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ON THE BEST CONSTANTS IN MARKOV–TYPE INEQUALITIES INVOLVING GEGENBAUER NORMS WITH DIFFERENT WEIGHTS

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Abstract. The paper is concerned with best constants in Markov-type inequalities between the norm of a higher derivative of a polynomial and the norm of the polynomial itself. The norm of the polynomial is taken in L^2 with the Gegenbauer weight corresponding to a parameter α, while the derivative is measured in L^2 with the Gegenbauer weight for a parameter β. Under the assumption that $\beta-\alpha$ is an integer, we determine the first order asymptotics of the best constants as the degree of the polynomial goes to infinity.

1. Introduction and main result

We denote by $\mathscr{P}_n$ the linear space of all algebraic polynomials of degree at most n with complex coefficients. For a real number $\alpha > -1$, the Gegenbauer norm $\|\cdot\|_\alpha$ is defined by

$$\|f\|_\alpha^2 = \int_{-1}^{1} |f(t)|^2 (1-t^2)^\alpha dt. \tag{1}$$

This paper is devoted to the best constant γ in inequalities of the form

$$\|f_n^{(\nu)}\|_\beta \leqslant \gamma \|f_n\|_\alpha \quad \text{for all} \quad f_n \in \mathscr{P}_n, \tag{2}$$

where $\alpha, \beta > -1$ are real numbers and $f_n^{(\nu)}$ is the νth derivative of f_n. The problem of finding the best constant in (2) with two different weights α and β was brought to our attention by Jürgen Prestin at the Workshop on Approximation Theory and Signal Analysis in Lindau, Lake Constance, in March 2009. The best constant in (2) depends on ν, n, α, β and will be denoted by $\gamma_n^{(\nu)}(\alpha,\beta)$. Our main result says that if $\beta-\alpha$ is an integer, then there exist numbers $G_\nu(\alpha,\beta)$ and $b_\nu(\alpha,\beta)$ such that

$$\gamma_n^{(\nu)}(\alpha,\beta) \sim G_\nu(\alpha,\beta) n^{b_\nu(\alpha,\beta)} \quad \text{as} \quad n \to \infty$$

and it provides explicit expressions for $G_\nu(\alpha,\beta)$ and $b_\nu(\alpha,\beta)$. Here and in the following we write $x_n \sim y_n$ if $x_n/y_n \to 1$ as $n \to \infty$.

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Inequalities of the form (2) are referred to as inequalities of the Markov type, and we refer to the books [11], [12], [13] for more on this big topic. The Markov brothers determined the best constant $\mu_n^{(\nu)}$ in the inequality $\|f_n^{(\nu)}\|_\infty \leqslant \mu_n^{(\nu)}\|f_n\|_\infty$ for $f_n \in \mathscr{P}_n$ in case $\|\cdot\|_\infty$ is the maximum norm on some segment of the real line. Erhard Schmidt [15], [16] considered the Legendre case ($\alpha = \beta = 0$) and discovered that

$$\gamma_n^{(1)}(0,0) \sim \frac{1}{\pi} n^2.$$

The very recent paper [10] is dedicated to the exact value of $\gamma_n^{(1)}(0,0)$. Shampine [17], [18] proved that

$$\gamma_n^{(2)}(0,0) \sim \frac{1}{4k_0^2} n^4,$$

where $k_0 = 1.8751041\ldots$ can be identified as the smallest positive solution of the equation $1+\cos k \cosh k = 0$. Konyagin [9] established that

$$\gamma_n^{(\nu)}(\alpha,\beta) \simeq \begin{cases} n^\nu & \text{if } \beta-\alpha \geqslant \nu, \\ n^{2\nu+\alpha-\beta} & \text{if } \beta-\alpha < \nu, \end{cases} \tag{3}$$

where $x_n \simeq y_n$ means that there are constants $0 < c_1 < c_2 < \infty$ such that $c_1 \leqslant x_n/y_n \leqslant c_2$ for all n. The estimates $\gamma_n^{(\nu)}(\alpha,\beta) \leqslant Cn^b$ with the correct exponent b were already obtained earlier by Daugavet and Rafal'son [6]. Guessab and Milovanović [7] showed that

$$\gamma_n^{(\nu)}(\alpha,\alpha+\nu) = \sqrt{\frac{n!}{(n-\nu)!}\,\frac{\Gamma(n+2\alpha+\nu+1)}{\Gamma(n+2\alpha+1)}} \sim n^\nu. \tag{4}$$

See also [1] and Section 6.1.8 of [11]. Finally, in [3], we proved that

$$\gamma_n^{(\nu)}(\alpha,\alpha) \sim \frac{1}{2^\nu}\,\|L_{\nu,\alpha,\alpha}^*\|_\infty n^{2\nu}, \tag{5}$$

where $L_{\nu,\alpha,\alpha}^*$ is the Volterra integral operator on $L^2(0,1)$ that is given by

$$(L_{\nu,\alpha,\alpha}^* f)(x) = \frac{1}{(\nu-1)!}\int_0^x x^{-\alpha/2} y^{\alpha/2} (x-y)^{\nu-1} f(y)\,dy$$

and $\|\cdot\|_\infty$ denotes the operator norm.

Our aim is to refine estimates (3) to asymptotic equalities, that is, we want to improve (3) to something like (5). In [4], we were able to solve the analogue of this problem for the Laguerre norms given by

$$|||f|||_\alpha^2 = \int_0^\infty |f(t)|^2 t^\alpha e^{-t}\,dt \quad (\alpha > -1).$$

Let $\lambda_n^{(\nu)}(\alpha,\beta)$ be the best constant for which $|||f_n^{(\nu)}|||_\beta \leqslant \lambda\,|||f_n|||_\alpha$ for all $f_n \in \mathscr{P}_n$. The result of [4] states that if $m = \beta-\alpha$ is an integer such that $\alpha+m > -1$, then

$$\lambda_n^{(\nu)}(\alpha,\alpha+m) \sim \begin{cases} 2^{m-\nu} n^{m/2} & \text{if } m \geqslant \nu, \\ \|L_{\nu,\alpha,\alpha+m}^*\|_\infty n^{\nu-m/2} & \text{if } m < \nu, \end{cases}$$

where $L^*_{\nu,\alpha,\alpha+m}$ is given on $L^2(0,1)$ by

$$(L^*_{\nu,\alpha,\alpha+m}f)(x) = \frac{1}{(\nu-m-1)!}\int_0^x x^{-\alpha/2}y^{(\alpha+m)/2}(x-y)^{\nu-m-1}f(y)\,dy.$$

Here is the main result of the present paper.

THEOREM 1.1. *Let $\alpha > -1$ be a real number and let m be an integer such that $\alpha+m > -1$. Then*

$$\gamma_n^{(\nu)}(\alpha,\alpha+m) \sim \begin{cases} n^\nu & \text{if } m \geqslant \nu, \\ \frac{1}{2^{\nu-m}}\|L^*_{\nu,\alpha,\alpha+m}\|_\infty n^{2\nu-m} & \text{if } m < \nu. \end{cases}$$

Tight estimates for the norms $\|L^*_{\nu,\alpha,\alpha+m}\|_\infty$ can be obtained as in [3]. In the case where $m = \nu-1$, we also have the following.

THEOREM 1.2. *If $\alpha > -1$, then*

$$\gamma_n^{(\nu)}(\alpha,\alpha+\nu-1) \sim G_\nu(\alpha,\alpha+\nu-1)\,n^{\nu+1},$$

where $G_\nu(\alpha,\alpha+\nu-1)$ is $1/(\nu+1)$ times the reciprocal of the smallest positive zero of the Bessel function $J_{(\alpha-1)/(\nu+1)}$.

Theorem 1.2 implies in particular that

$$\gamma_n^{(1)}(0,0) \sim \frac{1}{\pi}n^2, \qquad \gamma_n^{(1)}(2,2) \sim \frac{1}{2\pi}n^2,$$
$$\gamma_n^{(2)}\left(-\frac{1}{2},\frac{1}{2}\right) \sim \frac{2}{3\pi}n^3, \qquad \gamma_n^{(2)}\left(\frac{5}{2},\frac{7}{2}\right) \sim \frac{1}{3\pi}n^3,$$
$$\gamma_n^{(3)}(3,5) \sim \frac{1}{4\pi}n^4, \qquad \gamma_n^{(4)}\left(\frac{7}{2},\frac{13}{2}\right) \sim \frac{1}{5\pi}n^5.$$

The rest of the paper is devoted to the proofs of Theorems 1.1 and 1.2. In Section 2, we determine the matrix representation of the operator $f \mapsto f^{(\nu)}$ in orthonormal bases of Gegenbauer polynomials and in Sections 3 and 4, we then compute the asymptotics of the spectral norms of the matrices.

We want to remark that our restriction to the case where $\beta-\alpha$ is an integer comes from the techniques employed in Sections 3 and 4. The matrix representation of the differentiation operator $f \mapsto f^{(\nu)}$ is available without any restriction. But if $\beta-\alpha$ is an integer, we can fairly quickly derive asymptotic expressions for the matrix representation (Lemmas 2.1 and 2.2) and, in addition, an infinite matrix we will encounter in Section 3 is banded and the kernel $(y^2-x^2)^{\alpha+\nu-\beta-1}$, which will play a decisive role in Section 4, is a polynomial in x and y. All these circumstances simplify things essentially. The method of this paper in combination with more elaborate analysis will probably also work in the case where $\beta-\alpha$ is not an integer. However, the technical details would increase the volume of the paper drastically and would also to some extent obscure the two ideas underlying Sections 3 and 4. The objective of this paper is to present a method that allows us to find the asymptotics of the best constants in Markov-type inequalities with changing weights, and the restriction to the case where $\beta-\alpha$ is an integer allows us to do this in a very lucid fashion. We leave the analysis of the case where $\beta-\alpha$ is not an integer for future investigations.

2. Gegenbauer polynomials

Let $\mathscr{P}_n(\alpha)$ stand for the space $\mathscr{P}_n$ with the norm (1). The constant $\gamma_n^{(\nu)}(\alpha,\beta)$ is just the operator norm of the operator $D^\nu : \mathscr{P}_n(\alpha)\to\mathscr{P}_n(\beta)$, where D^ν is the operator of taking the νth derivative. We denote by

$$P_k(t,\alpha)=\frac{1}{2^k k!}(t^2-1)^{-\alpha}\frac{d^k}{dt^k}(t^2-1)^{\alpha+k}$$

the kth Gegenbauer polynomial and by

$$\hat{P}_k(t,\alpha)=\sqrt{\frac{k!(2k+2\alpha+1)\Gamma(k+2\alpha+1)}{2^{2\alpha+1}\Gamma(k+\alpha+1)^2}}\,P_k(t,\alpha)$$

the kth normalized Gegenbauer polynomial for the weight $(1-t^2)^\alpha$. We abbreviate $P_k(t,\alpha)$ and $\hat{P}_k(t,\alpha)$ to $P_k(\alpha)$ and $\hat{P}_k(\alpha)$. Then $\{\hat{P}_0(\alpha),\ldots,\hat{P}_n(\alpha)\}$ and $\{\hat{P}_0(\beta),\ldots,\hat{P}_n(\beta)\}$ are orthonormal bases in $\mathscr{P}_n(\alpha)$ and $\mathscr{P}_n(\beta)$, respectively. We are interested in the matrix representation $C_n=(c_{jk}^{(\nu)}(\alpha,\beta))_{j,k=0}^n$ of D^ν in this pair of bases. The entries $c_{jk}^{(\nu)}(\alpha,\beta)$ are zero for $j>k-\nu$ and are otherwise given by

$$\hat{P}_k^{(\nu)}(\alpha)=\sum_{j=0}^{k-\nu}c_{jk}^{(\nu)}(\alpha,\beta)\hat{P}_j(\beta). \tag{6}$$

It is well known that

$$\hat{P}_k^{(\nu)}(\alpha)=\omega_k^{(\nu)}(\alpha)\hat{P}_{k-\nu}(\alpha+\nu), \tag{7}$$

where

$$\omega_k^{(\nu)}(\alpha)=\sqrt{\frac{k!}{(k-\nu)!}\,\frac{\Gamma(k+2\alpha+\nu+1)}{\Gamma(k+2\alpha+1)}} \tag{8}$$

(see, e.g., [19, p. 282]). This implies at once that

$$c_{jk}^{(\nu)}(\alpha,\alpha+\nu)=\begin{cases}\omega_k^{(\nu)}(\alpha) & \text{for } j=k-\nu,\\ 0 & \text{otherwise.}\end{cases}$$

Hence, if $\beta=\alpha+\nu$ then C_n has only one nonzero diagonal and the operator norm ($=$ spectral norm) of C_n is the maximum of the absolute values of the entries on this diagonal. It follows that

$$\gamma_n^{(\nu)}(\alpha,\alpha+\nu)=\max_{\nu\leqslant k\leqslant n}\omega_k^{(\nu)}(\alpha)=\omega_n^{(\nu)}(\alpha),$$

which is just Guessab and Milovanivič's result (4).

Combining (6) and (7) we see that the coefficient $c_{jk}^{(\nu)}(\alpha,\beta)$ are determined by

$$\omega_k^{(\nu)}(\alpha)\hat{P}_{k-\nu}(\alpha+\nu)=\sum_{j=0}^{k-\nu}c_{jk}^{(\nu)}(\alpha,\beta)\hat{P}_j(\beta). \tag{9}$$

We are thus led to the so-called connection problem for Gegenbauer polynomials. The solution of this problem is known and can be found in [2, p. 360], [8, p. 291], [14, p. 587], for example. The formulas given there use the orthogonal polynomials with respect to the weight $(1-t^2)^{\alpha-1/2}$ and the normalization is that the kth polynomial assumes the value $(2\alpha)_k/k!$ at $t=1$, where $(z)_k:=\Gamma(z+k)/\Gamma(z)$. Our polynomials $P_k(\alpha)$ correspond to the weight $(1-t^2)^{\alpha}$ and take the value $(\alpha+1)_k/k!$ at $t=1$ (see, e.g., [19, p. 283]). Converting the formulas of [2], [8], [14] into our setting we obtain that $c_{jk}^{(\nu)}(\alpha,\beta)=0$ if $j+k-\nu$ is an odd number and that

$$\begin{aligned} c_{jk}^{(\nu)}(\alpha,\beta) &= \omega_k^{(\nu)}(\alpha)\,\frac{\delta_{k-\nu}^{(\alpha+\nu)}}{\delta_j^{(\beta)}}\,\frac{j+\beta+1/2}{(k+j-\nu)/2+\beta+1/2} \\ &\quad\times\frac{(\alpha+\nu-\beta)_{(k-j-\nu)/2}\,(\alpha+\nu+1/2)_{(k+j-\nu)/2}}{((k-j-\nu)/2)!\,(\beta+1/2)_{(k+j-\nu)/2}} \end{aligned} \tag{10}$$

with

$$\delta_\ell^{(\eta)}=\frac{\Gamma(2\eta+1)}{\Gamma(\eta+1)}\sqrt{\frac{\ell!(2\ell+2\eta+1)}{2^{2\eta+1}\Gamma(\ell+2\eta+1)}}$$

if $j+k-\nu$ is an even number.

Our proof of Theorem 1.1 splits into two parts, one for $\beta\geqslant\alpha+\nu$ and the other one for $\beta<\alpha+\nu$, and the arguments we will employ differ in the two parts. For $\beta\geqslant\alpha+\nu$, it will turn out to be more convenient to express things in slightly different terms. Let $\beta=\alpha+\nu+\mu$ with an integer $\mu\geqslant 0$. From (6) we infer that

$$\hat{P}_k(\alpha)=\sum_{j=0}^{k}c_{jk}^{(0)}(\alpha,\alpha+\nu+\mu)\hat{P}_j(\alpha+\nu+\mu),$$

and the connection formula implies that the sum is actually over $j=k-2\ell$ with $0\leqslant\ell\leqslant\mu$. Thus, we may write

$$\hat{P}_k(\alpha)=\sum_{\ell=0}^{\mu}q_{\ell k}(\alpha,\mu)\hat{P}_{k-2\ell}(\alpha+\nu+\mu) \tag{11}$$

with certain coefficients $q_{\ell k}(\alpha,\mu)$. Using (10) one can show the following.

LEMMA 2.1. *For each fixed* ℓ,

$$q_{\ell k}(\alpha,\mu)=(-1)^\ell\binom{\mu}{\ell}\frac{1}{2^\mu}+O\left(\frac{1}{k}\right)\quad as\quad k\to\infty.$$

In the case where $\beta < \alpha + \nu$, we will use the following result, which can be derived from (10).

LEMMA 2.2. *Let $\beta = \alpha + \nu - 1 - \mu$ with some integer $\mu \geqslant 0$ and suppose $\beta > -1$. If $j + k - \nu$ is even, then*

$$c_{jk}^{(\nu)}(\alpha, \beta) = \frac{1}{2^{\mu-1}\mu!} j^{1/2+\beta} k^{1/2-\alpha} R_{2\mu}(j,k) \left(1 + O\left(\frac{1}{j}\right)\right) \left(1 + O\left(\frac{1}{k}\right)\right)$$

where $R_{2\mu}(j,k)$ is a polynomial in j and k of the form

$$R_{2\mu}(j,k) = (k^2 - j^2)^{\mu} + \sum_{\sigma+\tau<2\mu} r_{\sigma\tau} j^{\sigma} k^{\tau}.$$

3. Proof of the main result for $m \geqslant \nu$

The case $m = \nu$ was already disposed of at the beginning of Section 2. So let $m \geqslant \nu + 1$, that is, $m = \nu + \mu$ with an integer $\mu \geqslant 1$. From (7) and (11) we obtain that

$$\begin{aligned} \hat{P}_k^{(\nu)}(\alpha) &= \omega_k^{(\nu)}(\alpha) \hat{P}_{k-\nu}(\alpha + \nu) \\ &= \omega_k^{(\nu)}(\alpha) \Big[q_{0,k-\nu}(\alpha+\nu,\mu) \hat{P}_{k-\nu}(\alpha+\nu+\mu) \\ &\quad + q_{1,k-\nu}(\alpha+\nu,\mu) \hat{P}_{k-\nu-2}(\alpha+\nu+\mu) + \ldots \\ &\quad + q_{\mu,k-\nu}(\alpha+\nu,\mu) \hat{P}_{k-\nu-2\mu}(\alpha+\nu+\mu) \Big]. \end{aligned}$$

The coefficient of $\hat{P}_j(\alpha + \nu + \mu)$ in this decomposition is the (j,k) entry of the matrix representation C_n of $D^{\nu} : \mathscr{P}_n(\alpha) \to \mathscr{P}_n(\alpha + \nu + \mu)$. Put $N = n - \nu + 1$. A little thought reveals that

$$C_n = \begin{pmatrix} 0 & A_n \\ 0 & 0 \end{pmatrix}, \tag{12}$$

where $A_n \in \mathbb{R}^{N \times N}$ is upper-triangular and banded. Let us abbreviate $\omega_k^{(\nu)}(\alpha)$ and $q_{jk}(\alpha + \nu, \mu)$ to ω_k and q_{jk}. The main diagonal of A_n is

$$\omega_{\nu} q_{00}, \quad \omega_{\nu+1} q_{01}, \quad \ldots, \quad \omega_{\nu+N-1} q_{0,N-1},$$

the first superdiagonal is zero, the second superdiagonal is

$$\omega_{\nu+2} q_{12}, \quad \omega_{\nu+3} q_{13}, \quad \ldots, \quad \omega_{\nu+N-1} q_{1,N-1},$$

the third superdiagonal is again zero and so on. The last nonzero superdiagonal is the 2μth, and its entries are

$$\omega_{\nu+2\mu} q_{\mu,2\mu}, \quad \omega_{\nu+2\mu+1} q_{\mu,2\mu+1}, \quad \ldots, \quad \omega_{\nu+N-1} q_{\mu,N-1}.$$

Thus, we may write $A_n = A_{n,0} + A_{n,1} + \ldots + A_{n,\mu}$ where $A_{n,\ell}$ has the entries

$$\omega_{\nu+2\ell} q_{\ell,2\ell}, \quad \omega_{\nu+2\ell+1} q_{\ell,2\ell+1}, \quad \ldots, \quad \omega_{\nu+N-1} q_{\ell,N-1}$$

in the 2ℓth superdiagonal and zeros elsewhere. Consequently,

$$\frac{1}{n^{\nu}}\|A_{n,\ell}\|_{\infty} = \frac{1}{n^{\nu}} \max_{0\leqslant s\leqslant N-2\ell-1} \left|\omega_{\nu+2\ell+s} q_{\ell,2\ell+s}\right|.$$

From (8) we infer that

$$\omega_{\nu+2\ell+s} = \omega_{\nu+2\ell+s}^{(\nu)}(\alpha) \leqslant (\nu+2\ell+s)^{\nu} + E_1(\nu+2\ell+s)^{\nu-1}$$

with some constant $E_1 < \infty$ depending only on α and ν, and since $\nu+2\ell+s \leqslant \nu+N-1 = n$, it follows that

$$\omega_{\nu+2\ell+s} \leqslant n^{\nu-1}(\nu+2\ell+s) + E_1 n^{\nu-1} = n^{\nu-1}(\nu+2\ell+s+E_1).$$

Lemma 2.1 tells us that

$$|q_{\ell,2\ell+s}| = |q_{\ell,2\ell+s}(\alpha+\nu,\mu)| \leqslant \binom{\mu}{\ell}\frac{1}{2^{\mu}} + \frac{E_2}{s+1}$$

where $E_2 < \infty$ depends only on α,ν,μ,ℓ. Thus,

$$\begin{aligned}\frac{1}{n^{\nu}}|\omega_{\nu+2\ell+s}q_{\ell,2\ell+s}| &\leqslant \frac{\nu+2\ell+s+E_1}{n}\left[\binom{\mu}{\ell}\frac{1}{2^{\mu}} + \frac{E_2}{s+1}\right] \\ &= \frac{\nu+2\ell+s}{n}\binom{\mu}{\ell}\frac{1}{2^{\mu}} + \frac{E_2(\nu+2\ell+s)}{n(s+1)} + \frac{E_1}{n}\binom{\mu}{\ell}\frac{1}{2^{\mu}} + \frac{E_1E_2}{n(s+1)}.\end{aligned}$$

As $\nu+2\ell+s \leqslant n$, this is at most

$$\binom{\mu}{\ell}\frac{1}{2^{\mu}} + O\left(\frac{1}{n}\right).$$

It results that

$$\begin{aligned}\frac{1}{n^{\nu}}\|A_n\|_{\infty} &\leqslant \frac{1}{n^{\nu}}\sum_{\ell=0}^{\mu}\|A_{n,\ell}\|_{\infty} \leqslant \sum_{\ell=0}^{\mu}\binom{\mu}{\ell}\frac{1}{2^{\mu}} + O\left(\frac{1}{n}\right) \\ &= \left(\frac{1}{2}+\frac{1}{2}\right)^{\mu} + O\left(\frac{1}{n}\right) = 1 + O\left(\frac{1}{n}\right). \qquad (13)\end{aligned}$$

To get a lower estimate for A_n consider

$$B_n := \frac{1}{n^{\nu}} J_N A_n J_N$$

where J_N is the $N\times N$ matrix with ones on the counterdiagonal and zeros elsewhere. We denote by π_N the projection

$$\pi_N : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+), \quad \{x_0,x_1,x_2,\ldots\} \mapsto \{x_0,\ldots,x_{N-1},0,\ldots\}.$$

Clearly, $\|B_n\pi_N\|_\infty = \|B_n\|_\infty = \|A_n\|_\infty/n^\nu$. Let B be the (simply) infinite lower-triangular Toeplitz matrix whose $2k$th subdiagonal is

$$(-1)^k \binom{\mu}{k} \frac{1}{2^\mu} \quad \text{for} \quad 0 \leqslant k \leqslant \mu$$

and all other entries of which are zero. The so-called symbol of B is the function $(1-z^2)^\mu/2^\mu$ and hence (see, for example, [5, page 10]) the operator norm of B on $\ell^2(\mathbb{Z}_+)$ is

$$\|B\|_\infty = \max_{|z|=1} \frac{|(1-z^2)^\mu|}{2^\mu} = 1.$$

We claim that $B_n\pi_N$ converges strongly to B as $n \to \infty$. The Banach-Steinhaus theorem then yields that

$$1 = \|B\|_\infty \leqslant \liminf_{n\to\infty} \|B_n\pi_N\|_\infty = \liminf_{n\to\infty} \frac{1}{n^\nu} \|A_n\|_\infty,$$

which together with (13) implies that

$$\gamma_n^{(\nu)}(\alpha, \alpha+\nu+\mu) = \|C_n\|_\infty = \|A_n\|_\infty \sim n^\nu$$

and thus completes the proof of Theorem 1.1 for $m = \nu + \mu \geqslant \nu$.

So let us prove the claim. From (13) we see that $\|B_n\pi_N\|_\infty = O(1)$. The strong convergence of $B_n\pi_N$ to B will therefore follow once we have shown that $B_n\pi_N e_k \to Be_k$ for every k, where $e_k \in \ell^2(\mathbb{Z}_+)$ is the sequence whose kth term is 1 and the remaining terms of which are zero. As $B_n\pi_N$ is banded with fixed bandwidth, it suffices to verify that the jth term of $B_n\pi_N e_k$ converges to the jth term of Be_k. Equivalently, it is enough to prove that the jk entry of $B_n\pi_N$ converges to the jk entry of B. The only nonzero subdiagonals of $B_n\pi_N$ and B are those with the numbers $0, 2, \ldots, 2\mu$. It remains to consider the entries in these subdiagonals. Let $0 \leqslant \ell \leqslant \mu$ and $0 \leqslant s \leqslant N - 2\ell - 1$. The 2ℓth subdiagonal of $B_n\pi_N$ is the 2ℓth superdiagonal of A_n/n^ν in reverse order. Thus, the sth entry of the 2ℓth subdiagonal of $B_n\pi_N$ is

$$\frac{1}{n^\nu}\,\omega_{\nu+2\ell+N-2\ell-1-s}\,q_{\ell,2\ell+N-2\ell-1-s} = \frac{1}{n^\nu}\,\omega_{\nu+N-1-s}\,q_{\ell,N-1-s} = \frac{1}{n^\nu}\,\omega_{n-s}\,q_{\ell,n-\nu-s}.$$

By virtue of (8) and Lemma 2.2, this is

$$\frac{1}{n^\nu}(n-s)^\nu \left(1 + O\left(\frac{1}{n-s}\right)\right) \left[(-1)^\ell \binom{\mu}{\ell} \frac{1}{2^\mu} + O\left(\frac{1}{n-\nu-s}\right)\right], \tag{14}$$

and since $\alpha, \nu, \mu, \ell, s$ are fixed, it follows that (14) converges to

$$(-1)^\ell \binom{\mu}{\ell} \frac{1}{2^\mu},$$

which is exactly the sth entry in the 2ℓth subdiagonal of the Toeplitz matrix B.

4. Proof of the main result for $m < \nu$

Let $m = \nu - 1 - \mu$ with an integer $\mu \geqslant 0$. From (6) we infer that the matrix representation of $D^\nu : \mathscr{P}_n(\alpha) \to \mathscr{P}_n(\alpha + \nu - 1 - \mu)$ is again of the form (12) with an $N \times N$ upper-triangular matrix A_n, where $N = n - \nu + 1$. The connection formula shows that A_n is of chessboard structure above the main diagonal. Since, obviously, $\gamma_{n-1}^{(\nu)} \leqslant \gamma_n^{(\nu)} \leqslant \gamma_{n+1}^{(\nu)}$, it suffices to prove Theorem 1.1 in the case where N is even. In that case there is a permutation matrix U_n such that

$$A_n = U_n \begin{pmatrix} E_n & 0 \\ 0 & F_n \end{pmatrix} U_n,$$

where $E_n = (e_{jk})_{j,k=0}^{N/2-1}$ and $F_n = (f_{jk})_{j,k=0}^{N/2-1}$ with

$$e_{jk} = c_{2j,\nu+2k}^{(\nu)}(\alpha, \alpha + \nu - 1 - \mu), \quad f_{jk} = c_{2j+1,\nu+2k+1}^{(\nu)}(\alpha, \alpha + \nu - 1 - \mu).$$

Clearly, $\|C_n\|_\infty = \|A_n\|_\infty = \max(\|E_n\|_\infty, \|F_n\|_\infty)$. From Lemma 2.2 we obtain

$$e_{jk} = \frac{2^{\nu+1}}{\mu!} j^{\alpha+\nu-\mu-1/2} k^{1/2-\alpha} (k^2 - j^2)^\mu \left(1 + O\left(\frac{1}{j}\right) + O\left(\frac{1}{k}\right)\right).$$

To find the asymptotics of $\|E_n\|_\infty$ we can now use an idea by Widom [20] and Shampine [17], [18]. Let K_n be the integral operator on $L^2(0,1)$ whose kernel is $e_{[xN/2],[yN/2]}$ where $[\cdot]$ denotes the integral part. One can show that

$$\|E_n\|_\infty = \frac{N}{2} \|K_n\|_\infty$$

(see, e.g., [3, Lemma 4.1]). For large N and for $y > x$, the kernel $e_{[xN/2],[yN/2]}$ behaves like

$$\begin{aligned} &\frac{2^{\nu+1}}{\mu!} \left(\frac{xN}{2}\right)^{\alpha+\nu-\mu-1/2} \left(\frac{yN}{2}\right)^{1/2-\alpha} \left(\left(\frac{yN}{2}\right)^2 - \left(\frac{xN}{2}\right)^2\right)^\mu \\ &= \frac{1}{2^{\mu-1}\mu!} N^{\nu+\mu} x^{\alpha+\nu-\mu-1/2} y^{1/2-\alpha} (y^2 - x^2)^\mu, \end{aligned}$$

which indicates that $N^{-\nu-\mu} K_n$ should converge to the operator K on $L^2(0,1)$ given by

$$(Kf)(x) = \frac{1}{2^{\mu-1}\mu!} \int_x^1 x^{\alpha+\nu-\mu-1/2} y^{1/2-\alpha} (y^2 - x^2)^\mu f(y)\, dy \tag{15}$$

and that therefore we should have

$$\|E_n\|_\infty \sim \frac{N}{2} N^{\nu+\mu} \|K\|_\infty = \frac{N^{\nu+\mu+1}}{2} \|K\|_\infty. \tag{16}$$

This can be founded rigorously: after expanding $(y^2 - x^2)^\mu$ by the binomial theorem, one can employ Theorem 4.2 of [3] to show that $N^{-\nu-\mu} K_n$ converges even in the

Hilbert-Schmidt norm to K and hence proves (16). Analogously one gets (16) with E_n replaced by F_n. In summary,

$$\gamma_n^{(\nu)}(\alpha,\alpha+\nu-1-\mu) \sim \frac{N^{\nu+\mu+1}}{2}\,\|K\|_\infty \sim \frac{n^{\nu+\mu+1}}{2}\,\|K\|_\infty. \tag{17}$$

Taking the adjoint of operator (15) and putting $\nu-1-\mu=m$, we can rewrite (17) in the form

$$\gamma_n^{(\nu)}(\alpha,\alpha+m) \sim \frac{n^{2\nu-m}}{2^{\nu-m-1}}\,\frac{1}{(\nu-m-1)!}\,\|G\|_\infty, \tag{18}$$

where G is given on $L^2(0,1)$ by

$$(Gf)(x) = \int_0^x x^{1/2-\alpha}\,y^{\alpha+m+1/2}\,(x^2-y^2)^{\nu-m-1}\,f(y)\,dy. \tag{19}$$

Let finally V be the unitary operator on $L^2(0,1)$ that is defined by

$$(Vf)(x) = 2^{1/2}x^{1/2}f(x^2), \quad (V^{-1}f)(x) = 2^{-1/2}x^{-1/4}f(x^{1/2}).$$

We have

$$\begin{aligned}(V^{-1}GVf)(x) &= 2^{-1/2}x^{-1/4}(GVf)(x^{1/2})\\ &= 2^{-1/2}x^{-1/4}\int_0^{x^{1/2}} x^{(1/2-\alpha)/2}\,t^{\alpha+m+1/2}\,(x-t^2)^{\nu-m-1}\,2^{1/2}t^{1/2}f(t^2)\,dt\end{aligned}$$

and after the substitution $t^2=y$ this becomes

$$\frac{1}{2}\int_0^x x^{-\alpha/2}\,y^{(\alpha+m)/2}\,(x-y)^{\nu-m-1}\,f(y)\,dy = \frac{(\nu-m-1)!}{2}(L^*_{\nu,\alpha,\alpha+m}f)(x),$$

where $L^*_{\nu,\alpha,\alpha+m}$ is as in Section 1. Consequently,

$$\gamma_n^{(\nu)}(\alpha,\alpha+m) \sim \frac{1}{2^{\nu-m}}\,\|L^*_{\nu,\alpha,\alpha+m}\|_\infty\, n^{2\nu-m},$$

which completes the proof of Theorem 1.1.

Theorem 1.2 is now an immediate consequence of Corollary 5.2 of [3], which states that if $2\delta+1>0$ and $\sigma:=\beta+\delta+1>0$, then the operator norm of the operator defined on $L^2(0,1)$ by

$$(T^*_{\beta,\delta}f)(x) = \int_0^x x^\beta y^\delta\, f(y)\,dy$$

is $1/\sigma$ times the reciprocal of the smallest positive zero of the classical Bessel function $J_{-(1+2\beta)/(2\sigma)}$. Indeed, in the case $m=\nu-1$, Theorem 1.1 yields

$$\gamma_n^{(\nu)}(\alpha,\alpha+\nu-1) \sim \frac{1}{2}\,\|T^*_{-\alpha/2,(\alpha+\nu-1)/2}\|_\infty\, n^{\nu+1}. \tag{20}$$

We have $\beta = -\alpha/2$, $\delta = (\alpha+\nu-1)/2$, $\sigma = (\nu+1)/2$, and hence the coefficient on the right of (20) is $1/(\nu+1)$ times the reciprocal of the first positive zero of $J_{(\alpha-1)/(\nu+1)}$, which completes the proof of Theorem 1.2. Finally, since

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x, \quad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x,$$

we get

$$\gamma_n^{(\nu)}\left(\frac{1-\nu}{2}, \frac{\nu-1}{2}\right) \sim \frac{2}{(\nu+1)\pi} n^{\nu+1} \quad \text{for} \quad \nu = 1, 2,$$
$$\gamma_n^{(\nu)}\left(\frac{\nu+3}{2}, \frac{3\nu+1}{2}\right) \sim \frac{1}{(\nu+1)\pi} n^{\nu+1} \quad \text{for} \quad \nu \geqslant 1.$$

REFERENCES

[1] R. P. AGARWAL AND G. V. MILOVANOVIĆ, *Extremal problems, inequalities, and classical orthogonal polynomials*, Appl. Math. Comput., **128** (2002), 151–166.

[2] G. E. ANDREWS, R. ASKEY, AND R. ROY, *Special Functions*, Encyclopedia of Mathematics and its Applications, Vol. 71, Cambridge University Press, Cambridge 1999.

[3] A. BÖTTCHER AND P. DÖRFLER, *Weighted Markov-type inequalities, norms of Volterra operators, and zeros of Bessel functions*, Math. Nachr., **283** (2010), 40–57.

[4] A. BÖTTCHER AND P. DÖRFLER, *On the best constants in Markov-type inequalities involving Laguerre norms with different weights*, Monatshefte f. Math., **161** (2010), 357–367.

[5] A. BÖTTCHER AND B. SILBERMANN, *Introduction to Large Truncated Toeplitz Matrices*, Springer-Verlag, New York 1999.

[6] I. K. DAUGAVET AND S. Z. RAFAL'SON, *Certain inequalities of Markov-Nikolskii type for algebraic polynomials*, Vestnik Leningrad. Univ., **1** (1972), 15–25 (Russian).

[7] A. GUESSAB AND G. V. MILOVANOVIĆ, *Weighted L^2-analogues of Bernstein's inequality and classical orthogonal polynomials*, J. Math. Anal. Appl., **182** (1994), 244–249.

[8] M. E. H. ISMAIL, *Classical and Quantum Orthogonal Polynomials in One Variable*, Encyclopedia of Mathematics and its Applications, Vol. 98, Cambridge University Press, Cambridge 2005.

[9] S. V. KONYAGIN, *Estimates of derivatives of polynomials*, Dokl. Akad. Nauk SSSR, **243** (1978), 1116–1118 (Russian).

[10] A. KROÓ, *On the exact constant in the L_2 Markov inequality*, J. Approx. Theory, **151** (2008), 208–211.

[11] G. V. MILOVANOVIĆ, D. S. MITRINOVIĆ, AND TH. M. RASSIAS, *Topics in Polynomials, Extremal Problems, Inequalities, Zeros*, World Scientific, Singapore, 1994.

[12] Q. I. RAHMAN AND G. SCHMEISSER, *Les inégalités de Markoff et de Bernstein*, Séminaire de Mathématiques Supérieures **86**, Presses de l'Université de Montréal, Montreal 1983.

[13] Q. I. RAHMAN AND G. SCHMEISSER, *Analytic Theory of Polynomials*, Clarendon Press, Oxford 2002.

[14] J. SÁNCHEZ-RUIZ AND J. S. DEHESA, *Some connection and linearization problems for polynomials in and beyond the Askey scheme*, J. Comput. Appl. Math., **133** (2001), 579–591.

[15] E. SCHMIDT, *Die asymptotische Bestimmung des Maximums des Integrals über das Quadrat der Ableitung eines normierten Polynoms, dessen Grad ins Unendliche wächst*, Sitzungsber. Preuss. Akad. Wiss. (1932), p. 287.

[16] E. SCHMIDT, *Über die nebst ihren Ableitungen orthogonalen Polynomensysteme und das zugehörige Extremum*, Math. Ann., **119** (1944), 165–204.
[17] L. F. SHAMPINE, *Some L_2 Markoff inequalities*, J. Res. Nat. Bur. Standards, **69B** (1965), 155–158.
[18] L. F. SHAMPINE, *An inequality of E. Schmidt*, Duke Math. J., **33** (1966), 145–150.
[19] P. K. SUETIN, *Classical Orthogonal Polynomials*, Second edition, Nauka, Moscow 1979 (Russian).
[20] H. WIDOM, *On the eigenvalues of certain Hermitian operators*, Trans. Amer. Math. Soc., **88** (1958), 491–522.

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