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INVERSE ELEMENTARY DIVISOR PROBLEMS FOR NONNEGATIVE MATRICES

B. NAGY

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Abstract. The aim of this paper is to answer three questions formulated by H. Minc in his two papers and book on the problem of prescribed elementary divisors for entrywise nonnegative or doubly stochastic matrices. They study the relation of the problem for a diagonalizable and for a general entrywise nonnegative matrix, respectively. One answer is in the positive, two are in the negative directions.

1. Introduction

The problem of prescribed elementary divisors for entrywise nonnegative and, in particular, for row stochastic and doubly stochastic matrices was in a more detailed way first considered by C.R. Johnson [4] and H. Minc [8], [7], and is more difficult than the inverse spectrum problem for the same classes. An overview of early results and problems is contained in Minc's monograph [9], a good recapitulation of later results together with new ones is in the recent paper by Soto and Ccapa [11].

The aim of this paper is to answer three open questions formulated by Minc in [8], [7] and [9]. They concern the relation of the problem for a diagonalizable and for a general entrywise nonnegative matrix, respectively, and will be precisely cited in Sections 2, 3 and 4. Here we present the basic terminology and notation.

A complex matrix is called *(doubly) quasi-stochastic* if all its row sums (and column sums) are 1. Equivalently, an $n \times n$ matrix A is (doubly) quasi-stochastic if and only if 1 is an eigenvalue of A and $(1 \ldots 1)^t$ (^{t} will always denote *transpose*) is a corresponding eigenvector for A (and A^t). A matrix is called *positive (nonnegative)* if all its entries are positive (nonnegative). A nonnegative (doubly) quasi-stochastic matrix is called *(doubly) stochastic* (in the latter case also *row stochastic*).

The inverse elementary divisor problem (in its general formulation) asks for necessary and sufficient conditions for a given matrix to be similar to an entrywise nonnegative (row stochastic, doubly stochastic) matrix. The inverse spectrum problem asks for necessary and sufficient conditions for a given list of n complex numbers to be the list of the eigenvalues (with algebraic multiplicities) of an $n \times n$ nonnegative (row

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stochastic, doubly stochastic) matrix. In the second group of problems remarkable results were obtained by Boyle and Handelman [2], and by many other researchers, but the first group of problems is apparently even more distant from a general solution.

We apply the usual notation $\mathbf{C},\mathbf{R},\mathbf{N}$, and define $\mathbf{N}_0 := \mathbf{N}\cup\{0\}$. Unless stated explicitly otherwise, a matrix in this paper is considered over $\mathbf{R}$. The $[i,j]$ entry of the matrix A will be denoted by $A[i,j]$ or a_{ij}. Vector without qualification will denote column vector, and the *Hadamard product* of $n\times k$ matrices A,B will be denoted by AHB, and defined as usual by $AHB[i,j] := A[i,j]\cdot B[i,j]$ $(1\leqslant i\leqslant n, 1\leqslant j\leqslant k)$. We use in $\mathbf{R}^n$ the usual inner product, and the corresponding norm will be denoted by $|\cdot|$. An (upper triangular) Jordan block with eigenvalue s and order j will be denoted by $J(s,j)$. $M-\lambda$ will stand for $M-\lambda I$, where I is the identity matrix. A matrix having a prescribed list of elementary divisors or eigenvalues will be called *a realization* of the given list. For a background on nonnegative matrices we refer to [1], [3] and [9].

2. Doubly stochastic matrices with prescribed elementary divisors

In [[7], Theorem 2] H. Minc proved that for any positive integers $e_2,e_3,\ldots,e_h$ with sum $N-1$, and any real number α satisfying $-1/(N-1)<\alpha<1$, there exists a doubly stochastic $N\times N$ matrix with elementary divisors $\lambda-1,(\lambda-\alpha)^{e_2},\ldots,(\lambda-\alpha)^{e_h}$. In [[7], Theorem 3] he showed that for each $N\geqslant 2$ there is *exactly one diagonalizable* doubly stochastic $N\times N$ matrix with elementary divisors $\lambda-1$ and $\lambda+1/(N-1)$ $(N-1$ *times*$)$. It is

$$D(N) := \frac{1}{N-1}[NJ_N - I_N],$$

where J_N denotes the $N\times N$ matrix with all entries $=1/N$, and I_N is the $N\times N$ identity matrix. Further, he proved that, surprisingly, *there is no* doubly stochastic 3×3 matrix with elementary divisors $\lambda-1$ and $(\lambda+\frac{1}{2})^2$.

Moreover, he exhibited a doubly stochastic matrix with elementary divisors $\lambda-1,\lambda+\frac{1}{3},(\lambda+\frac{1}{3})^2$ on [[7], p. 123], and asked what the situation is for the dimensions $N\geqslant 4$ and $\alpha := -\frac{1}{N-1}$ (see also [[7], pp. 122–123] and his book [[9], pp. 190–191]). We formulate the answer in the following

THEOREM 1. *For each $N\geqslant 4$, $\alpha := -\frac{1}{N-1}$ and for any positive integers $e_2,e_3,\ldots,e_h$ with sum $N-1$, there exists a doubly stochastic $N\times N$ matrix with elementary divisors $\lambda-1,(\lambda-\alpha)^{e_2},\ldots,(\lambda-\alpha)^{e_h}$. Moreover, if at least one $e_j>1$, then there exists a family (of the cardinality of the continuum) of doubly stochastic matrices, each with the prescribed elementary divisors, the entries of which are real affine functions of a finite number of variables.*

In the proof we shall need the following

LEMMA 1. *1. Let $s\in\mathbf{R}$, and $G := J(s,j_1)\oplus\ldots\oplus J(s,j_g)$ be the direct sum of Jordan blocks of the indicated orders. Let K be any modification of the matrix G by changing the entries $G[r,r+1]$ equaling 1 into $K[r,r+1] := k_r$, where k_r are arbitrary nonzero real numbers. Let M be any modification of the matrix G by changing the entries $G[1,r+1]$ into $M[1,r+1] := m_r$, where m_r are arbitrary nonzero real numbers,* *for some of those values r for which $G[r,r+1]=1$ (all the other entries of G remain unchanged). Then the matrices G,K,M are similar.*

2. Let the $2q\times 2q$ matrix L have nonzero entries outside the main diagonal at most in the submatrix S based on rows $1,\ldots,q$ and columns $q+1,\ldots,2q$. Let S have rank q, and let $L[i,i]=s$ $(i=1,2,\ldots,2q)$. Then L is similar to the direct sum of q copies of $J(s,2)$.

Proof of the Lemma. 1. It is easy to check that any Jordan *block* $J(s,j)$ and its any modification allowed above (for K) have the same elementary divisors. The elementary divisors of a direct sum of matrices are the collection of the elementary divisors of the matrices. Hence G is similar to K.

Assume now that M is a fixed modification allowed above. Apply the notation $x:=s-\lambda$, and add the penultimate row of $M-\lambda$ multiplied by $-(M-\lambda)[1,j_1+\ldots+j_g]$ to the first row of $M-\lambda$. We have changed then $(M-\lambda)[1,j_1+\ldots+j_g]$ into 0, and $(M-\lambda)[1,j_1+\ldots+j_g-1]$ into $(M-\lambda)[1,j_1+\ldots+j_g-1]-(M-\lambda)[1,j_1+\ldots+j_g]x$. Observe that continuing in this way with the last but two row multiplied by the new value of $-(M-\lambda)[1,j_1+\ldots+j_g-1]$, etc. upwards, the definitions of G and M imply that $G-\lambda$ is similar to ($\equiv$ has the same elementary divisors as) its modification having the first row

$$\begin{pmatrix} x\; v_1+v_2x\; v_3x\;\ldots\; v_{j_1+\ldots+j_g-1}x\; 0\end{pmatrix}\quad (v_r\in\mathbf{R})$$

(all the other rows of $G-\lambda$ [which are also rows of $M-\lambda$] remaining unchanged). Adding now appropriate multiples of the first column $\begin{pmatrix}x\;0\;0\;\ldots\;0\end{pmatrix}^t$ of $G-\lambda$, we can change the first row to $\begin{pmatrix}x\; v_1\; 0\;\ldots\;0\end{pmatrix}$, where the real number v_1 is nonzero exactly when $G[1,2]=1$. By the preceding paragraph, G is similar to M.

2. By assumption, $L-s=\begin{pmatrix}0 & S\\ 0 & 0\end{pmatrix}$, where 0 denotes zero matrix of order q. It follows that $rank(L-s)=rank(S)=q$ and $rank(L-s)^2=0$. Hence the order of the largest Jordan block of L is 2. The rank condition shows that there are at least (hence exactly) q blocks of order 2. $\square$

Proof of the Theorem. Let $n:=N-1$, and let

$$s_1:=\frac{1}{\sqrt{N}}\begin{pmatrix}1\ldots 1\end{pmatrix}^t,$$

a normalized eigenvector corresponding to 1 of (any) doubly stochastic matrix of order N. Let $S=\begin{pmatrix}s_1\;\ldots\;s_N\end{pmatrix}$ be its *arbitrary* complementation (by the column vectors $s_2,\ldots,s_N$) to a real orthogonal matrix of order N. Since $\sum_{k=1}^N s_ks_k^t$ is the spectral decomposition of the identity I_N, we have

$$s_1s_1^t-\frac{1}{n}[s_2s_2^t+\ldots+s_Ns_N^t]=\frac{N}{n}s_1s_1^t-\frac{1}{n}I_N=\frac{N}{n}J_N-\frac{1}{n}I_N.$$

By Minc's cited result [[7], Theorem 3, pp. 128–129], this matrix is the unique diagonalizable nonnegative $N\times N$ matrix $D(N)$ with the elementary divisors $\lambda-1$ and $\lambda+1/n$ $\quad$ (n *times*) (independently of the choice of S).

Let the matrix B be the direct sum of the 1×1 matrix $\mathbf{1}$ plus an upper triangular $n\times n$ matrix F with exclusively the numbers $-\frac{1}{n}$ in the main diagonal and with the prescribed (nonunit) elementary divisors. If $A\equiv A(S):=SBS^t$, then the matrix A will have the prescribed elementary divisors. Further,

$$A=SBS^t=\begin{pmatrix}s_1 & \dots & s_N\end{pmatrix}[\mathbf{1}\oplus F]\begin{pmatrix}s_1^t\\ \vdots\\ s_N^t\end{pmatrix}=s_1s_1^t-\frac{1}{n}[s_2s_2^t+\ldots+s_Ns_N^t]+\sum_{2\leqslant j<k\leqslant N}b_{jk}s_js_k^t$$
$$=D(N)+\sum_{2\leqslant j<k\leqslant N}b_{jk}s_js_k^t\equiv D(N)+\sum_{2\leqslant j<k\leqslant N}f_{jk}s_js_k^t.$$

It follows that the matrix A *is nonnegative if and only if* the right-hand side matrix above is a nonnegative matrix. In our case we have $trace(A)=0$, which is (if $A\geqslant 0$) equivalent to $A[r,r]=0\quad(r=1,\ldots,N)$. Since $[D(N)][r,r]=0\quad(r=1,\ldots,N)$, this means

$$\{\sum_{2\leqslant j<k\leqslant N}b_{jk}s_js_k^t\}[r,r]=0\quad(r=1,\ldots,N).$$

This condition is equivalent to

$$\sum_{2\leqslant j<k\leqslant N}f_{jk}s_jHs_k=0,\tag{$*$}$$

where the right-hand side is the $N\times 1$ zero matrix (column vector).

Consider now a fixed prescribed list of elementary divisors as in the statement of the Theorem, and one corresponding (upper triangular) Jordan matrix $M=\{m_{ik}:\ 2\leqslant i,k\leqslant N\}$ of order n of the form

$$M:=J(\alpha,e_2)\oplus J(\alpha,e_3)\oplus\ldots\oplus J(\alpha,e_h).$$

Let $J:=\{j_2,\ldots,j_r\}$ be the finite sequence of *all* subscripts satisfying

$$2\leqslant j_2<j_3<\ldots<j_r\leqslant N-1$$

and $M[j_k,j_k+1]=1$ $(k=2,3,\ldots,r)$. We shall modify some entries of the matrix M to obtain an upper triangular matrix $F=\{f_{ik}:\ 2\leqslant i,k\leqslant N\}$ of order n, with the same elementary divisors and satisfying $(*)$, *according to the cases distinguished below.*

Consider the $N\times N$ matrix

$$V:=\begin{pmatrix}1 & -n & 0 & 0 & 0 & \ldots & 0\\ 1 & 1 & 1-n & 0 & 0 & \ldots & 0\\ 1 & 1 & 1 & 2-n & 0 & \ldots & 0\\ 1 & 1 & 1 & 1 & 3-n & \ldots & 0\\ & & & \ldots & & & \\ 1 & 1 & 1 & 1 & 1 & \ldots & -1\\ 1 & 1 & 1 & 1 & 1 & \ldots & 1\end{pmatrix},$$

denote its column vectors by $v_1, v_2, \ldots, v_N$, and their normalized variants by $w_j := \frac{v_j}{|v_j|}$ $(j = 1, \ldots N)$. The vectors v_k are pairwise orthogonal, and their Hadamard products satisfy

$$v_k H v_j = v_{\max[k,j]} \quad (1 \leqslant k, j \leqslant N).$$

The vectors w_k form an orthonormal basis of $\mathbf{R}^N$, and their Hadamard products satisfy

$$w_k H w_j = \frac{1}{|v_k||v_j|} v_{\max[k,j]} = \frac{1}{|v_{\min[k,j]}|} w_{\max[k,j]} \quad (1 \leqslant k, j \leqslant N).$$

It is convenient to introduce the notation $z_j := \frac{1}{|v_j|}$. Then $0 < z_j < 1$, and we have

$$w_k H w_j = z_j w_k \quad (j < k).$$

With the help of the matrix V above form the vectors w_k, and define an orthogonal matrix S (corresponding to M) with columns s_j, *in the first two of the following 3 possible cases* (depending on M):

Case 1: If there is at least one $e_j = 1$ (i.e., *at least one Jordan block of order 1*), then we may and shall assume that $e_2 = 1$. Then $m_{23} = 0$, and we define

$$s_j := w_j \quad (j = 1, 2, 3, \ldots, N).$$

Case 2: If there is no Jordan block of order 1, but there is *at least one Jordan block of order greater than 2*, then we may and shall assume that precisely $e_2 > 2$, hence $m_{23} = m_{34} = 1$. In Case 2 we define

$$s_1 := w_1, s_2 := w_2, s_3 := w_4, s_4 := w_3, s_j := w_j \quad (j = 5, 6, \ldots, N).$$

(This is simply changing the order of w_3 and w_4, and its usefulness will become clear in what follows.)

Case 3: The single remaining case is when each e_k is equal to 2, i.e., M is the direct sum of blocks of order 2. Hence the order of M is even, say $n = 2q$, and we have $m_{k,k+1} = 0$ for k odd, and $m_{k,k+1} = 1$ for k even.

In Case 3 we shall define the orthogonal matrix S below.

Consider the linear homogeneous system $(*)$ of equations for the entries f_{jk} of the matrix F. We state that in each of the 3 cases above we can find (an S and) a solution matrix F which has exactly the prescribed elementary divisors.

In Case 1 : if $m_{k,k+1} = 1$ for some k, then we may define $f_{k,k+1}$ and $f_{2,k+1}$ to *be arbitrary nonzero real numbers* (note that for $k > 2$ we clearly have $m_{2,k+1} = 0$), and for all other entries we let $f_{ik} := m_{ik}$. The Lemma shows that *the elementary divisors of F so defined coincide with the elementary divisors of M*. Further, $(*)$ is then equivalent to

$$\sum_{j \in J} [f_{j,j+1} z_j + f_{2,j+1} z_2] w_{j+1} = 0.$$

Hence, e.g.,

$$f_{j,j+1} := z_2, \quad f_{2,j+1} := -z_j \quad (j \in J), \quad \text{and all the remaining } f_{jk} := 0 \ (j \neq k)$$

is a solution of the required type.

In Case 2 proceed similarly as in Case 1, but leave $f_{24} := m_{24} = 0$. Making use of the slight modification in the definition of the sequence $\{s_j\}$, $(*)$ is then equivalent to

$$[f_{23}z_2 + f_{34}z_3]w_4 + \sum_{j\in J, j>3} [f_{j,j+1}z_j + f_{2,j+1}z_2]w_{j+1} = 0,$$

and one (of many) possible definition of a matrix F of the wished type is clear.

In Case 3 consider the following matrix U of order $N = 2q+1$:

$$U := \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 1 & 1 & 1 & 1 \dots & 1 \\ 1 & -1 & 0 & 0 & \dots & 0 & 1 & 1 & 1 & 1 \dots & 1 \\ 1 & 0 & 1 & 0 & \dots & 0 & -1 & 1 & 1 & 1 \dots & 1 \\ 1 & 0 & -1 & 0 & \dots & 0 & -1 & 1 & 1 & 1 \dots & 1 \\ 1 & 0 & 0 & 1 & \dots & 0 & 0 & -2 & 1 & 1 \dots & 1 \\ 1 & 0 & 0 & -1 & \dots & 0 & 0 & -2 & 1 & 1 \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -3 & 1 \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -3 & 1 \dots & 1 \\ & & & & & \dots & & & & & \\ 1 & 0 & 0 & 0 & \dots & -1 & 0 & 0 & 0 & 0 \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \dots & -2q \end{pmatrix},$$

and denote its columns by u_k $(k = 1,2,\dots,2q+1)$. They are pairwise orthogonal. Normalize them by letting $s_k := \frac{u_k}{|u_k|}$, and define the orthogonal matrix S by its columns as

$$S := \begin{pmatrix} s_1 & s_2 & \dots & s_N \end{pmatrix}.$$

If $j \in \{2,3,\dots,q+1\}$, $k \in \{q+2,q+3,\dots,2q+1\}$, then

$$|u_j| = \sqrt{2}, \quad |u_k| = \sqrt{2(k-q)(k-q-1)} \ (k < 2q+1), \quad |u_{2q+1}| = \sqrt{(2q)^2 + 2q},$$

and we have

$$s_j H s_k = \frac{u_j H u_k}{|u_j||u_k|} = g_{jk} u_j.$$

It is easily seen that $g_{jk} = \frac{1}{|u_j||u_k|}\gamma_{jk}$, where $\gamma_{jk} = 1$ for $j \leqslant k-q$, $\gamma_{jk} = 0$ for $j \geqslant k-q+2$, and $\gamma_{jk} = -j+2$ for $j = k-q+1$. Hence the matrix $\gamma := (\gamma_{jk} : j \in \{2,3,\dots,q+1\}, k \in \{q+2,q+3,\dots,2q+1\})$ of order q may be represented as

$$\gamma \equiv \gamma(q) = \begin{pmatrix} 1 & 1 & 1 & 1 \dots & 1 & 1 \\ -1 & 1 & 1 & 1 \dots & 1 & 1 \\ 0 & -2 & 1 & 1 \dots & 1 & 1 \\ 0 & 0 & -3 & 1 \dots & 1 & 1 \\ \dots & & & & & \\ 0 & 0 & 0 & 0 \dots & 1-q & 1 \end{pmatrix}.$$

By Lemma 1, if $F = \{f_{jk} : \ j,k \in \{2,\dots,2q+1\}\}$ is an upper triangular real matrix of order $2q$ with main diagonal consisting of $2q$ copies of α and such that $f_{jk} \neq$

$0,\ j<k$ together imply that $j\in\{2,3,\dots,q+1\}$, $k\in\{q+2,q+3,\dots,2q+1\}$, and the corresponding submatrix $F_0:=\{f_{jk}:\ j\in\{2,3,\dots,q+1\},\ k\in\{q+2,q+3,\dots,2q+1\}\}$ of order q has rank q, then the Jordan form of F has exactly q blocks of order 2. If the orthogonal matrix S is defined as above, and F has these properties, then the system $(*)$ has the form

$$\sum_{j\in\{2,3,\dots,q+1\},\,k\in\{q+2,q+3,\dots,2q+1\}} f_{jk}g_{jk}u_j=0.$$

Since the vectors $\{u_j\}$ are pairwise orthogonal, this is equivalent to the system

$$\sum_{q+2\leqslant k\leqslant 2q+1} f_{jk}\frac{1}{|u_j||u_k|}\gamma_{jk}=\sum_{q+2\leqslant k\leqslant 2q+1} f_{jk}g_{jk}=0 \qquad (2\leqslant j\leqslant q+1).$$

The form of the matrix γ shows that this system has an infinity of real solutions $\{f_{jk}\}$ satisfying all the requirements formulated above (in particular, $\det(F_0)\neq 0$), if $q>1$. It is instructive to see how this argument *breaks down for* $q=1$ *(hence* $N=3$*), which is exactly the case of Minc's counterexample* cited in the first paragraph of this Section.

Indeed, if $q=1$, then we should have $f_{23}\frac{1}{|u_2||u_3|}=0$ *and* $f_{23}=\det(F_0)\neq 0$, a contradiction. If $q=2$, then the corresponding system of equations is

$$f_{24}\frac{1}{|u_2||u_4|}+f_{25}\frac{1}{|u_2||u_5|}=0, \qquad -f_{34}\frac{1}{|u_3||u_4|}+f_{35}\frac{1}{|u_3||u_5|}=0.$$

This implies $\det(F_0)=f_{24}f_{35}-f_{34}f_{25}=2f_{24}f_{34}|u_5|/|u_4|\neq 0$, whenever we pick $f_{24}\neq 0, f_{34}\neq 0$.

For any $q\in\mathbf{N}\setminus\{1\}$ we can construct a solution in the following way. Define

$$\phi_{jk}:=f_{jk}\frac{1}{|u_j||u_k|} \quad (j\in\{2,3,\dots,q+1\},\ k\in\{q+2,q+3,\dots,2q+1\}).$$

Then $(*)$ is equivalent to the system

$$\sum_{q+2\leqslant k\leqslant 2q+1}\phi_{jk}\gamma_{jk}=0 \qquad (2\leqslant j\leqslant q+1).$$

It follows that, e.g., the choice

$$\phi_{2,q+2}:=1,\ \phi_{2,q+3}:=-1,\ \phi_{j,j+q-1}:=1,\ \phi_{j,j+q}:=j-2 \quad (j=3,4,\dots,q+1),$$

(and every other $\phi_{jk}:=0$) satisfies the system. Hence the definition

$$f_{2,q+2}:=|u_2||u_{q+2}|,\quad f_{2,q+3}:=-|u_2||u_{q+3}|,$$

$$f_{j,j+q-1}:=|u_j||u_{j+q-1}|,\quad f_{j,j+q}:=(j-2)|u_j||u_{j+q}| \quad (j=3,4,\dots,q+1),$$

(and every other $f_{jk}:=0$) yields a solution matrix F. Indeed, we have

$$\det(F_0)=2(q-1)!|u_2||u_3|\dots|u_{2q}||u_{2q+1}|>0.$$

Assume now that that (*in any one of the 3 possible cases*) we *have determined one pair of matrices* (S,F) *as above, satisfying* $(*)$. Since this system of equations is homogeneous, we have for every $\rho \in \mathbf{R}$

$$\sum_{2 \leqslant j<k \leqslant N} \rho f_{jk} s_j H s_k = 0.$$

Let $F(\rho)$ denote the modification of the matrix F obtained by multiplying all the *nondiagonal* entries of F by the number $\rho > 0$. The elementary divisors of $F(\rho)$ are identical with those of F. In Cases 1 or 2 this is seen by applying Lemma 1, whereas in Case 3 we see that (with evident notation) $\det[F(\rho)_0] = \rho^q \det[F_0] \neq 0$. By choosing ρ sufficiently small, all the nondiagonal entries of $F(\rho)$ will have moduli less than any prescribed $\varepsilon > 0$. By choosing ε sufficiently small, and defining the matrix $B(\rho) := \mathbf{1} \oplus F(\rho)$ of order N, we obtain that each matrix

$$A(\rho) := SB(\rho)S^t = D(N) + \sum_{2 \leqslant j<k \leqslant N} \rho f_{jk} s_j s_k^t$$

is nonnegative, and has the prescribed elementary divisors. Finally, we show that for any real number ρ the matrix $A = A(\rho)$ is doubly quasistochastic, i.e. 1 is an eigenvalue of A and $e_1 := \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}^t$ is a corresponding eigenvector for A and A^t. We know that the matrix $D(N)$ is doubly stochastic, hence

$$Ae_1 = D(N)e_1 + \sum_{2 \leqslant j<k \leqslant N} \rho f_{jk} s_j s_k^t e_1 = e_1 + \sum_{2 \leqslant j<k \leqslant N} \rho f_{jk} s_j s_k^t e_1.$$

For each $k \geqslant 2$ the scalar product $s_k^t e_1$ of orthogonal vectors is 0, thus $Ae_1 = e_1$. Since every occurring number is real, we obtain similarly that $A^t e_1 = e_1$. Hence $A(\rho)$ *is doubly quasistochastic* for each ρ. The forms of $D(N)$ and A yield the last statement of the theorem. $\square$

EXAMPLE. Let $N = 5$, and the prescribed elementary divisors be $\lambda - 1, (\lambda + \frac{1}{4})^2, (\lambda + \frac{1}{4})^2$. We are then in Case 3 from above, and $q = 2$. The matrix U is given by

$$U := \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 & 1 \\ 1 & 0 & 1 & -1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 & -4 \end{pmatrix}.$$

Normalizing the columns u_j by $s_j := u_j/|u_j|$, taking the Hadamard products $s_j H s_k$, and introducing the notation

$$b := f_{24}, \quad c := f_{25}, \quad d := f_{34}, \quad e := f_{35},$$

we see that $(*)$ is in this case equivalent to the equalities

$$c = -\sqrt{5}b, \qquad e = \sqrt{5}d.$$

Note that (with the notation of the proof of the theorem)

$$det(F_0) = be - cd = 2\sqrt{5}bd \neq 0 \qquad \text{iff} \qquad bd \neq 0,$$

which we assume from now on. Forming the orthogonal matrix S as in the proof of the theorem, and applying the notation $\sigma := 4c/\sqrt{10}$, $\tau := 4e/\sqrt{10}$, we see that each matrix of the form

$$A \equiv A(\sigma,\tau) := \frac{1}{4}\begin{pmatrix} 0 & 1 & 1+\sigma & 1+\sigma & 1-2\sigma \\ 1 & 0 & 1-\sigma & 1-\sigma & 1+2\sigma \\ 1+\tau & 1+\tau & 0 & 1 & 1-2\tau \\ 1-\tau & 1-\tau & 1 & 0 & 1+2\tau \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \qquad (\sigma\tau \neq 0)$$

is doubly quasistochastic and has the prescribed elementary divisors. If $0 < |\sigma|, |\tau|$ are sufficiently small, then $A(\sigma,\tau)$ is also nonnegative, hence doubly stochastic. In particular, the choice $\sigma = \tau = 2/5$ yields the doubly stochastic matrix

$$A(2/5, 2/5) = \frac{1}{20}\begin{pmatrix} 0 & 5 & 7 & 7 & 1 \\ 5 & 0 & 3 & 3 & 9 \\ 7 & 7 & 0 & 5 & 1 \\ 3 & 3 & 5 & 0 & 9 \\ 5 & 5 & 5 & 5 & 0 \end{pmatrix}$$

with the prescribed elementary divisors.

3. No nonnegative matrix with the given elementary divisors

In [[8], Theorem 1] H. Minc proved that for every *positive* diagonalizable matrix there is a positive matrix with the *same spectrum* [$\equiv$ spectral list, allowing for algebraic multiplicities] but with *arbitrarily prescribed elementary divisors*, subject to the condition that elementary divisors corresponding to nonreal eigenvalues occur in conjugate pairs.

In his paper [[8], p. 665] and book [[9], p. 188] Minc posed the similar question on the *nonnegative* inverse elementary divisor problem: "whether for every nonnegative diagonalizable matrix there exists a nonnegative matrix with the *same spectral list* but with *arbitrarily prescribed elementary divisors*, subject to the condition that elementary divisors corresponding to nonreal eigenvalues occur in conjugate pairs. In the case of the doubly stochastic matrices, the answer to this problem is in the negative" [in the sense that there may *not exist any doubly stochastic* matrix with the prescribed elementary divisors as he showed by his Example 3.1 in [[9], pp. 188–189]]. We show that the answer to Minc's problem is in the negative in the following precise sense:

THEOREM 2. *There exist a diagonalizable doubly stochastic matrix M and a prescribed list of elementary divisors (subject to the condition above) with the same spectral list as that of M such that there is no nonnegative matrix with the prescribed list of elementary divisors.*

Proof. Consider the following doubly stochastic matrix

$$M := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

It is clearly diagonalizable, and has the spectral list $L = (1, 1, -1, -1)$ (with linear elementary divisors). Any nonnegative matrix with the given spectral list cannot be irreducible, and is cogredient ($\equiv$ permutationally similar) to its *reducible normal form*. The trace condition evidently shows that it is a nonnegative matrix N of the block structure

$$N = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}.$$

Here every block is a 2×2 submatrix, and A and C are irreducible, each with the spectral list $(1, -1)$, hence with the elementary divisors $\lambda - 1$ and $\lambda + 1$. If $B = 0$, then the matrix N has the 4 linear elementary divisors. If $B \neq 0$, we can apply a result of Rothblum [10] (cf. also [[1], Chapter 2, (3.28), p. 45]) implying that the elementary divisors of N then necessarily include $(\lambda - 1)^2$. Whichever the case for B, it follows that the list

$$\lambda - 1, \lambda - 1, (\lambda + 1)^2$$

cannot be the list of the elementary divisors of any matrix N, which proves the theorem. $\square$

REMARK. If we want to avoid invoking Rothblum's result, it can be shown that the (nonnegative) entries of any N are as follows:

$$N \equiv N(b, f, x, y, w, z) = \begin{pmatrix} 0 & b & 0 & 0 \\ b^{-1} & 0 & 0 & 0 \\ x & y & 0 & f \\ w & z & f^{-1} & 0 \end{pmatrix}, \qquad bf > 0.$$

Thus it is possible to determine directly the cases of the distinct elementary divisor lists. It will turn out that each list distinct from that given above can be the list of a suitable nonnegative matrix N.

4. No nonnegative diagonalizable matrix with the given spectrum

Recall that in his book [[9], p. 188] H. Minc posed the following question on a variant of the nonnegative inverse elementary divisor problem: whether for every nonnegative (or even positive) matrix there exists a diagonalizable nonnegative (positive) matrix with the same spectrum. Before we answer this question, it may be interesting to note that an ingenious result of Johnson, Laffey and Loewy [5] and its proof on *nonnegative realizations of nonzero lists* can be slightly modified to yield the following result for the *positive* case.

THEOREM 3. *If A is a positive matrix of order m and of rank r, then there is a positive integer $q \leqslant r^2$ and a positive matrix A_1 of order q having the same nonzero spectrum (i.e. list of nonzero eigenvalues counting algebraic multiplicities) as A (in signs: $A_1 \sim A$), and having rank not greater than r.*

Proof. We can clearly assume $r^2 < m$. Since A has rank r, there are $r \times m$ real matrices B, C with columns as follows:

$$B = \begin{pmatrix} b_1 \dots b_m \end{pmatrix}, \qquad C = \begin{pmatrix} c_1 \dots c_m \end{pmatrix}$$

satisfying $A = B^t C$. We have then $A \sim CB^t = \sum_{i=1}^m c_i b_i^t$, and the dyads $c_i b_i^t$ are in the r^2-dimensional vector space of $r \times r$ real matrices. Applying a classical result of Caratheodory's (see also [5]), there are nonnegative numbers $d_1, \dots, d_q, \dots, d_{r^2}$ such that

$$CB^t = \sum_{j=1}^{r^2} d_j c_{i_j} b_{i_j}^t.$$

We may and shall assume that exactly the first $q \leqslant r^2$ numbers d_j are positive, hence the upper boundary of the summation can and will be q. Let now

$$B_1 := \begin{pmatrix} b_{i_1} \; b_{i_2} \dots b_{i_q} \end{pmatrix}, \quad C_1 := \begin{pmatrix} c_{i_1} \; c_{i_2} \dots c_{i_q} \end{pmatrix}, \quad D := diag(d_1, \dots, d_q),$$

i.e., D be the diagonal matrix of order q with the indicated main diagonal. Then

$$CB^t = C_1 D B_1^t \sim B_1^t C_1 D =: A_1.$$

The matrix A_1 is of order q, and its entry (j,k) is $d_k b_{i_j}^t c_{i_k}$. Since $d_k > 0$, and the product of the last two factors is the entry (i_j, i_k) of the positive matrix A, the matrix A_1 is positive, and we clearly have $A_1 \sim A$. The last assertion follows simply from the size of, say, C_1. $\square$

COROLLARY. *Assume that the list $L \in \mathbf{C}^n$ is the nonzero spectrum of a nonnegative (or positive) matrix. Then there is a nonnegative (or positive, respectively) matrix A with nonzero spectrum L and with α copies of zeros in the spectrum, where the number α satisfies*

$$\alpha \leqslant n(n-1) + d(d+2n).$$

Here d denotes the dimension of the quotient G/K, where G, K denote the subspaces of all generalized and proper eigenvectors of A corresponding to the eigenvalue 0, respectively. Equivalently, $d = \alpha - \gamma$, where γ is the geometric (and α is the algebraic) multiplicity of 0 in the spectrum of A.

Proof. Consider the family of nonnegative (positive) matrices with nonzero spectrum L. It contains a matrix A of the least order m. Let r denote the rank of A. [5] or Theorem 3 above shows that then $m \leqslant r^2$.

Considering the Jordan form of A, we see that $r = n + \alpha - \gamma$. Hence

$$\alpha + n = m \leqslant r^2 = (n + \alpha - \gamma)^2 = (n+d)^2.$$

Reordering we obtain

$$\alpha \leqslant n^2 - n + 2dn + d^2,$$

i.e. the stated inequality. $\square$

The following result (with the help and with the notation of the Corollary above) answers Minc's problem at the beginning of this Section in the negative.

THEOREM 4. *Let $D \in \mathbf{N}_0$. There is a positive matrix P (with diagonalizable Jordan direct summand corresponding to the nonzero eigenvalues) such that there is no nonnegative matrix N with the same nonzero spectrum and having the dimension*

$$d(N) := \dim[G(N)/K(N)] \leqslant D.$$

Proof. Let $0 < \varepsilon < \sqrt{2}$, and consider the list $L(\varepsilon) := \{\sqrt{2}, i, -i, \varepsilon\}$. From results by Johnson [[4], Theorem 4] and by Loewy and London [6] (cf. also [[2], pp. 310–314]), it follows that for every such ε there is a *positive* matrix $P(\varepsilon)$ whose nonzero spectrum is the list $L(\varepsilon)$, and also that the minimal size $s(\varepsilon)$ among such *nonnegative* matrices satisfies the inequality

$$s(\varepsilon) > \frac{2}{\varepsilon^2}.$$

Note that for each $P(\varepsilon)$ (with the notation of the above Corollary) we have $n = 4$. Any nonnegative matrix N with the same nonzero spectrum and satisfying $d(N) \leqslant D$ has at most rank $r = n + D$. By the Corollary, there is such a nonnegative matrix N of size $\leqslant (4+D)^2$. However, if $\varepsilon > 0$ is so small that

$$\frac{2}{\varepsilon^2} > (4+D)^2,$$

there cannot exist such a nonnegative matrix. The case $D = 0$ yields the negative answer to Minc's question above. $\square$

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REFERENCES

[1] A. BERMAN AND R. J. PLEMMONS, *Nonnegative matrices in the mathematical sciences*, Academic Press, New York, 1979. MR0544666 (82b:15013)

[2] M. BOYLE AND D. HANDELMAN, *The spectra of nonnegative matrices via symbolic dynamics*, Ann. of Math., **2**, 133 (1991), 249–316. MR1097240 (92d:58057)

[3] R. A. HORN AND C. R. JOHNSON, *Matrix analysis*, Cambridge Univ. Press, Cambridge, 1986. MR0832183 (87e:15001)

[4] C. R. JOHNSON, *Row stochastic matrices similar to doubly stochastic matrices*, Linear Multilinear Alg., **10** (1981), 113–130. MR0618581 (82g:15016)

[5] C. R. JOHNSON, T. LAFFEY AND R. LOEWY, *The real and the symmetric nonnnegative inverse eigenvalue problems are different*, Proc. Amer. Math. Soc., **124** (1996), 3647–3651. MR1350951 (97b:15009)

[6] R. LOEWY AND D. LONDON, *A note on an inverse problem for nonnegative matrices*, Linear Multilinear Alg., **6** (1978), 83–90. MR0480563 (58 #722)

[7] H. MINC, *Inverse elementary divisor problem for doubly stochastic matrices*, Linear Multilinear Alg., **11** (1982), 121–131. MR0650726 (84h:15024)

[8] H. MINC, *Inverse elementary divisor problem for nonnegative matrices*, Proc. Amer. Math. Soc., **83** (1981), 665–670. MR0630033 (82j:15012)

[9] H. MINC, *Nonnegative matrices*, Wiley-Interscience, New York, 1988. MR0932967 (89i:15001)

[10] U. G. ROTHBLUM, *Algebraic eigenspaces of nonnegative matrices*, Linear Algebra Appl., **12** (1975), 281–292. MR0404298 (53 #8100)

[11] R. L. SOTO AND J. CCAPA, *Nonnegative matrices with prescribed elementary divisors*, Electronic J. Linear Alg., **17** (2008), 287–303. MR2420977 (2009e:15057)

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B. Nagy
Department of Analysis, Institute of Mathematics
University of Technology and Economics
Budapest H-1521
Hungary
e-mail: bnagy@math.bme.hu