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UNITARY EQUIVALENCE TO A COMPLEX SYMMETRIC MATRIX: A MODULUS CRITERION

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Abstract. We develop a procedure for determining whether a square complex matrix is unitarily equivalent to a complex symmetric (i.e., self-transpose) matrix. We compare our approach to several existing methods [1, 19, 20] and present a number of examples.

1. Introduction

Following [19], we say that a matrix $T \in M_n(\mathbb{C})$ is *UECSM* if it is unitarily equivalent to a complex symmetric (i.e., self-transpose) matrix. Here we use the term *unitarily equivalent* in the sense of operator theory: we say that two matrices A and B are unitarily equivalent if $A = UBU^*$ for some unitary matrix U. In contrast, the term *unitarily similar* is frequently used in the matrix-theory literature.

Since every $n \times n$ complex matrix is *similar* to a complex symmetric matrix [15, Thm. 4.4.9] (see also [7, Ex. 4] and [6, Thm. 2.3]), it is often difficult to tell whether or not a given matrix is UECSM. For instance, exactly one of the following matrices is UECSM:

$$\begin{pmatrix} 5 & 1 & 1 & 3 \\ 1 & 1 & 1 & -1 \\ 1 & -3 & 5 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 5 & -1 & 3 & 3 \\ 1 & 3 & -1 & -1 \\ 1 & -1 & 3 & -1 \\ -1 & 1 & -3 & 1 \end{pmatrix}. \tag{1}$$

In particular, there are no readily apparent features which suggest that either of these matrices possesses more "symmetry" than the other (see Section 4).

Our primary motivation stems from the emerging theory of complex symmetric operators on Hilbert space [2, 3, 5, 6, 7, 8, 10, 12, 17, 21, 22]. To be more specific, a bounded operator T on a separable complex Hilbert space $\mathscr{H}$ is called a *complex symmetric operator* if $T = CT^*C$ for some conjugation C (a conjugate-linear, isometric involution) on $\mathscr{H}$. The terminology stems from the fact that the preceding condition is equivalent to insisting that T has a complex symmetric matrix representation with respect to some orthonormal basis [6, Sect. 2.4-2.5]. Thus the problem of determining

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whether a given matrix is UECSM is equivalent to determining whether that matrix represents a complex symmetric operator with respect to some orthonormal basis. In other words, T is UECSM if and only if T belongs to the unitary orbit of the complex symmetric matrices in $M_n(\mathbb{C})$.

Although ad-hoc methods sometimes suffice for specific examples (e.g., [7, Ex. 5,7], [10, Ex. 1, Thm. 4], [20, Ex. 3]), the first general approach was due to Vermeer, who proved that T is UECSM if and only if $T = ST^tS^*$ for some symmetric unitary matrix S [20, Thm. 3]. Although highly effective in many situations, this procedure (which we denote `VTest`) does not quite fit our needs since it is basis-dependent and does not immediately adapt to the abstract Hilbert space setting.

Another approach is due to J. Tener [19] who developed a procedure (`UECSMTest`) based upon the diagonalization of the selfadjoint components A and B in the Cartesian decomposition $T = A + iB$. More recently, L. Balayan and the first author developed another procedure (`StrongAngleTest`) based upon a careful analysis of the eigenstructure of T itself [1].

In this note, we pursue a different approach, based upon the diagonalization of T^*T and TT^*. Before discussing our main result, we require a few definitions. Recall that the singular values of a matrix $T \in M_n(\mathbb{C})$ are defined to be the eigenvalues of the positive semidefinite matrix $|T| = \sqrt{T^*T}$, the so-called *modulus* of T. We also remark that T^*T and TT^* share the same eigenvalues [14, Pr. 101].

THEOREM 1. *If $T \in M_n(\mathbb{C})$ has distinct singular values,*

(1) *$u_1, u_2, \ldots, u_n$ are unit eigenvectors of T^*T corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively,*

(2) *$v_1, v_2, \ldots, v_n$ are unit eigenvectors of TT^* corresponding to the eigenvalues λ_1, $\lambda_2, \ldots, \lambda_n$, respectively,*

then T is UECSM if and only if

$$|\langle u_i, v_j \rangle| = |\langle u_j, v_i \rangle|, \tag{2}$$

$$\langle u_i, v_j \rangle \langle u_j, v_k \rangle \langle u_k, v_i \rangle = \langle u_i, v_k \rangle \langle u_k, v_j \rangle \langle u_j, v_i \rangle, \tag{3}$$

holds for $1 \leqslant i \leqslant j \leqslant k \leqslant n$.

The procedure suggested by the preceding theorem can easily be implemented in `Mathematica` (we refer to this procedure as `ModulusTest`). We should also remark that T.Y. Tam has recently obtained another proof of Theorem 1 and several other related results based upon the singular value decomposition of T [18].

Although the eigenvectors of T^*T and TT^* are determined only up to *unimodular* (i.e., unit modulus) constant factors, it is easy to see that (2) and (3) do not depend upon these constants. In particular, Theorem 1 is basis-independent and can in principle be applied to operators on Hilbert space (see Section 5). Moreover, as a byproduct of our method we are able to construct the symmetric unitary matrix S of Vermeer's condition based upon the data required by Theorem 1.

The structure of this paper is as follows. The proof of Theorem 1 is the subject of Section 2. Section 3 contains a number of instructive examples. In Section 4 we compare `ModulusTest` to the procedures `UECSMTest` [19], `StrongAngleTest` [1], and `VTest` [20, Thm. 3]. In Section 5 we discuss applications of our results to compact operators. As an illustration, we reveal a "hidden symmetry" of the Volterra integration operator.

EXAMPLE 1. Before we proceed, we list several matrices which are UECSM and their corresponding complex symmetric matrices. These matrices were tested by `ModulusTest` and the unitary equivalences exhibited using the procedures outlined in Section 3. In particular, we have selected relatively simple matrices which enjoy no apparent "symmetry" whatsoever. The symbol $\cong$ denotes unitary equivalence.

$$\begin{pmatrix} 5 & 2 & 2 \\ 7 & 0 & 0 \\ 7 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} \frac{1}{2}\left(5-\sqrt{187}\right) & -5i\sqrt{\frac{561+5\sqrt{187}}{1658}} & -i\sqrt{\frac{3350}{829}-\frac{125\sqrt{187}}{1658}} \\ -5i\sqrt{\frac{561+5\sqrt{187}}{1658}} & \frac{1}{829}\left(1870+293\sqrt{187}\right) & \frac{9}{829}\sqrt{\frac{1}{2}\left(173723+7075\sqrt{187}\right)} \\ -i\sqrt{\frac{3350}{829}-\frac{125\sqrt{187}}{1658}} & \frac{9}{829}\sqrt{\frac{1}{2}\left(173723+7075\sqrt{187}\right)} & \frac{81}{-5+3\sqrt{187}} \end{pmatrix},$$

$$\begin{pmatrix} 9 & 8 & 9 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix} \cong \begin{pmatrix} 8-\frac{\sqrt{149}}{2} & \frac{9}{2}i\sqrt{\frac{16837+64\sqrt{149}}{13093}} & i\sqrt{\frac{133672}{13093}-\frac{1296\sqrt{149}}{13093}} \\ \frac{9}{2}i\sqrt{\frac{16837+64\sqrt{149}}{13093}} & \frac{207440+9477\sqrt{149}}{26186} & \frac{18\sqrt{3978002+82324\sqrt{149}}}{13093} \\ i\sqrt{\frac{133672}{13093}-\frac{1296\sqrt{149}}{13093}} & \frac{18\sqrt{3978002+82324\sqrt{149}}}{13093} & \frac{92675+1808\sqrt{149}}{13093} \end{pmatrix}.$$

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2. Proof of Theorem 1

2.1. Preliminary lemmas

Recall that a conjugation C on $\mathbb{C}^n$ is a conjugate-linear involution (i.e., $C^2 = I$) which is also isometric (i.e., $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathbb{C}^n$). It is easy to see that each conjugation C on $\mathbb{C}^n$ is of the form $C = SJ$ where S is a complex symmetric unitary matrix and J is the canonical conjugation

$$J(z_1, z_2, \ldots, z_n) = (\overline{z_1}, \overline{z_2}, \ldots, \overline{z_n}) \tag{4}$$

on $\mathbb{C}^n$. The relevance of conjugations to our endeavor lies in the following lemma.

LEMMA 1. *$T \in M_n(\mathbb{C})$ is UECSM if and only if there exists a conjugation C on $\mathbb{C}^n$ such that $T = CT^*C$.*

Proof. Suppose that $T = CT^*C$ for some conjugation C on $\mathbb{C}^n$. By [7, Lem. 1] there exists an orthonormal basis $e_1, e_2, \ldots, e_n$ such that $Ce_i = e_i$ for $i = 1, 2, \ldots, n$. Let $Q = (e_1|e_2|\cdots|e_n)$ be the unitary matrix whose columns are these basis vectors.

The matrix $M = Q^*TQ$ is complex symmetric since the ijth entry $[M]_{ij}$ of M satisfies $[M]_{ij} = \langle Te_j, e_i\rangle = \langle CT^*Ce_j, e_i\rangle = \langle e_i, T^*e_j\rangle = \langle Te_i, e_j\rangle = [M]_{ji}$. □

Our next result shows that, under the hypotheses of Theorem 1, T is UECSM if and only if there is a conjugation intertwining T^*T and TT^*.

LEMMA 2. *If C is a conjugation on $\mathbb{C}^n$ and $T \in M_n(\mathbb{C})$ has distinct singular values, then*

$$T = CT^*C \quad \Leftrightarrow \quad T^*T = C(TT^*)C. \tag{5}$$

Proof. The $(\Rightarrow)$ implication of (5) follows immediately, regardless of any hypotheses on the singular values of T. The implication $(\Leftarrow)$ is considerably more involved. Suppose that $T^*T = CTT^*C$. Write $T = U(T^*T)^{\frac{1}{2}}$ where U is unitary and observe that $TT^* = UT^*TU^*$ whence $UT^*T = TT^*U$. It follows that $UT^*T = CT^*TCU$ which implies that

$$CU(T^*T) = (T^*T)CU. \tag{6}$$

Let $e_1, e_2, \ldots, e_n$ denote unit eigenvectors of T^*T corresponding to the (necessarily non-negative) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of T^*T.

In light of (6), we see that $T^*Te_i = \lambda_i e_i$ if and only if $(T^*T)(CUe_i) = \lambda_i(CUe_i)$. In other words, the conjugate-linear operator CU maps each eigenspace of T^*T into itself. Since CU is isometric and since the eigenspaces of T^*T are one-dimensional, it follows that $CUe_i = \zeta_i^2 e_i$ for some unimodular constants $\zeta_1, \zeta_2, \ldots, \zeta_n$. Using the fact that C is conjugate-linear we find that the unit vectors $w_i = \zeta_i e_i$ satisfy $CUw_i = w_i$ and $T^*Tw_i = \lambda_i w_i$.

We claim that the conjugate-linear operator $K = CU$ is a conjugation on $\mathbb{C}^n$. Indeed, since U is unitary and C is a conjugation it is clear that K is isometric. Moreover, since $K^2 w_i = CUCUw_i = CUw_i = w_i$ for $i = 1, 2, \ldots, n$ it follows that $K^2 = I$ whence K is a conjugation. By (6) it follows that $K(T^*T)K = T^*T$ whence $K|T|K = |T|$ (since $|T| = p(T^*T)$ for some polynomial $p(x) \in \mathbb{R}[x]$).

Putting this all together, we find that $T = CK|T|$ where K is a conjugation that commutes with $|T|$. In particular, the unitary matrix U factors as $U = CK$ and satisfies $U^* = KC$. We therefore conclude that $T = CK|T| = C|T|K = C(|T|KC)C = C(|T|U^*)C = CT^*C$. □

We remark that the implication $(\Leftarrow)$ of Lemma 2 is false if one drops the hypothesis that the singular values of T are distinct. For instance, let T be unitary matrix which is not complex symmetric (i.e., $T \neq JT^*J$ where J denotes the canonical conjugation (4) on $\mathbb{C}^n$). In this case, $T^*T = I = TT^*$ (i.e., all of the singular vales of T are 1) and hence the condition on the right-hand side of (5) obviously holds. On the other hand, $T \neq JT^*J$ by hypothesis.

From here on, we maintain the notation and conventions of Theorem 1, namely that $u_1, u_2, \ldots, u_n$ are unit eigenvectors of T^*T and $v_1, v_2, \ldots, v_n$ are unit eigenvectors of TT^* corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively.

LEMMA 3. *If C is a conjugation on $\mathbb{C}^n$ and $T \in M_n(\mathbb{C})$ has distinct singular values, then $T^*T = CTT^*C$ if and only if $Cu_i = \alpha_i v_i$ for some unimodular constants $\alpha_1, \alpha_2, \ldots, \alpha_n$.*

Proof. For the forward implication, observe that $\lambda_i u_i = T^*Tu_i = CTT^*Cu_i$ whence $TT^*(Cu_i) = \lambda_i(Cu_i)$. Since the eigenspaces of TT^* are one-dimensional and C is isometric, it follows that $Cu_i = \alpha_i v_i$ for some unimodular constants $\alpha_1, \alpha_2, \ldots, \alpha_n$.

On the other hand, suppose that there exist unimodular constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $Cu_i = \alpha_i v_i$ for $i = 1, 2, \ldots, n$. Since C is a conjugation, it follows that $Cv_i = \alpha_i u_i$ for $i = 1, 2, \ldots, n$. It follows that $CTT^*Cu_i = CTT^*\alpha_i v_i = \overline{\alpha_i}CTT^*v_i = \overline{\alpha_i}\lambda_i Cv_i = \overline{\alpha_i}\alpha_i\lambda_i u_i = \lambda_i u_i$ for $i = 1, 2, \ldots, n$. Since the linear operators CTT^*C and T^*T agree on the orthonormal basis $u_1, u_2, \ldots, u_n$, we conclude that $T^*T = CTT^*C$. □

LEMMA 4. *There exists a conjugation C and unimodular constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $Cu_i = \alpha_i v_i$ for $i = 1, 2, \ldots, n$ if and only if*

$$\langle u_i, v_j \rangle = \alpha_j \overline{\alpha_i} \langle u_j, v_i \rangle \tag{7}$$

holds for $1 \leqslant i, j \leqslant n$.

Proof. For the forward implication, simply note that if $Cu_i = \alpha_i v_i$ for $i = 1, 2, \ldots, n$, then (7) follows immediately from the fact that C is isometric and conjugate-linear. Conversely, suppose that (7) holds for $1 \leqslant i, j \leqslant n$. We claim that the definition $Cu_i = \alpha_i v_i$ for $1 \leqslant i \leqslant n$ extends by conjugate-linearity to a conjugation on all of $\mathbb{C}^n$. Since $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ are orthonormal bases of $\mathbb{C}^n$ and since the constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ are unimodular, it follows that C is isometric. It therefore suffices to prove that $C^2 = I$. To this end, we need only show that $Cv_i = \alpha_i u_i$ for $1 \leqslant i \leqslant n$. This follows from a straightforward computation:

$$\begin{aligned} Cv_i = C\left(\sum_{j=1}^n \langle v_i, u_j \rangle u_j \right) &= \sum_{j=1}^n \langle u_j, v_i \rangle Cu_j = \sum_{j=1}^n \langle u_j, v_i \rangle \alpha_j v_j \\ &= \sum_{j=1}^n \alpha_i \overline{\alpha_j} \langle u_i, v_j \rangle \alpha_j v_j = \alpha_i \sum_{j=1}^n \langle u_i, v_j \rangle v_j = \alpha_i u_i. \end{aligned}$$

Thus C is a conjugation on $\mathbb{C}^n$, as desired. □

We can interpret the condition (7) in terms of matrices. Let $U = (u_1|u_2|\cdots|u_n)$ and $V = (v_1|v_2|\cdots|v_n)$ denote the $n \times n$ unitary matrices whose columns are the orthonormal bases $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$, respectively. Now observe that (7) is equivalent to asserting that

$$(V^*U)^t = A^*(V^*U)A \tag{8}$$

holds where $A = \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ denotes the diagonal unitary matrix having the unimodular constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ along the main diagonal.

Putting Lemmas 2, 3, and 4 together, we obtain the following important lemma.

LEMMA 5. *There exist unimodular constants* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *such that* (7) *holds if and only if* T *is UECSM.*

With these preliminaries in hand, we are now ready to complete the proof of Theorem 1.

2.2. Proof of the implication $(\Rightarrow)$

Suppose that T is UECSM. By Lemma 5, there exist unimodular constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ so that (7) holds for $1 \leqslant i, j \leqslant n$. The desired conditions (2) and (3) from the statement of Theorem 1 then follow immediately.

2.3. Proof of the implication $(\Leftarrow)$

The proof that conditions (2) and (3) are sufficient for T to be UECSM is somewhat more complicated. Fortunately, the proof of [1, Thm. 2] goes through with minor notational changes and we refer the reader there for the details. We sketch the main idea below.

Suppose that $\langle u_j, v_i \rangle \neq 0$ for $1 \leqslant i, j \leqslant n$ (the proof of [1, Thm. 2] explains how to get around this restriction) and observe that (2) ensures that the constants

$$\beta_{ij} = \frac{\langle u_i, v_j \rangle}{\langle u_j, v_i \rangle}$$

are unimodular. The condition (3) then implies that $\beta_{ij}\beta_{jk} = \beta_{ik}$, from which it follows that the unimodular constants $\alpha_i = \beta_{1i}$ satisfy (7). We therefore conclude that T is UECSM by Lemma 5. $\square$

3. Examples and computations

Before considering several examples, let us first remark that Theorem 1 is constructive. Maintaining the notation and conventions established in the proof of Theorem 1, define the unitary matrices U, V, and A as in (8). Let $s_1, s_2, \ldots, s_n$ denote the standard basis of $\mathbb{C}^n$ and let J denote the canonical conjugation (4) on $\mathbb{C}^n$. In particular, observe that $Js_i = s_i$ for $i = 1, 2, \ldots, n$. The proof of Theorem 1 tells us that if T satisfies (2) and (3) (e.g., "T passes `ModulusTest`"), then there exist unimodular constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $Cu_i = \alpha_i v_i$ for $i = 1, 2, \ldots, n$. Letting $A = \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ we see that

$$\begin{aligned} VAU^t Ju_i &= VAJU^* u_i = VAJs_i \\ &= VAs_i = \alpha_i V s_i \\ &= \alpha_i v_i. \end{aligned}$$

Thus the conjugate-linear operators C and $(VAU^t)J$ agree on the orthonormal basis $u_1, u_2, \ldots, u_n$ whence they agree on all of $\mathbb{C}^n$. Although it is not immediately obvious, the unitary matrix $S = VAU^t$ is complex symmetric. Indeed, the condition $S = S^t$ is equivalent to (8).

Once the conjugation $C = SJ$ has been obtained it is a simple matter of finding an orthonormal basis with respect to which T has a complex symmetric matrix representation (see Lemma 1). To find such a basis, observe that since $S = CJ$ is a C-symmetric unitary operator, each of its eigenspaces are fixed by C [6, Lem. 8.3]. Let us also note that $T = CT^*C = SJT^*SJ = ST^t\overline{S}$ so that $T = ST^tS^*$. In other words, the matrix S constructed above is the symmetric unitary matrix whose existence is guaranteed by Vermeer's criterion [20, Thm. 3].

EXAMPLE 2. Although at this point many different proofs of the fact that every 2×2 matrix is UECSM exist (see [1, Cor. 3], [3, Cor. 3.3], [7, Ex. 6], [9], [10, Cor. 1], [16, p. 477], [19, Cor. 3]), for the sake of illustration we give yet another.

By Schur's Theorem on unitary triangularization, we need only consider upper triangular 2×2 matrices. If T is such a matrix and has repeated eigenvalues, then upon subtracting a multiple of the identity we may assume that

$$T = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}. \tag{9}$$

A routine computation now shows that $T = UAU^*$ where

$$A = \begin{pmatrix} \frac{a}{2} & \frac{ia}{2} \\ \frac{ia}{2} & -\frac{a}{2} \end{pmatrix}, \qquad U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}.$$

Thus it suffices to consider the case where T has distinct eigenvalues. Upon subtracting a multiple of the identity and then scaling, we may assume that

$$T = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}. \tag{10}$$

Moreover, we may also assume that $a \geqslant 0$ since this may be obtained by conjugating T by an appropriate diagonal unitary matrix. Thus we have

$$T^*T = \begin{pmatrix} 1 & a \\ a & a^2 \end{pmatrix}, \quad TT^* = \begin{pmatrix} 1+a^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

The eigenvalues of T^*T and TT^* are $\lambda_1 = 1+a^2$ and $\lambda_2 = 0$ and corresponding unit eigenvectors are

$$u_1 = \begin{pmatrix} \frac{1}{\sqrt{1+a^2}} \\ \frac{a}{\sqrt{1+a^2}} \end{pmatrix}, \quad u_2 = \begin{pmatrix} \frac{-a}{\sqrt{1+a^2}} \\ \frac{1}{\sqrt{1+a^2}} \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let us first consider the condition (2) of the procedure `ModulusTest`. For $i = j$ it holds trivially and for $i \neq j$ we have

$$|\langle u_1, v_2\rangle| = \frac{a}{\sqrt{1+a^2}} = |\langle u_2, v_1\rangle|.$$

Now let us consider the second condition (3). Since $n=2$, at least two of i,j,k must be equal whence (3) holds trivially. By Theorem 1, it follows that T is UECSM.

Let us now explicitly construct a complex symmetric matrix which T is unitarily equivalent to. Since the equation

$$\frac{a}{\sqrt{1+a^2}} = \langle u_1, v_2\rangle = \overline{\alpha_1}\alpha_2\langle u_2, v_1\rangle = \overline{\alpha_1}\alpha_2\frac{-a}{\sqrt{1+a^2}}$$

is satisfied by $\alpha = 1$ and $\alpha_2 = -1$, we let

$$S = \underbrace{\left(\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array}\right)}_{V} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{1+a^2}} & \frac{a}{\sqrt{1+a^2}} \\ \frac{-a}{\sqrt{1+a^2}} & \frac{1}{\sqrt{1+a^2}} \end{pmatrix}}_{U^t} = \begin{pmatrix} \frac{1}{\sqrt{1+a^2}} & \frac{a}{\sqrt{1+a^2}} \\ \frac{a}{\sqrt{1+a^2}} & -\frac{1}{\sqrt{1+a^2}} \end{pmatrix}$$

and note that the conjugation $C = SJ$ satisfies $T = CT^*C$. An orthonormal basis e_1, e_2 of $\mathbb{C}^2$ whose elements are fixed by C is given by

$$e_1 = \begin{pmatrix} \frac{1-\sqrt{1+a^2}}{\sqrt{2+2a^2-2\sqrt{1+a^2}}} \\ \frac{a}{\sqrt{2+2a^2-2\sqrt{1+a^2}}} \end{pmatrix} \qquad e_2 = \begin{pmatrix} \frac{-ia}{\sqrt{2+2a^2-2\sqrt{1+a^2}}} \\ \frac{i(1-\sqrt{1+a^2})}{\sqrt{2+2a^2-2\sqrt{1+a^2}}} \end{pmatrix}.$$

Note that these are certain normalized eigenvectors of S, corresponding to the eigenvalues 1 and -1, respectively, whose phases are selected so that $Ce_1 = SJe_1 = Se_1 = e_1$ and $Ce_2 = SJe_2 = S(-e_2) = -Se_2 = e_2$. Letting $Q = (e_1|e_2)$ denote the unitary matrix whose columns are e_1 and e_2, we find that

$$Q^*TQ = \begin{pmatrix} \frac{1}{2}(1-\sqrt{1+a^2}) & \frac{ia}{2} \\ \frac{ia}{2} & \frac{1}{2}(1+\sqrt{1+a^2}) \end{pmatrix}.$$

As predicted by Lemma 1, this matrix is complex symmetric.

The following simple example was first considered, using ad-hoc methods, in [10, Ex. 1]. Note that the procedure `StrongAngleTest` of [1] cannot be applied to this matrix due to the repeated eigenvalue 0.

EXAMPLE 3. Suppose that $ab \neq 0$ and $|a| \neq |b|$. In this case, the singular values of

$$T = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$$

are distinct. Normalized eigenvectors u_1, u_2, u_3 of T^*T and v_1, v_2, v_3 of TT^* corresponding to the eigenvalues $0, |a|^2, |b|^2$, respectively are given by

$$u_1 = v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad u_2 = v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad u_3 = v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since $\langle u_1, v_2\rangle = 0$ and $\langle u_2, v_1\rangle = 1$, condition (2) fails from which we conclude that T is not UECSM. On the other hand, if either $a = 0$ or $b = 0$, then T is the direct sum of a 1×1 with a 2×2 matrix whence T is UECSM by Example 2. Moreover, if $|a| = |b|$, then T is unitarily equivalent to a Toeplitz matrix and thus UECSM by [6, Sect. 2.2].

EXAMPLE 4. We claim that the lower-triangular matrix

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

is UECSM. Normalized eigenvectors u_1, u_2, u_3 of T^*T and v_1, v_2, v_3 of TT^* corresponding to the eigenvalues $\lambda_1 = 0$, $\lambda_2 = 4$, and $\lambda_3 = 6$ are given by

$$u_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \qquad u_2 = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad u_3 = \begin{pmatrix} -\frac{4}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix},$$

and

$$v_1 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

respectively. Since

$$\begin{aligned} \langle u_1, v_2\rangle &= \langle u_2, v_1\rangle = 0, \\ \langle u_2, v_3\rangle &= \langle u_3, v_2\rangle = 0, \\ \langle u_3, v_1\rangle &= \langle u_1, v_3\rangle = \tfrac{1}{\sqrt{3}}, \end{aligned}$$

conditions (2) and (3) are obviously satisfied. By Theorem 1, we conclude that T is UECSM. Let us now construct a complex symmetric matrix which T is unitarily equivalent to.

By inspection, we find that $\alpha_1 = \alpha_2 = \alpha_3 = 1$ is a solution to (7). Maintaining the notation established at the beginning of this section, we observe that the matrix

$$S = \underbrace{\left(\begin{array}{c|c|c} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{array}\right)}_{V} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{A} \underbrace{\left(\begin{array}{ccc} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \hline 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \hline -\frac{4}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{array}\right)}_{U^t}$$

$$= \begin{pmatrix} -\frac{4}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{2} + \frac{1}{\sqrt{6}} & -\frac{1}{2} + \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{2} + \frac{1}{\sqrt{6}} & \frac{1}{2} + \frac{1}{\sqrt{6}} \end{pmatrix}$$

is symmetric and unitary (i.e., S is the matrix from Vermeer's criterion). We then find an orthonormal basis e_1, e_2, e_3 whose elements are fixed by the conjugation $C = SJ$. Following Lemma 1, we encode one such example as the columns of the unitary matrix

$$Q = \left(\begin{array}{c|c|c} -i\sqrt{\frac{1}{2}+\frac{1}{\sqrt{6}}} & \frac{1}{5}\sqrt{11-4\sqrt{6}} & \frac{1}{\sqrt{2(9+\sqrt{6})}} \\ \frac{i}{2\sqrt{3+\sqrt{6}}} & 0 & \frac{1}{2}\sqrt{3+\sqrt{\frac{2}{3}}} \\ \frac{i}{2\sqrt{3+\sqrt{6}}} & \frac{1}{5}\left(\sqrt{2}+2\sqrt{3}\right) & -\frac{1}{10}\sqrt{19-23\sqrt{\frac{2}{3}}} \end{array} \right)$$

and note that Q^*TQ is complex symmetric:

$$\begin{pmatrix} 1-\sqrt{\frac{3}{2}} & -\frac{1}{5}i\sqrt{9-\sqrt{6}} & -\frac{1}{5}i\sqrt{\frac{7}{2}+\sqrt{6}} \\ -\frac{1}{5}i\sqrt{9-\sqrt{6}} & \frac{1}{25}\left(26+11\sqrt{6}\right) & \frac{1}{25}\sqrt{123-47\sqrt{6}} \\ -\frac{1}{5}i\sqrt{\frac{7}{2}+\sqrt{6}} & \frac{1}{25}\sqrt{123-47\sqrt{6}} & \frac{1}{50}\left(98+3\sqrt{6}\right) \end{pmatrix}.$$

Independent confirmation that T is UECSM is obtained from `VTest` or simply by noting that $T-2I$ has rank one (every rank-one matrix is UECSM by [10, Cor. 5]).

4. Comparison with other methods

With the addition of `ModulusTest` there are now four general procedures for determining whether a matrix T is UECSM. Each has its own restrictions:

1. `ModulusTest` (this article) requires that T has distinct singular values,

2. `StrongAngleTest` [1] requires that T has distinct eigenvalues,

3. `UECSMTest` [19] requires that the selfadjoint matrices A, B in the Cartesian decomposition $T = A + iB$ (where $A = A^*$, $B = B^*$) both have distinct eigenvalues. However, this restriction can be removed in the 3×3 case.

4. `VTest` [20, Thm. 3] has no theoretical restrictions. On the other hand, this approach requires one to solve nonlinear equations in several complex variables and their conjugates.

Table 1 provides a number of examples indicating that `ModulusTest` is not subsumed by either `StrongAngleTest` or `UECSMTest`. At this point we should also remark that the two matrices (1) from the introduction are unitarily equivalent to constant multiples of the corresponding matrices in Table 1. In particular, the first matrix in (1)

T	$\sigma(T^*T)$	$\sigma(T)$	$\sigma(A)$	$\sigma(B)$	UECSM?
$\begin{pmatrix} 1&0&1&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&1&0&0 \end{pmatrix}$	$0,2,\frac{3\pm\sqrt{5}}{2}$	$0,1,1,1$	$\frac{1}{2},\frac{3}{2},\frac{1\pm\sqrt{2}}{2}$	$-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\frac{1}{2}$	YES
$\begin{pmatrix} 1&0&1&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&1&0 \end{pmatrix}$	$0,1,2\pm\sqrt{2}$	$0,1,1,1$	distinct	$0,0,\pm\frac{\sqrt{2}}{2}$	NO

Table 1: *Matrices which satisfy the hypotheses of* `ModulusTest` *but not those of* `UECSMTest` *or* `StrongAngleTest` *(the notation* $\sigma(\cdot)$ *denotes the spectrum of a matrix). Whether or not these matrices are UECSM can be determined by* `ModulusTest`*. In the second row, the eigenvalues of* A *are distinct but cannot be displayed exactly in the confines of the table.*

is unitarily equivalent to

$$\begin{pmatrix} \frac{2}{17}\left(23+16\sqrt{2}\right) & \frac{4}{17}\sqrt{50-31\sqrt{2}} & -2i\sqrt{\frac{1}{17}\left(5+2\sqrt{2}\right)} & -i\sqrt{\frac{48}{17}-\frac{8\sqrt{2}}{17}} \\ \frac{4}{17}\sqrt{50-31\sqrt{2}} & \frac{2}{17}\left(45+\sqrt{2}\right) & -i\sqrt{\frac{48}{17}-\frac{8\sqrt{2}}{17}} & 2i\sqrt{\frac{1}{17}\left(5+2\sqrt{2}\right)} \\ -2i\sqrt{\frac{1}{17}\left(5+2\sqrt{2}\right)} & -i\sqrt{\frac{48}{17}-\frac{8\sqrt{2}}{17}} & 2 & 0 \\ -i\sqrt{\frac{48}{17}-\frac{8\sqrt{2}}{17}} & 2i\sqrt{\frac{1}{17}\left(5+2\sqrt{2}\right)} & 0 & 2-2\sqrt{2} \end{pmatrix}.$$

One advantage that `ModulusTest` has over `UECSMTest` and `StrongAngleTest` is due to the nonlinear nature of the map $X \mapsto X^*X$ on $M_n(\mathbb{C})$. First note that the property of being UECSM is invariant under translation $X \mapsto X + cI$ for $c \in \mathbb{C}$. Next observe that if T does not satisfy the hypotheses of `UECSMTest` or `StrongAngleTest`, then neither does $T + cI$ for any value of c. On the other hand, $T + cI$ will often satisfy the hypotheses of `ModulusTest` even if T itself does not.

T	$\sigma(T^*T)$	$\sigma(T)$	$\sigma(A)$	$\sigma(B)$	UECSM?
$\begin{pmatrix} 1&0&0&0\\ 0&0&2&0\\ 0&0&0&2\\ 0&0&0&0 \end{pmatrix}$	$0,1,4,4$	$0,0,0,1$	$0,1,\pm\sqrt{2}$	$0,0,\pm\sqrt{2}$	YES
$\begin{pmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&0&0&2\\ 0&0&0&0 \end{pmatrix}$	$0,1,1,4$	$0,0,0,1$	$0,1,\pm\frac{\sqrt{5}}{2}$	$0,0,\pm\frac{\sqrt{5}}{2}$	NO

Table 2: *Matrices which cannot be tested by* `UECSMTest`*,* `StrongAngleTest`*, or* `ModulusTest`*. However,* `ModulusTest` does *apply to* $T+I$ *and hence* `ModulusTest` *can be used indirectly to test the original matrix* T.

Table 2 displays two matrices which do not satisfy the hypotheses of `UECSMTest`, `StrongAngleTest`, or `ModulusTest`. Nevertheless, the translation trick described above renders these matrices indirectly susceptible to `ModulusTest`. For instance, the first matrix in Table 2 is unitarily equivalent to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\sqrt{2} \\ 0 & 0 & 0 & \sqrt{2} \\ 0 & i\sqrt{2} & \sqrt{2} & 0 \end{pmatrix}.$$

Rather than grind through the computational details, we can use simple ad-hoc means to independently confirm the results listed in Table 2. The first matrix in Table 2 is the direct sum of a 1×1 matrix and a Toeplitz matrix and is therefore UECSM by [6, Sect. 2.2]. On the other hand, the second matrix in Table 2 is not UECSM. To see this requires a little additional work. First note that the lower right 3×3 block is not UECSM (see Example 3 or [10, Ex. 1]). We next use the fact that a matrix T is UECSM if and only if the external direct sum $0 \oplus T$ is UECSM [11, Lem. 1].

5. Testing compact operators

Our final example indicates that the natural infinite-dimensional generalization of `ModulusTest` can sometimes be used to detect hidden symmetries in Hilbert space operators. For instance if T is compact, then T^*T and TT^* are diagonalizable selfadjoint operators having the same spectrum [13, Pr. 76] and hence the proofs of our results go through *mutatis mutandis*.

EXAMPLE 5. We claim that the *Volterra integration operator* $T : L^2[0,1] \to L^2[0,1]$, defined by

$$[Tf](x) = \int_0^x f(y)\,dy,$$

is unitarily equivalent to a complex symmetric matrix acting on $l^2(\mathbb{Z})$. Before explicitly demonstrating this with `ModulusTest`, let us note that neither of the other procedures previously available (`StrongAngleTest` [1], `UECSMTest` [19], or `VTest` [20, Thm. 3]) are capable of showing this.

1. The Volterra operator has no eigenvalues at all (indeed, it is quasinilpotent) and hence no straightforward generalization of `StrongAngleTest` can possibly apply.

2. Since $[T^*f](x) = \int_x^1 f(y)\,dy$, we find that $A = \frac{1}{2}(T+T^*)$ equals $\frac{1}{2}$ times the orthogonal projection onto the one-dimensional subspace of $L^2[0,1]$ spanned by the constant function 1. In particular, the operator A has the eigenvalue 0 with infinite multiplicity whence no direct generalization of Tener's `UECSMTest` can possibly apply.

3. The Volterra operator does not come equipped with a convenient matrix representation. Even after computing a matrix representation, an application of `VTest` would require solving nonlinear equations (arising from the unitarity condition) in infinitely many complex variables.

On the other hand, the singular values of the Volterra operator are distinct and thus `ModulusTest` applies. In fact, the eigenvalues of T^*T and TT^* are

$$\lambda_n = \frac{2}{(2n+1)\pi},$$

for $n = 0,1,2,\ldots$ and corresponding normalized eigenvectors are

$$u_n = \sqrt{2}\cos[(n+\tfrac{1}{2})\pi x], \quad v_n = \sqrt{2}\sin[(n+\tfrac{1}{2})\pi x].$$

These computations are well-known [13, Pr. 188] and left to the reader (a different derivation of these facts can be found in [8, Ex. 6]). An elementary computation now reveals that

$$\langle u_i, v_j\rangle = \begin{cases} \dfrac{(-1)^{i+j}(2i+1)-(2j+1)}{\pi(i-j+i^2-j^2)} & \text{if } i \neq j, \\ \dfrac{2}{\pi(1+2i)} & \text{if } i = j, \end{cases}$$

from which it is clear that

$$\langle u_i, v_j\rangle = (-1)^{i+j}\langle u_j, v_i\rangle. \tag{11}$$

Taking absolute values of the preceding, we see that (2) is satisfied. Moreover,

$$\begin{aligned} \langle u_i, v_j\rangle\langle u_j, v_k\rangle\langle u_k, v_i\rangle &= (-1)^{2(i+j+k)}\langle u_i, v_k\rangle\langle u_k, v_j\rangle\langle u_j, v_i\rangle \\ &= \langle u_i, v_k\rangle\langle u_k, v_j\rangle\langle u_j, v_i\rangle, \end{aligned}$$

whence (3) is satisfied. By Theorem 1, it follows that the Volterra operator T has a complex symmetric matrix representation with respect to some orthonormal basis of $L^2[0,1]$. Let us exhibit this explicitly.

Looking at (11) we define $\alpha_n = (-1)^n$ and note that (7) is satisfied for all i and j. We now wish to concretely identify the conjugation C on $L^2[0,1]$ which satisfies

$$C(\underbrace{\cos[(n+\tfrac{1}{2})\pi x]}_{u_n}) = \underbrace{(-1)^n}_{\alpha_n}\underbrace{\sin[(n+\tfrac{1}{2})\pi x]}_{v_n}$$

for $n = 0,1,2,\ldots$. Basic trigonometry tells us that

$$\begin{aligned} u_n(1-x) &= \cos[(n+\tfrac{1}{2})\pi(1-x)] \\ &= \cos(n+\tfrac{1}{2})\pi\cos(n+\tfrac{1}{2})\pi x + \sin(n+\tfrac{1}{2})\pi\sin(n+\tfrac{1}{2})\pi x \\ &= (-1)^n\sin[(n+\tfrac{1}{2})\pi x] = \alpha_n v_n(x) \\ &= [Cu_n](x) \end{aligned}$$

whence $[Cf](x) = \overline{f(1-x)}$ for $f \in L^2[0,1]$. In particular, it is readily verified that $T = CT^*C$ (see also [6, Lem. 4.3], [7, Sect. 4.3]).

Now observe that C fixes each element of the orthonormal basis

$$e_n = \exp[2\pi i n(x-\tfrac{1}{2})], \qquad (n \in \mathbb{Z})$$

of $L^2[0,1]$ and that the matrix for T with respect to this basis is

$$\left(\begin{array}{cccc|c|cccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & \frac{i}{6\pi} & 0 & 0 & \frac{i}{6\pi} & 0 & 0 & 0 & \cdots \\
\cdots & 0 & \frac{i}{4\pi} & 0 & -\frac{i}{4\pi} & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & \frac{i}{2\pi} & \frac{i}{2\pi} & 0 & 0 & 0 & \cdots \\
\hline
\cdots & \frac{i}{6\pi} & -\frac{i}{4\pi} & \frac{i}{2\pi} & \frac{1}{2} & -\frac{i}{2\pi} & \frac{i}{4\pi} & -\frac{i}{6\pi} & \cdots \\
\hline
\cdots & 0 & 0 & 0 & -\frac{i}{2\pi} & -\frac{i}{2\pi} & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & \frac{i}{4\pi} & 0 & -\frac{i}{4\pi} & 0 & \cdots \\
\cdots & 0 & 0 & 0 & -\frac{i}{6\pi} & 0 & 0 & -\frac{i}{6\pi} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}\right).$$

The Cartesian components A and B of the Volterra operator are clearly visible in the preceding matrix representation.

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