambda = -1$ *and there exists* $R = R^* = R^2 \in \mathscr{B}(\mathscr{H})$ *such that* P *or* $\overline{P}$ *has the form* $X \mapsto RXR^t + (I-R)X(I-R^t)$.

Proof. According to [9, Theorem 2.3], there are no nontrivial bicircular projections on $\mathscr{S}(\mathscr{H})$, so [5, Theorem 2.1] implies $\lambda = -1$ and $P = \frac{1}{2}(I+\varphi)$ for some surjective linear isometry $\varphi : \mathscr{S}(\mathscr{H}) \to \mathscr{S}(\mathscr{H})$ such that $\varphi^2 = I$. According to Theorem 2.5, there exists a unitary $U \in \mathscr{B}(\mathscr{H})$ such that $\varphi(X) = UXU^t$ for all $X \in \mathscr{S}(\mathscr{H})$. Since $\varphi^2 = I$, we get

$$U^2X(U^t)^2 = X$$

for all $X \in \mathscr{S}(\mathscr{H})$. Lemma 2.3 implies $U^2 = I$ or $U^2 = -I$.

If $U^2 = I$, then P has the form $X \mapsto RXR^t + (I-R)X(I-R^t)$ with $R = \frac{1}{2}(I-U)$. If $U^2 = -I$, then $\overline{P}$ has the form $X \mapsto RXR^t + (I-R)X(I-R^t)$ with $R = \frac{1}{2}(I-iU)$. In both cases, $R = R^* = R^2$. $\square$

3. Generalized bicircular projections on $\mathscr{A}(\mathscr{H})$

In the setting of $\mathscr{A}(\mathscr{H})$ we follow the same pattern as in the setting of $\mathscr{S}(\mathscr{H})$, but the role of rank one operators is now played by rank two operators. However, as expected, it is more difficult to deal with rank two operators than with rank one operators.

First recall some well-known facts on rank two operators. If $X \in \mathscr{B}(\mathscr{H})$ is rank two, then its range is a two-dimensional Hilbert space. Let $\{e, f\}$ be an orthonormal basis for the range of X. There exist linear functionals f, g on $\mathscr{H}$ such that $X\xi = f(\xi)e + g(\xi)f$ for all $\xi \in \mathscr{H}$. Since $f(\xi) = \langle X\xi, e\rangle$, we conclude that f is bounded. If we define $X_1\xi = f(\xi)e$ for all $\xi \in \mathscr{H}$, then $X_1 \in \mathscr{B}(\mathscr{H})$ is rank one. Analogously, $X_2\xi = g(\xi)f$ defines a rank one $X_2 \in \mathscr{B}(\mathscr{H})$. Hence, X is a sum of two rank one operators in $\mathscr{B}(\mathscr{H})$.

LEMMA 3.1. *If* $X \in \mathscr{A}(\mathscr{H})$ *is a rank two operator, then there exist* $x, y \in \mathscr{H}$ *such that* $X = x \otimes \overline{y} - y \otimes \overline{x}$.

Proof. Let $X = x \otimes \overline{u} + v \otimes \overline{w}$ with $x, u, v, w \in \mathscr{H}$ and x, v linearly independent. Since $X \in \mathscr{A}(\mathscr{H})$,

$$x \otimes \overline{u} + v \otimes \overline{w} = -u \otimes \overline{x} - w \otimes \overline{v}.$$

This implies

$$\langle u, \xi\rangle x + \langle w, \xi\rangle v + \langle x, \xi\rangle u + \langle v, \xi\rangle w = 0 \tag{7}$$

for all $\xi \in \mathscr{H}$. Since x and v are linearly independent, there exists $\xi_1 \in \mathscr{H}$ such that $\langle v, \xi_1 \rangle = 0$ and $\langle x, \xi_1 \rangle = 1$. Then (7) implies $u = \alpha x + \beta v$ for some $\alpha, \beta \in \mathbb{C}$. Furthermore, there exists $\xi_2 \in \mathscr{H}$ such that $\langle x, \xi_2 \rangle = 0$ and $\langle v, \xi_2 \rangle = 1$, so (7) implies $w = \gamma x + \delta v$ for some $\gamma, \delta \in \mathbb{C}$. Now (7) can be written as

$$\langle 2\alpha x + (\beta+\gamma)v, \xi \rangle x + \langle (\beta+\gamma)x + 2\delta v, \xi \rangle v = 0$$

for all $\xi \in \mathscr{H}$. Since x and v are linearly independent,

$$2\alpha x + (\beta+\gamma)v = (\beta+\gamma)x + 2\delta v = 0$$

and finally $\alpha = \delta = 0$ and $\gamma = -\beta$. Hence, $u = \beta v$ and $w = -\beta x$, thus

$$X = x \otimes \overline{\beta v} - v \otimes \overline{\beta x} = x \otimes \overline{\beta v} - \beta v \otimes \overline{x}.$$

It remains to put $y = \beta v$. $\square$

REMARK 3.1. It is clear that for a rank two $X \in \mathscr{A}(\mathscr{H})$ the representation $X = x \otimes \overline{y} - y \otimes \overline{x}$ from Lemma 3.1 is not unique. Namely, if $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are such that $\alpha\delta - \beta\gamma = 1$ and $z = \alpha x + \beta y$, $w = \gamma x + \delta y$, then $z \otimes \overline{w} - w \otimes \overline{z} = x \otimes \overline{y} - y \otimes \overline{x}$.

Let us prove the converse. Let $x, y \in \mathscr{H}$ be linearly independent and let $z, w \in \mathscr{H}$ be linearly independent. Assume that

$$x \otimes \overline{y} - y \otimes \overline{x} = z \otimes \overline{w} - w \otimes \overline{z}.$$

Let $\mathscr{H}_1$ be the linear subspace of $\mathscr{H}$ generated by x and y, and let $\mathscr{H}_2$ be the linear subspace of $\mathscr{H}$ generated by z and w. For $\xi \in \mathscr{H}_1{}^{\perp}$ we have

$$\langle w, \xi \rangle z - \langle z, \xi \rangle w = 0.$$

This implies $\langle w, \xi \rangle = \langle z, \xi \rangle = 0$, so $\xi \in \mathscr{H}_2{}^{\perp}$. Hence, $\mathscr{H}_1{}^{\perp} \subseteq \mathscr{H}_2{}^{\perp}$, which yields $\mathscr{H}_2 \subseteq \mathscr{H}_1$. Thus, $z = \alpha x + \beta y$ and $w = \gamma x + \delta y$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Then

$$x \otimes \overline{y} - y \otimes \overline{x} = (\alpha\delta - \beta\gamma)(x \otimes \overline{y} - y \otimes \overline{x}),$$

which implies $\alpha\delta - \beta\gamma = 1$.

In particular, if $x, y, z \in \mathscr{H}$ are such that $x \otimes \overline{y} - y \otimes \overline{x} = x \otimes \overline{z} - z \otimes \overline{x}$, then the above consideration yields $z = \lambda x + y$ for some $\lambda \in \mathbb{C}$. The converse is trivial.

LEMMA 3.2. *Let $x, y \in \mathscr{H}$ be linearly independent and let $z, w \in \mathscr{H}$ be linearly independent. Let $T = x \otimes \overline{y} - y \otimes \overline{x}$ and $S = z \otimes \overline{w} - w \otimes \overline{z}$. If $T + S$ is rank two, then the set $\{x, y, z, w\}$ is linearly dependent.*

Proof. According to Lemma 3.1, there exist $u, v \in \mathscr{H}$ such that $T + S = u \otimes \overline{v} - v \otimes \overline{u}$. Then

$$x \otimes \overline{y} - y \otimes \overline{x} + z \otimes \overline{w} - w \otimes \overline{z} = u \otimes \overline{v} - v \otimes \overline{u}.$$

Let $\mathscr{H}_0$ be the linear subspace of $\mathscr{H}$ generated by $\{x,y,z\}$. For all $\xi \in \mathscr{H}_0^{\perp}$ we have

$$\langle w,\xi\rangle z = \langle v,\xi\rangle u - \langle u,\xi\rangle v.$$

We consider three cases.

(i) If $\langle u,\xi\rangle = \langle v,\xi\rangle = 0$ for all $\xi \in \mathscr{H}_0^{\perp}$, then $w \in \mathscr{H}_0$. Hence, $w = \alpha x + \beta y + \gamma z$ for some $\alpha,\beta,\gamma \in \mathbb{C}$.

(ii) If there exists $\xi \in \mathscr{H}_0^{\perp}$ such that $\langle u,\xi\rangle \neq 0$, then $v = \alpha u + \beta z$ for some $\alpha,\beta \in \mathbb{C}$. Then $T+S = \beta u \otimes \overline{z} - z \otimes \overline{\beta u}$, which implies $T = (\beta u + w) \otimes \overline{z} - z \otimes \overline{(\beta u + w)}$. According to Remark 3.1, z is a linear combination of x and y. Hence, the set $\{x,y,z\}$ is linearly dependent, and so is $\{x,y,z,w\}$.

(iii) The case when $\langle v,\xi\rangle \neq 0$ for some $\xi \in \mathscr{H}_0^{\perp}$ is analogous to the case (ii). $\square$

LEMMA 3.3. *Let $T,S \in \mathscr{A}(\mathscr{H})$ be rank two operators such that $T+S$ is also rank two. Then there exists $x \in \mathscr{H}$ such that $T = x\otimes\overline{y} - y\otimes\overline{x}$ and $S = x\otimes\overline{z} - z\otimes\overline{x}$ for some $y,z \in \mathscr{H}$. If T and S are linearly independent, such $x \in \mathscr{H}$ is unique, up to a scalar multiple.*

Proof. According to Lemma 3.1, there exist $a,b,c,d \in \mathscr{H}$ such that $T = a\otimes\overline{b} - b\otimes\overline{a}$ and $S = c\otimes\overline{d} - d\otimes\overline{c}$. Lemma 3.2 implies that the set $\{a,b,c,d\}$ is linearly dependent. Without loss of generality, we may assume that there exist $\alpha,\beta,\gamma \in \mathbb{C}$ such that $d = \alpha a + \beta b + \gamma c$, thus $S = c\otimes\overline{(\alpha a + \beta b)} - (\alpha a + \beta b)\otimes\overline{c}$.

If $\beta = 0$, then $d = \alpha a + \gamma c$ and $S = a\otimes\overline{(-\alpha c)} - (-\alpha c)\otimes\overline{a}$. It remains to define $x = a,\ y = b,\ z = -\alpha c$.

If $\beta \neq 0$, we have $T = \frac{1}{\beta}a\otimes\overline{(\alpha a + \beta b)} - (\alpha a + \beta b)\otimes\overline{\frac{1}{\beta}a}$. We define $x = \alpha a + \beta b,\ y = -\frac{1}{\beta}a,\ z = -c$.

Assume that there exist $u,v,w \in \mathscr{H}$ such that $T = u\otimes\overline{v} - v\otimes\overline{u}$ and $S = u\otimes\overline{w} - w\otimes\overline{u}$. By Remark 3.1, $u = \alpha x + \beta y = \gamma x + \delta z$ for some $\alpha,\beta,\gamma,\delta \in \mathbb{C}$. If T and S are linearly independent, $\{x,y,z\}$ is linearly independent. Thus, $\alpha = \gamma$ and $\beta = \delta = 0$. This implies $u = \alpha x$. $\square$

REMARK 3.2. Let $\{x,y,z\}$ be a linearly independent subset of $\mathscr{H}$. Define $T = x\otimes\overline{y} - y\otimes\overline{x}$, $S = x\otimes\overline{z} - z\otimes\overline{x}$, $W = y\otimes\overline{z} - z\otimes\overline{y}$. Let $\lambda,\mu,\nu \in \mathbb{C}$ be such that $\lambda T + \mu S + \nu W = 0$. Then, for all $\xi \in \mathscr{H}$,

$$\langle \lambda y + \mu z,\xi\rangle x + \langle -\lambda x + \nu z,\xi\rangle y + \langle -\mu x - \nu y,\xi\rangle z = 0,$$

which implies $\lambda y + \mu z = -\lambda x + \nu z = -\mu x - \nu y = 0$, and finally $\lambda = \mu = \nu = 0$. Hence, $\{T,S,W\}$ is a linearly independent subset of $\mathscr{A}(\mathscr{H})$.

LEMMA 3.4. *Let $x \in \mathscr{H}$ be nonzero and let $\Omega = \{x\otimes\overline{y} - y\otimes\overline{x}\ :\ y \in \mathscr{H}\}$. Let $\{T,S,W\}$ be a linearly independent set of rank two operators in $\mathscr{A}(\mathscr{H})$ such that $T,S \in \Omega$, $W \notin \Omega$, and such that $T+W$ and $S+W$ are rank two. Then there exist unique $y,z \in \mathscr{H}$ and $\mu \in \mathbb{C}$ such that*

$$T = x\otimes\overline{y} - y\otimes\overline{x}, \quad S = x\otimes\overline{z} - z\otimes\overline{x}, \quad W = \mu(y\otimes\overline{z} - z\otimes\overline{y}).$$

Proof. Let $T = x\otimes\overline{u} - u\otimes\overline{x}$, $S = x\otimes\overline{v} - v\otimes\overline{x}$ for some $u,v\in\mathscr{H}$. Since $T+W$ is rank two, Lemma 3.3 implies the existence of $p,q,r\in\mathscr{H}$ such that $T = p\otimes\overline{q} - q\otimes\overline{p}$, $W = p\otimes\overline{r} - r\otimes\overline{p}$. By Remark 3.1, $p = \alpha_1 x + \beta_1 u$ for some $\alpha_1,\beta_1\in\mathbb{C}$, $\beta_1\neq 0$. Since $S+W$ is also rank two, Lemma 3.2 implies that the set $\{x,v,p,r\}$ is linearly dependent. Then the set $\{x,u,v,r\}$ is also linearly dependent. Since $\{x,u,v\}$ is linearly independent, we may write $r = \alpha_2 x + \beta_2 u + \gamma_2 v$ for some $\alpha_2,\beta_2,\gamma_2\in\mathbb{C}$. Then

$$W = (\alpha_1\beta_2 - \alpha_2\beta_1)(x\otimes\overline{u} - u\otimes\overline{x}) + \alpha_1\gamma_2(x\otimes\overline{v} - v\otimes\overline{x}) + \beta_1\gamma_2(u\otimes\overline{v} - v\otimes\overline{u}).$$

If we define $V = u\otimes\overline{v} - v\otimes\overline{u}$, then $W = \alpha T + \beta S + \gamma V$ for some $\alpha,\beta,\gamma\in\mathbb{C}$. Clearly, $\gamma\neq 0$. We write

$$W = \gamma\left(\left(\frac{\beta}{\gamma}x + u\right)\otimes\overline{\left(-\frac{\alpha}{\gamma}x + v\right)} - \left(-\frac{\alpha}{\gamma}x + v\right)\otimes\overline{\left(\frac{\beta}{\gamma}x + u\right)}\right).$$

It remains to define $\mu = \gamma$, $y = \frac{\beta}{\gamma}x + u$, $z = -\frac{\alpha}{\gamma}x + v$.

Let us prove that this representation is unique. Assume that there exist $y_1,z_1\in\mathscr{H}$ and $\mu_1\in\mathbb{C}$ such that

$$T = x\otimes\overline{y_1} - y_1\otimes\overline{x}, \quad S = x\otimes\overline{z_1} - z_1\otimes\overline{x}, \quad W = \mu_1(y_1\otimes\overline{z_1} - z_1\otimes\overline{y_1}).$$

According to Remark 3.1, there exist $\alpha,\beta\in\mathbb{C}$ such that $y_1 = \alpha x + y$, $z_1 = \beta x + z$. Thus,

$$\begin{aligned} W &= \mu_1\big(\alpha(x\otimes\overline{z} - z\otimes\overline{x}) - \beta(x\otimes\overline{y} - y\otimes\overline{x}) + (y\otimes\overline{z} - z\otimes\overline{y})\big)\\ &= -\beta\mu_1 T + \alpha\mu_1 S + \frac{\mu_1}{\mu}W. \end{aligned}$$

Since the set $\{T,S,W\}$ is linearly independent, $\mu_1 = \mu$, $\alpha = \beta = 0$. $\square$

LEMMA 3.5. *Let $\Gamma\subseteq\mathscr{A}(\mathscr{H})$ be a set of rank two operators such that $T+S$ is either zero or a rank two operator for all $T,S\in\Gamma$. If Γ is not contained in a three-dimensional subspace of $\mathscr{A}(\mathscr{H})$, then there exists a unique (up to a scalar multiple) $x\in\mathscr{H}$ such that $\Gamma\subseteq\{x\otimes\overline{y} - y\otimes\overline{x} : y\in\mathscr{H}\}$.*

Proof. Let $T,S\in\Gamma$ be linearly independent. By Lemma 3.3, there exists a unique, up to a scalar multiple, $x\in\mathscr{H}$ such that $T,S\in\{x\otimes\overline{y} - y\otimes\overline{x} : y\in\mathscr{H}\}\stackrel{def}{=}\Omega$.

Suppose that there exists V in Γ which is not in Ω. Then $T+V$ and $S+V$ are rank two. By Lemma 3.4, there exist unique $y,z\in\mathscr{H}$ and $\mu\in\mathbb{C}$ such that

$$T = x\otimes\overline{y} - y\otimes\overline{x}, \quad S = x\otimes\overline{z} - z\otimes\overline{x}, \quad V = \mu(y\otimes\overline{z} - z\otimes\overline{y}).$$

Since T and S are linearly independent, $\{x,y,z\}$ is linearly independent, thus $\{T,S,V\}$ is linearly independent by Remark 3.2. Since Γ is not contained in a three-dimensional subspace of $\mathscr{A}(\mathscr{H})$, there exists $W\in\Gamma$ such that $\{T,S,V,W\}$ is linearly independent.

If $W \notin \Omega$, then Lemma 3.4 implies the existence of unique $u, v \in \mathscr{H}$ and $\nu \in \mathbb{C}$ such that

$$T = x \otimes \overline{u} - u \otimes \overline{x}, \quad S = x \otimes \overline{v} - v \otimes \overline{x}, \quad W = \nu(u \otimes \overline{v} - v \otimes \overline{u}).$$

Remark 3.1 yields $u = \alpha x + y$ and $v = \beta x + z$ for some $\alpha, \beta \in \mathbb{C}$. Then $W = \nu(-\beta T + \alpha S + \frac{1}{\mu} V)$, which is in contradiction with linear independence of $\{T, S, V, W\}$.

If $W \in \Omega$, then Lemma 3.4 applied to T, W and V implies the existence of unique $p, q \in \mathscr{H}$ and $\tau \in \mathbb{C}$ such that

$$T = x \otimes \overline{p} - p \otimes \overline{x}, \quad W = x \otimes \overline{q} - q \otimes \overline{x}, \quad V = \tau(p \otimes \overline{q} - q \otimes \overline{p}).$$

Now we have $V = \mu(y \otimes \overline{z} - z \otimes \overline{y}) = \tau(p \otimes \overline{q} - q \otimes \overline{p})$. According to Remark 3.1, $q = \gamma y + \delta z$ for some $\gamma, \delta \in \mathbb{C}$. Then $W = x \otimes \overline{q} - q \otimes \overline{x} = \gamma T + \delta S$, which contradicts the fact that $\{T, S, W\}$ is linearly independent.

Hence, $\Gamma \subseteq \Omega$. $\square$

LEMMA 3.6. *A nonzero operator $X \in \mathscr{A}(\mathscr{H})$ is rank two if and only if $X\mathscr{A}(\mathscr{H})X = \mathbb{C}X$.*

Proof. If $X \in \mathscr{A}(\mathscr{H})$ is rank two, then there exist $x, y \in \mathscr{H}$ such that $X = x \otimes \overline{y} - y \otimes \overline{x}$. For all $Y \in \mathscr{A}(\mathscr{H})$ and $z \in \mathscr{H}$ we have

$$\begin{aligned}
XYXz &= (x \otimes \overline{y} - y \otimes \overline{x})Y(x \otimes \overline{y} - y \otimes \overline{x})z \\
&= (x \otimes \overline{y} - y \otimes \overline{x})(\langle z, \overline{y}\rangle Yx - \langle z, \overline{x}\rangle Yy) \\
&= \langle z, \overline{y}\rangle(\langle Yx, \overline{y}\rangle x - \langle Yx, \overline{x}\rangle y) - \langle z, \overline{x}\rangle(\langle Yy, \overline{y}\rangle x - \langle Yy, \overline{x}\rangle y) \\
&= \langle z, \overline{y}\rangle\langle Yx, \overline{y}\rangle x + \langle z, \overline{x}\rangle\langle Yy, \overline{x}\rangle y \\
&= \langle Yx, \overline{y}\rangle(\langle z, \overline{y}\rangle x - \langle z, \overline{x}\rangle y) \\
&= \langle Yx, \overline{y}\rangle(x \otimes \overline{y} - y \otimes \overline{x})z = \langle Yx, \overline{y}\rangle Xz.
\end{aligned}$$

Hence, $X\mathscr{A}(\mathscr{H})X \subseteq \mathbb{C}X$. Analogously as in the proof of Lemma 2.1 we get $\mathbb{C}X \subseteq X\mathscr{A}(\mathscr{H})X$.

Let us assume that a nonzero $X \in \mathscr{A}(\mathscr{H})$ has the property $X\mathscr{A}(\mathscr{H})X = \mathbb{C}X$. There exists $\lambda \in \Lambda$ such that $Xe_\lambda \neq 0$. Then there exists $\mu \in \Lambda$ such that $\langle Xe_\lambda, e_\mu\rangle \neq 0$. Since $X \in \mathscr{A}(\mathscr{H})$, $\mu \neq \lambda$. If we assume that $Xe_\mu = 0$, then $\langle Xe_\lambda, e_\mu\rangle = -\langle Xe_\mu, e_\lambda\rangle = 0$; a contradiction. Hence, $Xe_\mu \neq 0$ as well. In particular, there exists $\alpha \in \mathbb{C}$ such that $X(e_\lambda \otimes e_\mu - e_\mu \otimes e_\lambda)X = \alpha X$. This implies, for all $x \in \mathscr{H}$,

$$\langle Xx, e_\mu\rangle Xe_\lambda - \langle Xx, e_\lambda\rangle Xe_\mu = \alpha Xx. \tag{8}$$

If $\alpha = 0$, then

$$\langle Xx, e_\mu\rangle Xe_\lambda = \langle Xx, e_\lambda\rangle Xe_\mu$$

for all $x \in \mathscr{H}$. In particular, for $x = e_\lambda$ we get

$$\langle Xe_\lambda, e_\mu\rangle Xe_\lambda = 0,$$

a contradiction. Therefore, $\alpha \neq 0$. Hence, (8) implies

$$Xx = \frac{1}{\alpha}\langle Xx, e_\mu\rangle Xe_\lambda - \frac{1}{\alpha}\langle Xx, e_\lambda\rangle Xe_\mu.$$

Let $\delta, \varepsilon \in \mathbb{C}$ be such that $\delta Xe_\lambda + \varepsilon Xe_\mu = 0$. Then

$$0 = \delta\langle Xe_\lambda, e_\mu\rangle + \varepsilon\langle Xe_\mu, e_\mu\rangle = \delta\langle Xe_\lambda, e_\mu\rangle,$$

which implies $\delta = 0$, thus $\varepsilon = 0$ as well. Hence, the set $\{Xe_\lambda, Xe_\mu\}$ is linearly independent. Thus, X is rank two. $\square$

LEMMA 3.7. *If $A \in \mathscr{A}(\mathscr{H})$ is such that $XAX = 0$ for all rank two operators $X \in \mathscr{A}(\mathscr{H})$, then $A = 0$.*

Proof. If $A \neq 0$, then there exist $x, y \in \mathscr{H}$ such that $\langle Ax, \overline{y}\rangle \neq 0$. Since $A \in \mathscr{A}(\mathscr{H})$, x and y are linearly independent. In the same manner as in the proof of Lemma 3.6, for $X = x \otimes \overline{y} - y \otimes \overline{x}$ we get

$$0 = XAX = \langle Ax, \overline{y}\rangle X,$$

a contradiction. Hence, $A = 0$. $\square$

LEMMA 3.8. *If $A \in \mathscr{B}(\mathscr{H})$ is such that $AXA^t = X$ for all rank two operators $X \in \mathscr{A}(\mathscr{H})$, then $A = I$ or $A = -I$.*

Proof. For all $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$, we have

$$A(e_\lambda \otimes e_\mu - e_\mu \otimes e_\lambda)A^t e_\mu = (e_\lambda \otimes e_\mu - e_\mu \otimes e_\lambda)e_\mu,$$

which implies

$$\langle A^t e_\mu, e_\mu\rangle Ae_\lambda - \langle A^t e_\mu, e_\lambda\rangle Ae_\mu = e_\lambda,$$

that is

$$\langle Ae_\mu, e_\mu\rangle Ae_\lambda - \langle Ae_\lambda, e_\mu\rangle Ae_\mu = e_\lambda. \tag{9}$$

Let $\eta \in \Lambda$ be such that $\eta \neq \lambda$ and $\eta \neq \mu$. Replacing λ with η in (9), we get

$$\langle Ae_\mu, e_\mu\rangle Ae_\eta - \langle Ae_\eta, e_\mu\rangle Ae_\mu = e_\eta. \tag{10}$$

Then $\langle Ae_\mu, e_\mu\rangle \neq 0$ or $\langle Ae_\eta, e_\mu\rangle \neq 0$. From (9) we get

$$\langle Ae_\mu, e_\mu\rangle\langle Ae_\lambda, e_\eta\rangle = \langle Ae_\lambda, e_\mu\rangle\langle Ae_\mu, e_\eta\rangle. \tag{11}$$

Interchanging the roles of μ and η, (11) becomes

$$\langle Ae_\eta, e_\eta\rangle\langle Ae_\lambda, e_\mu\rangle = \langle Ae_\lambda, e_\eta\rangle\langle Ae_\eta, e_\mu\rangle. \tag{12}$$

If $\langle Ae_\mu, e_\mu\rangle \neq 0$, divide (11) by $\langle Ae_\mu, e_\mu\rangle$ and substitute $\langle Ae_\lambda, e_\eta\rangle$ into (12). If $\langle Ae_\eta, e_\mu\rangle \neq 0$, divide (12) by $\langle Ae_\eta, e_\mu\rangle$ and substitute $\langle Ae_\lambda, e_\eta\rangle$ into (11). In both cases we get

$$\langle Ae_\lambda, e_\mu\rangle(\langle Ae_\eta, e_\eta\rangle\langle Ae_\mu, e_\mu\rangle - \langle Ae_\mu, e_\eta\rangle\langle Ae_\eta, e_\mu\rangle) = 0,$$

that is

$$\langle Ae_\lambda, e_\mu\rangle\big\langle\langle Ae_\mu, e_\mu\rangle Ae_\eta - \langle Ae_\eta, e_\mu\rangle Ae_\mu, e_\eta\big\rangle = 0,$$

which implies, by (10), $\langle Ae_\lambda, e_\mu\rangle = 0$ for all $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$. Substituting this into (9) we get

$$\langle Ae_\mu, e_\mu\rangle Ae_\lambda = e_\lambda, \tag{13}$$

which implies

$$\langle Ae_\mu, e_\mu\rangle\langle Ae_\lambda, e_\lambda\rangle = 1 \tag{14}$$

for all $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$. Let $\eta \in \Lambda$, $\eta \neq \lambda$, $\eta \neq \mu$. We have

$$\begin{aligned}\langle Ae_\mu, e_\mu\rangle^2 &= \langle Ae_\mu, e_\mu\rangle^2(\langle Ae_\lambda, e_\lambda\rangle\langle Ae_\eta, e_\eta\rangle)\\ &= (\langle Ae_\lambda, e_\lambda\rangle\langle Ae_\mu, e_\mu\rangle)(\langle Ae_\mu, e_\mu\rangle\langle Ae_\eta, e_\eta\rangle) = 1\end{aligned}$$

for all $\mu \in \Lambda$. By (14), either $\langle Ae_\mu, e_\mu\rangle = 1$ for all $\mu \in \Lambda$, or $\langle Ae_\mu, e_\mu\rangle = -1$ for all $\mu \in \Lambda$. By (13), either $Ae_\lambda = e_\lambda$ for all $\lambda \in \Lambda$, or $Ae_\lambda = -e_\lambda$ for all $\lambda \in \Lambda$, and the lemma follows. $\square$

PROPOSITION 3.9. *Let $\varphi : \mathscr{A}(\mathscr{H}) \to \mathscr{A}(\mathscr{H})$ be a bounded injective linear map preserving rank two operators in both directions. Then there exists a unique (up to a sign) $T \in \mathscr{B}(\mathscr{H})$ such that*

$$\varphi(X) = TXT^t$$

for all rank two $X \in \mathscr{A}(\mathscr{H})$. Furthermore, T is injective.

Proof. Let $x \in \mathscr{H}$ be nonzero and let $\Gamma_x = \{\varphi(x\otimes\overline{y} - y\otimes\overline{x}) : y \in \mathscr{H}\}$. Then $\Gamma_x \subseteq \mathscr{A}(\mathscr{H})$ is a set of rank two operators and zero, and the sum of any two operators in Γ_x is either zero or rank two. Assume that Γ_x is contained in a three-dimensional subspace of $\mathscr{A}(\mathscr{H})$. Let $y, z, w \in \mathscr{H}$ be such that the set $\{x, y, z, w\}$ is linearly independent. Suppose that

$$\alpha\varphi(x\otimes\overline{y} - y\otimes\overline{x}) + \beta\varphi(x\otimes\overline{z} - z\otimes\overline{x}) + \gamma\varphi(x\otimes\overline{w} - w\otimes\overline{x}) = 0$$

for some $\alpha, \beta, \gamma \in \mathbb{C}$. Linearity and injectivity of φ then imply

$$x\otimes\overline{(\alpha y + \beta z + \gamma w)} - (\alpha y + \beta z + \gamma w)\otimes\overline{x} = 0.$$

Then there exists $\delta \in \mathbb{C}$ such that $\alpha y + \beta z + \gamma w = \delta x$. Since $\{x, y, z, w\}$ is linearly independent, $\alpha = \beta = \gamma = \delta = 0$. Hence,

$$\{\varphi(x\otimes\overline{y} - y\otimes\overline{x}),\, \varphi(x\otimes\overline{z} - z\otimes\overline{x}),\, \varphi(x\otimes\overline{w} - w\otimes\overline{x})\} \subseteq \Gamma_x$$

is linearly independent. Then for every $v \in \mathscr{H}$ there exist $\alpha(v),\ \beta(v),\ \gamma(v) \in \mathbb{C}$ such that

$$\varphi(x\otimes\overline{v}-v\otimes\overline{x}) = \alpha(v)\varphi(x\otimes\overline{y}-y\otimes\overline{x})+\beta(v)\varphi(x\otimes\overline{z}-z\otimes\overline{x})+\gamma(v)\varphi(x\otimes\overline{w}-w\otimes\overline{x}),$$

which yields

$$x\otimes\overline{v}-v\otimes\overline{x} = x\otimes\overline{(\alpha(v)y+\beta(v)z+\gamma(v)w)}-(\alpha(v)y+\beta(v)z+\gamma(v)w)\otimes\overline{x}.$$

Remark 3.1 implies the existence of $\delta(v) \in \mathbb{C}$ such that

$$v = \alpha(v)y+\beta(v)z+\gamma(v)w+\delta(v)x.$$

Hence, $\{x,y,z,w\}$ is a basis for $\mathscr{H}$, which is impossible.

According to Lemma 3.5, for every $x \in \mathscr{H}$ there exists a unique, up to a scalar multiple, $u_x \in \mathscr{H}$ such that $\Gamma_x \subseteq \{u_x\otimes\overline{v}-v\otimes\overline{u_x} : v \in \mathscr{H}\}$. Let us fix such a u_x and define $Sx = u_x,\ x \in \mathscr{H}$. Let $\Omega_x = \{Sx\otimes\overline{v}-v\otimes\overline{Sx} : v \in \mathscr{H}\}$.

Let $x,y \in \mathscr{H}$ be linearly independent. Since $\varphi(x\otimes\overline{y}-y\otimes\overline{x}) \in \Omega_x\cap\Omega_y$, there exist $u_{x,y}, v_{x,y} \in \mathbb{C}$ such that

$$\varphi(x\otimes\overline{y}-y\otimes\overline{x}) = Sx\otimes\overline{u_{x,y}}-u_{x,y}\otimes\overline{Sx} = Sy\otimes\overline{v_{x,y}}-v_{x,y}\otimes\overline{Sy}.$$

By Remark 3.1, there exist $\alpha(x,y),\beta(x,y) \in \mathbb{C}$ such that

$$Sy = \alpha(x,y)Sx+\beta(x,y)u_{x,y}.$$

This implies

$$Sx\otimes\overline{Sy}-Sy\otimes\overline{Sx} = \beta(x,y)\varphi(x\otimes\overline{y}-y\otimes\overline{x}). \tag{15}$$

Assume $\beta(x,y) = 0$. Then $Sy = \alpha(x,y)Sx$. Let $z \in \mathscr{H}$ be such that $\{x,y,z\}$ is linearly independent. Then

$$\begin{aligned}\beta(y,z)\varphi(y\otimes\overline{z}-z\otimes\overline{y}) &= Sy\otimes\overline{Sz}-Sz\otimes\overline{Sy}\\ &= \alpha(x,y)(Sx\otimes\overline{Sz}-Sz\otimes\overline{Sx})\\ &= \alpha(x,y)\beta(x,z)\varphi(x\otimes\overline{z}-z\otimes\overline{x}),\end{aligned}$$

which implies $\beta(y,z) = \beta(x,z) = 0$. Hence, $Sz = \alpha(x,z)Sx$. Let $w \in \mathscr{H}$ be such that $\{x,y,z,w\}$ is linearly independent. Since $\varphi(z\otimes\overline{w}-w\otimes\overline{z}) \in \Omega_x$, the operator $\varphi(x\otimes\overline{y}-y\otimes\overline{x}+z\otimes\overline{w}-w\otimes\overline{z})$ is in Ω_x as well, in particular, it is either zero or rank two. If it is rank two, since φ is rank two preserving in both directions, Lemma 3.2 implies that $\{x,y,z,w\}$ is linearly dependent; if it is zero, the same conclusion follows from injectivity of φ and Remark 3.1. Since this contradicts the fact that $\{x,y,z,w\}$ is linearly independent, $\beta(x,y) \neq 0$. If we put $\mu(x,y) = \frac{1}{\beta(x,y)}$, then (15) implies

$$\varphi(x\otimes\overline{y}-y\otimes\overline{x}) = \mu(x,y)(Sx\otimes\overline{Sy}-Sy\otimes\overline{Sx}) \tag{16}$$

for all linearly independent $x,y \in \mathscr{H}$. Obviously, $\mu(x,y) = \mu(y,x)$.

Let $x,y\in\mathscr{H}$ be linearly independent. Inserting $x+y$ instead of x in (16), we get

$$\varphi(x\otimes\overline{y}-y\otimes\overline{x})=\mu(x+y,y)(S(x+y)\otimes\overline{Sy}-Sy\otimes\overline{S(x+y)}). \tag{17}$$

Comparing (16) and (17), we conclude, by Remark 3.1,

$$\mu(x+y,y)S(x+y)=\mu(x,y)Sx+\nu(x,y)Sy \tag{18}$$

for some $\nu(x,y)\in\mathbb{C}$. Let $w\in\mathscr{H}$ be such that the set $\{x,y,w\}$ is linearly independent. Then (18) implies

$$\begin{aligned}&\mu(x+y,y)\big(S(x+y)\otimes\overline{Sw}-Sw\otimes\overline{S(x+y)}\big)\\ &=\mu(x,y)(Sx\otimes\overline{Sw}-Sw\otimes\overline{Sx})+\nu(x,y)(Sy\otimes\overline{Sw}-Sw\otimes\overline{Sy}),\end{aligned}$$

that is

$$\begin{aligned}&\frac{\mu(x+y,y)}{\mu(x+y,w)}\big(\varphi(x\otimes\overline{w}-w\otimes\overline{x})+\varphi(y\otimes\overline{w}-w\otimes\overline{y})\big)\\ &=\frac{\mu(x,y)}{\mu(x,w)}\varphi(x\otimes\overline{w}-w\otimes\overline{x})+\frac{\nu(x,y)}{\mu(y,w)}\varphi(y\otimes\overline{w}-w\otimes\overline{y}),\end{aligned}$$

and finally

$$\frac{\mu(x+y,y)}{\mu(x+y,w)}=\frac{\mu(x,y)}{\mu(x,w)}=\frac{\nu(x,y)}{\mu(y,w)}.$$

Hence,

$$\frac{\mu(x,w)}{\mu(y,w)}=\frac{\mu(x,y)}{\nu(x,y)} \tag{19}$$

for all $w\in\mathscr{H}$ such that $\{x,y,w\}$ is linearly independent. Let $\alpha,\beta\in\mathbb{C}\setminus\{0\}$. If $t\in\mathscr{H}$ is such that $\{x,y,t\}$ is linearly independent, then the sets $\{x,t,\alpha x+\beta y\}$ and $\{y,t,\alpha x+\beta y\}$ are linearly independent as well. Thus, using (19) several times, we get

$$\begin{aligned}\frac{\mu(x,\alpha x+\beta y)}{\mu(y,\alpha x+\beta y)}&=\frac{\mu(x,\alpha x+\beta y)}{\mu(t,\alpha x+\beta y)}\cdot\frac{\mu(t,\alpha x+\beta y)}{\mu(y,\alpha x+\beta y)}\\ &=\frac{\mu(x,t)}{\nu(x,t)}\cdot\frac{\nu(y,t)}{\mu(y,t)}=\frac{\mu(x,t)}{\mu(y,t)}\cdot\frac{\nu(y,t)}{\nu(x,t)}\\ &=\frac{\mu(x,t)}{\mu(y,t)}\cdot\frac{\nu(y,t)\mu(y,x)}{\nu(x,t)\mu(x,y)}\\ &=\frac{\mu(x,t)}{\mu(y,t)}\cdot\frac{\mu(y,t)\mu(t,x)}{\mu(x,t)\mu(t,y)}=\frac{\mu(x,t)}{\mu(y,t)}.\end{aligned}$$

By (19), this yields

$$\frac{\mu(x,\alpha x+\beta y)}{\mu(y,\alpha x+\beta y)}=\frac{\mu(x,y)}{\nu(x,y)}. \tag{20}$$

Now (19) and (20) imply

$$\frac{\mu(x,w)}{\mu(y,w)} = \frac{\mu(x,y)}{\nu(x,y)} \tag{21}$$

for all $w \in \mathscr{H}$ such that $\{x,w\}$ and $\{y,w\}$ are linearly independent.

From now on, z is a fixed nonzero element in $\mathscr{H}$. Define $\nu(x) = \nu(x,z)$ for all $x \in \mathscr{H}$ linearly independent with z. Then

$$\nu(x) = \frac{\mu(x,z)\mu(z,y)}{\mu(x,y)} \tag{22}$$

for all $x,y \in \mathscr{H}$ such that $\{x,y\}$, $\{x,z\}$, $\{y,z\}$ are linearly independent. Notice that, for all such $x,y \in \mathscr{H}$, we have

$$\nu(y) = \frac{\mu(y,z)\mu(z,x)}{\mu(y,x)} = \frac{\mu(x,z)\mu(z,y)}{\mu(x,y)} = \nu(x). \tag{23}$$

Fix a nonzero $\alpha \in \mathbb{C}$. Inserting αx instead of x in (16), we get

$$\alpha\varphi(x\otimes\overline{y} - y\otimes\overline{x}) = \mu(\alpha x,y)(S(\alpha x)\otimes\overline{Sy} - Sy\otimes\overline{S(\alpha x)}). \tag{24}$$

Comparing (16) and (24) and using Remark 3.1, we get

$$\mu(\alpha x,y)S(\alpha x) = \alpha\mu(x,y)Sx + \tau(x,y)Sy \tag{25}$$

for some $\tau(x,y) \in \mathbb{C}$. Let $w \in \mathscr{H}$ be linearly independent with x. Then

$$\mu(\alpha x,w)S(\alpha x) = \alpha\mu(x,w)Sx + \tau(x,w)Sw. \tag{26}$$

Comparing (25) and (26), we get

$$\alpha\frac{\mu(x,y)}{\mu(\alpha x,y)}Sx + \frac{\tau(x,y)}{\mu(\alpha x,y)}Sy = \alpha\frac{\mu(x,w)}{\mu(\alpha x,w)}Sx + \frac{\tau(x,w)}{\mu(\alpha x,w)}Sw,$$

which implies

$$\frac{\tau(x,y)}{\mu(\alpha x,y)}(Sx\otimes\overline{Sy} - Sy\otimes\overline{Sx}) = \frac{\tau(x,w)}{\mu(\alpha x,w)}(Sx\otimes\overline{Sw} - Sw\otimes\overline{Sx}).$$

This yields, by (16),

$$\frac{\tau(x,y)}{\mu(\alpha x,y)\mu(x,y)}\varphi(x\otimes\overline{y} - y\otimes\overline{x}) = \frac{\tau(x,w)}{\mu(\alpha x,w)\mu(x,w)}\varphi(x\otimes\overline{w} - w\otimes\overline{x}).$$

In particular, for $w \in \mathscr{H}$ such that $\{x,y,w\}$ is linearly independent, we conclude $\tau(x,y) = 0$. Hence, (25) can be written as

$$\mu(\alpha x,y)S(\alpha x) = \alpha\mu(x,y)Sx. \tag{27}$$

Therefore,

$$\frac{\mu(\alpha x,y)}{\mu(x,y)}=\frac{\alpha Sx}{S(\alpha x)}$$

for all $x,y\in\mathscr{H}$ such that $\{x,y\}$ is linearly independent. This implies

$$\frac{\mu(\alpha x,y)}{\mu(x,y)}=\frac{\mu(\alpha x,w)}{\mu(x,w)} \tag{28}$$

for all $x,y,w\in\mathscr{H}$ such that $\{x,y\},\ \{x,w\}$ are linearly independent.

Let $\alpha\in\mathbb{C}$ be nonzero and let $x\in\mathscr{H}$ be linearly independent with z. Let $y\in\mathscr{H}$ be such that $\{x,y\},\ \{y,z\}$ are linearly independent. By (22) and (28), we have

$$\nu(\alpha x)=\frac{\mu(\alpha x,z)\mu(z,y)}{\mu(\alpha x,y)}=\frac{\mu(x,z)\mu(z,y)}{\mu(x,y)}=\nu(x). \tag{29}$$

By (23) and (29), $\nu(x)$ is a constant for all $x\in\mathscr{H}$ linearly independent with z; let us denote it by ν. Let $w\in\mathscr{H}$ be linearly independent with z and define, for all nonzero $\lambda\in\mathbb{C}$,

$$\mu(\lambda z,z)=\nu\frac{\mu(\lambda z,w)}{\mu(z,w)}.$$

By (28), $\mu(\lambda z,z)$ does not depend on the choice of w. We have

$$\nu=\frac{\mu(\lambda z,z)\mu(z,w)}{\mu(\lambda z,w)} \tag{30}$$

for all $w\in\mathscr{H}$ linearly independent with z. According to (22) and (30), for all linearly independent $x,y\in\mathscr{H}$ we have

$$\nu=\frac{\mu(x,z)\mu(z,y)}{\mu(x,y)}. \tag{31}$$

Define $Tx=\nu^{-1/2}\mu(x,z)Sx$ for all nonzero $x\in\mathscr{H}$, and $Tx=0$ for $x=0$.

Let $x,y\in\mathscr{H}$ be linearly independent. Using (31) and (16), we get

$$\begin{aligned}&Tx\otimes\overline{Ty}-Ty\otimes\overline{Tx}=\nu^{-1}\mu(x,z)\mu(z,y)(Sx\otimes\overline{Sy}-Sy\otimes\overline{Sx})\\&=\mu(x,y)(Sx\otimes\overline{Sy}-Sy\otimes\overline{Sx})=\varphi(x\otimes\overline{y}-y\otimes\overline{x}).\end{aligned}$$

Let us prove linearity of T. Let $x,y\in\mathscr{H}$ be linearly independent. By (31),

$$\frac{\mu(x,y)}{\mu(x,z)}=\frac{\mu(z,y)}{\nu},\quad \frac{\mu(x+y,y)}{\mu(x+y,z)}=\frac{\mu(z,y)}{\nu},$$

hence

$$\frac{\mu(x,y)}{\mu(x+y,y)}=\frac{\mu(x,z)}{\mu(x+y,z)}. \tag{32}$$

According to (21),

$$\nu(x,y)=\frac{\mu(x,y)\mu(x+y,y)}{\mu(x+y,x)},$$

which, together with (32), implies

$$\frac{\nu(x,y)}{\mu(x+y,y)} = \frac{\mu(y,z)}{\mu(x+y,z)}. \tag{33}$$

Then (18) implies, using (32) and (33),

$$\begin{aligned} S(x+y) &= \frac{\mu(x,y)}{\mu(x+y,y)}Sx + \frac{\nu(x,y)}{\mu(x+y,y)}Sy \\ &= \frac{\mu(x,z)}{\mu(x+y,z)}Sx + \frac{\mu(y,z)}{\mu(x+y,z)}Sy \\ &= \frac{1}{\nu^{-1/2}\mu(x+y,z)}(Tx+Ty) \end{aligned}$$

and finally $T(x+y) = Tx+Ty$. Let $\alpha \in \mathbb{C}$ and $x \in \mathscr{H}$ be nonzero. By (31),

$$\frac{\mu(x,y)}{\mu(x,z)} = \frac{\mu(z,y)}{\nu}, \quad \frac{\mu(\alpha x,y)}{\mu(\alpha x,z)} = \frac{\mu(z,y)}{\nu},$$

which implies

$$\frac{\mu(x,y)}{\mu(\alpha x,y)} = \frac{\mu(x,z)}{\mu(\alpha x,z)}.$$

According to (27),

$$S(\alpha x) = \alpha\frac{\mu(x,y)}{\mu(\alpha x,y)}Sx = \alpha\frac{\mu(x,z)}{\mu(\alpha x,z)}Sx = \frac{1}{\nu^{-1/2}\mu(\alpha x,z)}\alpha Tx$$

and finally $T(\alpha x) = \alpha Tx$.

For all $x,y \in \mathscr{H}$ and all $\lambda,\mu \in \Lambda$ we have

$$\begin{aligned} \langle T(x\otimes\overline{y})T^t e_\lambda, e_\mu\rangle &= \langle T^t e_\lambda, \overline{y}\rangle\langle Tx, e_\mu\rangle = \langle Ty, e_\lambda\rangle\langle Tx, e_\mu\rangle \\ &= \langle e_\lambda, \overline{Ty}\rangle\langle Tx, e_\mu\rangle = \langle (Tx\otimes\overline{Ty})e_\lambda, e_\mu\rangle. \end{aligned}$$

Hence, $T(x\otimes\overline{y})T^t = Tx\otimes\overline{Ty}$. Since T is additive, we conclude, for all linearly independent $x,y \in \mathscr{H}$,

$$\varphi(x\otimes\overline{y} - y\otimes\overline{x}) = T(x\otimes\overline{y} - y\otimes\overline{x})T^t.$$

Thus, $\varphi(X) = TXT^t$ for all rank two $X \in \mathscr{A}(\mathscr{H})$.

Furthermore, for all linearly independent $x,y \in \mathscr{H}$, we have

$$\|Tx\otimes\overline{Ty} - Ty\otimes\overline{Tx}\| = \|\varphi(x\otimes\overline{y} - y\otimes\overline{x})\| \leqslant \|\varphi\|\cdot\|x\otimes\overline{y} - y\otimes\overline{x}\|.$$

If $\{x,y\}$ is linearly dependent, then linearity of T yields that $\{Tx,Ty\}$ is linearly dependent as well. Hence,

$$\|Tx\otimes\overline{Ty} - Ty\otimes\overline{Tx}\| \leqslant \|\varphi\|\cdot\|x\otimes\overline{y} - y\otimes\overline{x}\|$$

holds for all $x,y \in \mathscr{H}$. Let (y_n) be a sequence in $\mathscr{H}$ such that $y_n \to 0$ and $Ty_n \to w$ for some $w \in \mathscr{H}$. Then

$$\|Tx \otimes \overline{Ty_n} - Ty_n \otimes \overline{Tx}\| \leqslant \|\varphi\| \cdot \|x \otimes \overline{y_n} - y_n \otimes \overline{x}\|.$$

After taking the limits, we conclude

$$Tx \otimes \overline{w} - w \otimes \overline{Tx} = 0$$

for all $x \in \mathscr{H}$. Assume $w \neq 0$. Then for every $x \in \mathscr{H}$ there exists $\gamma(x) \in \mathbb{C}$ such that $Tx = \gamma(x)w$. If x and y are linearly independent, then

$$\begin{aligned}\varphi(x \otimes \overline{y} - y \otimes \overline{x}) &= Tx \otimes \overline{Ty} - Ty \otimes \overline{Tx} \\ &= \gamma(x)w \otimes \overline{\gamma(y)w} - \gamma(y)w \otimes \overline{\gamma(x)w} = 0.\end{aligned}$$

Since φ is injective, this is impossible. Hence, $w = 0$. According to the closed graph theorem, T is bounded.

Let us prove that T is injective. Suppose that there exists a nonzero $x \in \mathscr{H}$ such that $Tx = 0$. Then $\varphi(x \otimes \overline{y} - y \otimes \overline{x}) = 0$ for all $y \in \mathscr{H}$. Since φ is injective, this yields that $\{x,y\}$ is a linearly dependent set for all $y \in \mathscr{H}$; a contradiction.

Let $V \in \mathscr{B}(\mathscr{H})$ be such that $\varphi(X) = VXV^t$ for all rank two $X \in \mathscr{A}(\mathscr{H})$. Let $x \in \mathscr{H}$ be nonzero and let $y \in \mathscr{H}$ be linearly independent with x. Then we have

$$Vx \otimes \overline{Vy} - Vy \otimes \overline{Vx} = Tx \otimes \overline{Ty} - Ty \otimes \overline{Tx}, \tag{34}$$

and Remark 3.1 implies

$$Vx = \alpha(x,y)Tx + \beta(x,y)Ty. \tag{35}$$

Analogously, if $w \in \mathscr{H}$ is such that $\{x,y,w\}$ is linearly independent, we get

$$Vx = \alpha(x,w)Tx + \beta(x,w)Tw. \tag{36}$$

Since T is linear and injective, comparing of (35) and (36) yields

$$\big(\alpha(x,y) - \alpha(x,w)\big)x + \beta(x,y)y - \beta(x,w)w = 0.$$

Since $\{x,y,w\}$ is linearly independent, $\alpha(x,y) = \alpha(x,w)$ and $\beta(x,y) = \beta(x,w) = 0$. Then $Vx = \alpha(x,y)Tx$. Analogously, $Vy = \alpha(y,x)Ty$. Inserting this in (34), we get $\alpha(x,y)\alpha(y,x) = 1$. Hence,

$$\begin{aligned}1 &= \big(\alpha(x,y)\alpha(y,x)\big)\big(\alpha(x,w)\alpha(w,x)\big)\big(\alpha(y,w)\alpha(w,y)\big) \\ &= \big(\alpha(x,y)\alpha(x,w)\big)\big(\alpha(y,x)\alpha(y,w)\big)\big(\alpha(w,x)\alpha(w,y)\big) \\ &= \alpha(x,y)^2\alpha(y,w)^2\alpha(w,y)^2 \\ &= \alpha(x,y)^2\big(\alpha(y,w)\alpha(w,y)\big)^2 = \alpha(x,y)^2,\end{aligned}$$

which implies $\alpha(x,y) = 1$ or $\alpha(x,y) = -1$. Hence, for every $x \in \mathscr{H}$, $Vx = Tx$ or $Vx = -Tx$. If $Vx = Tx$ for some $x \in \mathscr{H}$, then (34) yields $Vx = Tx$ for all $x \in \mathscr{H}$, hence $V = T$. Analogously, if $Vx = -Tx$ for some $x \in \mathscr{H}$, then $V = -T$. $\square$

REMARK 3.3. As far as we know, linear maps on $\mathscr{A}(\mathscr{H})$ preserving rank two operators have been studied only in the finite dimensional (matrix) case, see [2, Theorem 2.4].

Based on Lemmas 3.6, 3.7, 3.8 and Proposition 3.9 one can prove the following result in an analogous way as Theorem 2.5. Let us emphasize that the assumption that φ is a surjective linear isometry yields not only that φ preserves rank two operators, but that it preserves rank two operators in both directions.

THEOREM 3.10. *Let $\varphi : \mathscr{A}(\mathscr{H}) \to \mathscr{A}(\mathscr{H})$ be a surjective linear isometry. Then there exists a unitary $U \in \mathscr{B}(\mathscr{H})$ such that*

$$\varphi(X) = UXU^t$$

for all $X \in \mathscr{A}(\mathscr{H})$.

Theorem 3.10 together with [9, Theorem 2.5] and [5, Theorem 2.1] yields the following corollary.

COROLLARY 3.11. *Let $P : \mathscr{A}(\mathscr{H}) \to \mathscr{A}(\mathscr{H})$ be a nontrivial linear projection and $\lambda \neq 1$ a modulus one complex number. Then $P + \lambda\overline{P}$ is an isometry if and only if one of the following holds:*

(i) $\lambda = -1$ *and there exists* $R = R^* = R^2 \in \mathscr{B}(\mathscr{H})$ *such that* P *or* $\overline{P}$ *has the form* $X \mapsto RXR^t + (I-R)X(I-R^t)$,

(ii) P *or* $\overline{P}$ *has the form* $X \mapsto RX + XR^t$, *where* $R = x \otimes x$ *for some norm one* $x \in \mathscr{H}$.

Acknowledgements. The authors wish to express their gratitude to Chi-Kwong Li and Lajos Molnár for useful suggestions concerning the approach to the problems investigated in this paper.

REFERENCES

[1] R. AN, J. HOU, AND L. ZHAO, *Adjacency preserving maps on the space of symmetric operators*, Linear Algebra Appl., **405** (2005), 311–324.

[2] C. CAO AND X. TANG, *Linear maps preserving rank 2 on the space of alternate matrices and their applications*, Int. J. Math. Math. Sci., **61–64** (2004), 3409–3417.

[3] T. DANG, Y. FRIEDMAN, B. RUSSO, *Affine geometric proofs of the Banach Stone theorems of Kadison and Kaup*, Rocky Mountain J. Math., **20** (1990), 409–428.

[4] M. FOŠNER, D. ILIŠEVIĆ, AND C.K. LI, *G-invariant norms and bicircular projections*, Linear Algebra Appl., **420** (2007), 596–608.

[5] D. ILIŠEVIĆ, *Generalized bicircular projections on JB*-triples*, Linear Algebra Appl., **432** (2010), 1267–1276.

[6] W. KAUP, *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces*, Math. Z., **183** (1983), 503–529.

[7] L. MOLNÁR, *Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces*, Springer-Verlag, Berlin-Heidelberg, 2007.
[8] J. G. MURPHY, , *C*-Algebras and Operator Theory*, Academic Press, Boston, 1990.
[9] L. L. STACHÓ AND B. ZALAR, *Bicircular projections on some matrix and operator spaces*, Linear Algebra Appl., **384** (2004), 9–20.

(Received December 21, 2009)

Ajda Fošner
Faculty of Management
University of Primorska
Cankarjeva 5
SI-6104 Koper
Slovenia
e-mail: ajda.fosner@fm-kp.si

Dijana Ilišević
Department of Mathematics
University of Zagreb
Bijenička 30
P.O. Box 335
10002 Zagreb
Croatia
e-mail: ilisevic@math.hr