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COHERENCE OF THE REAL SYMMETRIC HARDY ALGEBRA

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Abstract. Let $\mathbb{D}$ denote the open unit disk in $\mathbb{C}$ centered at 0. Let $H^\infty_{\mathbb{R}}$ denote the set of all bounded and holomorphic functions defined in $\mathbb{D}$ that also satisfy $f(z) = \overline{f(\overline{z})}$ for all $z \in \mathbb{D}$. It is shown that $H^\infty_{\mathbb{R}}$ is a coherent ring.

1. Introduction

In this paper, we prove that the real symmetric Hardy algebra is a coherent ring. The definitions of these objects and that of coherence of a ring are given below.

We will denote the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ by $\mathbb{D}$, and the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ by $\mathbb{T}$.

DEFINITION 1.1. A holomorphic function $f : \mathbb{D} \to \mathbb{C}$ is said to be *real symmetric* if

$$f(z) = \overline{f(\overline{z})} \quad \text{for all } z \in \mathbb{D}.$$

The *real symmetric Hardy algebra* $H^\infty_{\mathbb{R}}$ is the set of all real symmetric holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ that are bounded, that is, functions $f \in H^\infty$ such that

$$\|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)| < \infty.$$

Then $H^\infty_{\mathbb{R}}$ is a *real* Banach algebra with pointwise operations and this norm.

We denote by $H^2_{\mathbb{R}}$ the space of real symmetric holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_2 := \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right)^{\frac{1}{2}} < \infty.$$

Then $H^2_{\mathbb{R}}$ is a real Hilbert space.

We denote by $H^1_{\mathbb{R}}$ the space of real symmetric holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_1 := \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| d\theta \right) < \infty.$$

Then $H^1_{\mathbb{R}}$ is a real Banach space.

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We now recall the definition of a coherent ring.

DEFINITION 1.2. A commutative ring R is said to be *coherent* if for all $n \in \mathbb{N}$ and all n-tuples $(a_1, \dots, a_n)$ of elements from R, the module of all relations between the a_i's, namely the following R-module

$$\{(b_1, \dots, b_n) \in R^n : a_1 b_1 + \dots + a_n b_n = 0\},$$

is a finitely generated submodule of R^n.

McVoy and Rubel proved that the Hardy algebra H^∞ is coherent [3]. In [6], McVoy and Rubel's result on the coherence of H^∞ of the disk $\mathbb{D}$ was extended to more general domains of finite connectivity in the complex plane.

There has been interest in algebraic properties of the real symmetric Hardy algebra and the real symmetric disk algebra, since these arise naturally as classes of stable transfer functions in control theory; see for example [7]. Whether or not the real symmetric ring $H^\infty_\mathbb{R}$ is coherent is also a relevant question in control theory; see [5]. Our main result is the following:

THEOREM 1.3. *$H^\infty_\mathbb{R}$ is a coherent ring.*

The proof of Theorem 1.3 is a modification of the proof in [6], where the Hardy algebra is replaced by the real symmetric Hardy algebra. In particular, we will show a version of the weak-$*$ Beurling-Lax-Halmos theorem for the real symmetric Hardy algebra, which is given below:

THEOREM 1.4. *Let $\mathbf{M} \in (H^\infty_\mathbb{R})^{m \times n}$. Then there exists a $k \in \mathbb{N}$ and a $\mathbf{W} \in (H^\infty_\mathbb{R})^{n \times k}$ such that $\ker_{H^\infty_\mathbb{R}} \mathbf{M} = \mathbf{W}(H^\infty_\mathbb{R})^k$.*

In the above, by $\ker_{H^\infty_\mathbb{R}} \mathbf{M}$ we mean the set $\{\mathbf{v} \in (H^\infty_\mathbb{R})^n : \mathbf{M}\mathbf{v} = 0\}$.

2. Kernels of multiplication maps are images

We will need the following technical lemma:

LEMMA 2.1. *Suppose that M is a $H^\infty_\mathbb{R}$-submodule of $(H^\infty_\mathbb{R})^n$, equipped with the norm induced from $(H^2_\mathbb{R})^n$. Let $\mathbf{w} : M \to (H^2_\mathbb{R})^\ell$ be a $H^\infty_\mathbb{R}$-module morphism. If $\mathbf{w}$ is continuous, then $\mathbf{w}(M) \subset (H^\infty_\mathbb{R})^\ell$.*

Proof. Let $f \in M$ and $\lambda > 0$ be such that $|\,(\mathbf{w}(f))(z)\,| > \lambda$ on a subset S of $\mathbb{T}$ of positive measure. (If such a set does not exist, then $\mathbf{w}(f) = 0 \in (H^\infty_\mathbb{R})^\ell$, and we are done.) Now we can arrange S to be real symmetric, that is, $S = \overline{S}$. Here $|\cdot|$ denotes the Euclidean norm in $\mathbb{C}^\ell$. For any $k \in \mathbb{N}$, let $\varphi_k \in H^\infty_\mathbb{R}$ be such that

$$|\varphi_k| = \begin{cases} k \text{ almost everywhere on } S, \\ 1 \text{ almost everywhere on } \mathbb{T} \setminus S. \end{cases}$$

For example, we can take φ_k to be the outer function

$$\varphi_k(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\kappa(e^{i\theta})d\theta\right) \quad (z \in \mathbb{D}),$$

where

$$\kappa(e^{i\theta}) = \begin{cases} \log k & \text{for } e^{i\theta} \in S, \\ 0 & \text{for } e^{i\theta} \in \mathbb{T}\setminus S. \end{cases}$$

Since $\kappa(e^{i\theta}) = \kappa(e^{-i\theta})$ for all θ, it follows that φ_k satisfies $f(z) = \overline{f(\overline{z})}$ for all $z \in \mathbb{D}$.

Then

$$\|\varphi_k f\|_2^2 \leqslant m(\mathbb{T})\|f\|_\infty^2 + k^2\|f\|_\infty^2 m(S),$$

and

$$\|\mathbf{w}(\varphi_k f)\|_2^2 = \|\varphi_k \mathbf{w}(f)\|_2^2 \geqslant k^2\lambda^2 m(S).$$

If the induced operator norm of $\mathbf{w}$ is denoted by $\|\mathbf{w}\|$, then we have, from the above, that for all k,

$$k^2\lambda^2 m(S) \leqslant \|\mathbf{w}(\varphi_k f)\|_2^2 \leqslant \|\mathbf{w}\|^2\|\varphi_k f\|_2^2 \leqslant \|\mathbf{w}\|^2\left(m(\mathbb{T})\|f\|_\infty^2 + k^2\|f\|_\infty^2 m(S)\right).$$

Dividing by $k^2 m(S)$, we obtain

$$\lambda^2 \leqslant \|w\|^2 \frac{m(\mathbb{T})\|f\|_\infty^2}{k^2 m(S)} + \|\mathbf{w}\|^2\|f\|_\infty^2.$$

Passing the limit as $k \to \infty$, we have

$$\lambda^2 \leqslant \|\mathbf{w}\|^2\|f\|_\infty^2.$$

So $\mathbf{w}(f) \in (H^\infty_{\mathbb{R}})^\ell$. $\square$

We will also need the following result (see for example [4]):

PROPOSITION 2.2. (Wold decomposition) *Let H be a Hilbert space and suppose that $T: H \to H$ is an isometry. Let $V \subset H$ be a subspace satisfying $TV \subset V$. Let V_0 be the orthogonal complement of TV in V, that is $V_0 = V \ominus TV$. Then V_0 is a wandering subspace of V (that is, $T^n V_0 \perp T^m V_0$ for all distinct $n, m \geqslant 0$), and*

$$V = \left(\bigoplus_{n\geqslant 0} T^n V_0\right) \oplus \left(\bigcap_{n\geqslant 0} T^n V\right).$$

Proof of Theorem 1.4. Let

$$V = \{f \in (H^2_{\mathbb{R}})^n : \mathbf{M}f = 0\}.$$

Then V is a real Hilbert space with the induced norm from $(H^2_{\mathbb{R}})^n$. Consider the map $\mathbf{S}: V \to V$ of multiplication by z on V, that is, $(\mathbf{S}f)(z) = zf(z)$, $z \in \mathbb{D}$. Suppose V_0 is the orthogonal complement of $\mathbf{S}V$ in V. By the Wold decomposition, we have

$$V = V_0 \oplus \mathbf{S}V_0 \oplus \mathbf{S}^2 V_0 \oplus \mathbf{S}^3 V_0 \oplus \ldots.$$

Consider the map $\mathbf{\Pi}: V \to \mathbb{C}^n$ given by

$$\mathbf{\Pi} f = f(0).$$

Then $\ker \mathbf{\Pi} = \mathbf{S}V$. So there is an isomorphism of $V/\mathbf{S}V$ onto $\operatorname{ran} \mathbf{\Pi}$. But $V/\mathbf{S}V$ is isomorphic to V_0, and $\operatorname{ran} \mathbf{\Pi}$, being a subspace of $\mathbb{C}^n$, is a finite dimensional real vector space of dimension at most $2n$. Thus V_0 is a finite dimensional real Hilbert space of dimension $k \leqslant 2n$. Suppose $\mathbf{E}$ is the embedding of V in $H^2_{\mathbb{R}}(V_0)$ defined in the following manner: if $f = (f_1, \dots, f_n) \in V$, we first use the Wold decomposition in order to write

$$f = g_0 + \mathbf{S} g_1 + \mathbf{S}^2 g_2 + \dots,$$

with $g_0, g_1, g_2, \dots \in V_0$ and

$$\|g_0\|_2^2 + \|g_1\|_2^2 + \|g_2\|_2^2 + \dots < \infty.$$

Next, we set

$$(\mathbf{E}(f))(z) = g_0 + z g_1 + z^2 g_2 + \dots.$$

Then $\mathbf{E} : V \to H^2_{\mathbb{R}}(V_0)$ is an invertible isometry.

We note that for f belonging to V (and with the same notation above),

$$\begin{aligned}(\mathbf{E}(\mathbf{S}f))(z) &= (\mathbf{E}(\mathbf{S}g_0 + \mathbf{S}^2 g_1 + \mathbf{S}^3 g_2 + \dots))(z) \\ &= z g_0 + z^2 g_1 + z^3 g_2 + \dots \\ &= z(g_0 + z g_1 + z^2 g_2 + \dots) \\ &= (\mathbf{S}(\mathbf{E}f))(z),\end{aligned}$$

and so $\mathbf{E}(\mathbf{S}f) = \mathbf{S}(\mathbf{E}f)$. By induction, $\mathbf{E}(\mathbf{S}^i f) = \mathbf{S}^i(\mathbf{E}f)$ for all $i \in \mathbb{N}$. So now we have that

$$\mathbf{E}(pf) = p\mathbf{E}(f) \tag{2.1}$$

for all real symmetric polynomials p.

We will show that the restriction of $\mathbf{E}$ to $V \cap (H^\infty_{\mathbb{R}})^n$ is a $H^\infty_{\mathbb{R}}$-module morphism onto $H^\infty_{\mathbb{R}}(V_0)$. Suppose that $\varphi \in H^\infty_{\mathbb{R}}$. Then φ belongs to $H^2_{\mathbb{R}}$, and let $(p_n)_n$ be a sequence of polynomials in $H^2_{\mathbb{R}}$ that converge to φ. Let $f \in V$. Then by the Cauchy-Schwarz inequality it follows that $p_n \mathbf{E}(f) \to \varphi \mathbf{E}(f)$ in $(H^1_{\mathbb{R}})^n$. But since $f \in (H^\infty_{\mathbb{R}})^n$, we have that $p_n f \to \varphi f$ in $(H^2_{\mathbb{R}})^n$. Since $\mathbf{E}$ is continuous, we can conclude that $\lim_n p_n \mathbf{E}(f)$ exists:

$$\mathbf{E}(\varphi f) = \mathbf{E}(\lim_n p_n f) = \lim_n \mathbf{E}(p_n f) = \lim_n p_n \mathbf{E}(f).$$

Suppose that $g \in H^2_{\mathbb{R}}(V_0)$ is such that $p_n \mathbf{E}(f) \to g$ in $H^2_{\mathbb{R}}(V_0)$. Then we also have that $p_n \mathbf{E}(f) \to g$ in $H^1_{\mathbb{R}}(V_0)$. Since the limit is unique, we conclude that

$$g = \varphi \mathbf{E}(f).$$

Thus $\mathbf{E}(\varphi f) = \varphi \mathbf{E}(f)$. So $\mathbf{E} : V \cap (H^\infty_{\mathbb{R}})^n \to H^2_{\mathbb{R}}(V_0)$ is a $H^\infty_{\mathbb{R}}$-module morphism. By Lemma 2.1, it follows that

$$\mathbf{E}(V \cap (H^\infty_{\mathbb{R}})^n) \subset H^\infty_{\mathbb{R}}(V_0).$$

Now we will show that the restriction of $\mathbf{E}^{-1}: H^2_{\mathbb{R}}(V_0) \to (H^2_{\mathbb{R}})^n$ to $H^\infty_{\mathbb{R}}(V_0)$ is a $H^\infty_{\mathbb{R}}$-module morphism. The proof is the same, mutatis mutandis, as the above. First of all, we note that $\mathbf{E}^{-1}$ is continuous since $\mathbf{E}: V \to H^2_{\mathbb{R}}(V_0)$ was an invertible isometry. The property

$$\mathbf{E}^{-1}(pg) = p\mathbf{E}^{-1}(g)$$

for all $g \in H^2_{\mathbb{R}}(V_0)$ and real symmetric polynomials p, follows from (2.1) by writing $g = \mathbf{E}f$, $f \in V$, and applying $\mathbf{E}^{-1}$. The rest is done again by approximating in $H^2_{\mathbb{R}}$ a given $\varphi \in H^\infty_{\mathbb{R}}$ by a sequence $(p_n)_n$ of real symmetric polynomials, and with a similar argument as provided above, we see that for $g \in H^\infty_{\mathbb{R}}(V_0)$,

$$\mathbf{E}^{-1}(\varphi g) = \mathbf{E}^{-1}(\lim_n p_n g) = \lim_n \mathbf{E}^{-1}(p_n g) = \lim_n p_n \mathbf{E}^{-1}(g) = \varphi \mathbf{E}^{-1}(g).$$

So again by Lemma 2.1, it follows that

$$\mathbf{E}^{-1}(H^\infty_{\mathbb{R}}(V_0)) \subset (H^\infty_{\mathbb{R}})^n.$$

But $\mathbf{E}^{-1}(H^2_{\mathbb{R}}(V_0)) \subset V$. Consequently,

$$\mathbf{E}^{-1}(H^\infty_{\mathbb{R}}(V_0)) \subset V \cap (H^\infty_{\mathbb{R}})^n.$$

This shows that $\mathbf{E}: V \cap (H^\infty_{\mathbb{R}})^n \to H^\infty_{\mathbb{R}}(V_0)$ is a $H^\infty_{\mathbb{R}}$-module isomorphism. But since $\dim V_0 < \infty$, this finishes the proof of the claim in the statement of this theorem. □

3. Coherence of the real symmetric Hardy algebra

Proof of Theorem 1.3. Suppose that $n \in \mathbb{N}$ and that $(f_1, \dots, f_n) \in (H^\infty_{\mathbb{R}})^n$. Then with

$$\mathbf{M} := \begin{bmatrix} f_1 & \dots & f_n \end{bmatrix} \in (H^\infty_{\mathbb{R}})^{1\times n},$$

we have that the module of relations between the f_i's is precisely $\ker_{H^\infty_{\mathbb{R}}} \mathbf{M}$. By Theorem 1.4, there exists a $k \in \mathbb{N}$ and a $\mathbf{W} \in (H^\infty_{\mathbb{R}})^{n\times k}$ such that $\ker_{H^\infty_{\mathbb{R}}} \mathbf{M} = \mathbf{W}(H^\infty_{\mathbb{R}})^k$. Clearly the columns of $\mathbf{W}$ generate the module of relations between the f_i's. Hence the ring $H^\infty_{\mathbb{R}}$ is coherent. □

REMARK 3.1. In fact from the proof of Theorem 1.4, we see that the module of relations on $(f_1, \dots, f_n) \in (H^\infty_{\mathbb{R}})^n$ has a set of generators (as a $H^\infty_{\mathbb{R}}$-module) of cardinality $\leqslant 2n$.

A characterization of coherent rings is the following; see [1]:

PROPOSITION 3.2. *A commutative ring R is coherent if and only if the intersection of any two finitely generated ideals in R is finitely generated and the annihilator of any element is finitely generated.*

Since we have shown that $H^\infty_{\mathbb{R}}$ is coherent, it follows that the intersection of any two finitely generated ideals in $H^\infty_{\mathbb{R}}$ is finitely generated. In fact, one can obtain a bound on the number of generators of the intersection. To do this, we recall the following result from [3, Lemma 1.17] (the proof of which uses the diagram chasing method from [2]).

PROPOSITION 3.3. *If R is a commutative ring and if the module of relations on each n-tuple of elements of R is generated by $N(n)$ elements of R for each n, then whenever I and J are ideals of R generated by m and n elements of R, respectively, it follows that $I \cap J$ has a set of generators of cardinality $\leqslant N(n+m)$.*

In light of Remark 3.1, we obtain the following consequence.

COROLLARY 3.4. *If I and J are two ideals in $H^\infty_{\mathbb{R}}$ with m and n generators, respectively, then $I \cap J$ has a set of generators of cardinality $\leqslant 2(m+n)$.*

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