Operators
and
Matrices
Volume 5, Number 2 (2011), 309–326

# ALGEBRAS OF TRUNCATED TOEPLITZ OPERATORS

N. A. SEDLOCK

(*Communicated by I. M. Spitkovsky*)

*Abstract.* We find necessary and sufficient conditions for the product of two truncated Toeplitz operators on a model space to itself be a truncated Toeplitz operator, and as a result find a characterization for the maximal algebras of bounded truncated Toeplitz operators.

## 1. Introduction

Let $\mathbb{C}$ denote the complex plane, $\mathbb{C}^*$ the Riemann sphere, $\mathbb{D}$ denote the unit disc, and let $\mathbb{T}$ denote the unit circle. $H^2$ is the usual Hardy space, the subspace of $L^2(\mathbb{T})$ of normalized Lebesgue measure $m$ on $\mathbb{T}$ whose harmonic extensions to $\mathbb{D}$ are holomorphic (or, whose negative indexed Fourier coefficients are all zero). $H^2$ will interchangably refer to both the boundary functions and the functions on $\mathbb{D}$. Let $P$ denote the projection from $L^2(\mathbb{T})$ to $H^2$, which is given explicitly by the Cauchy integral:

$$(Pf)(\lambda) = \int_{\mathbb{T}} \frac{f(\zeta)}{1-\lambda\overline{\zeta}}\, dm(\zeta), \lambda \in \mathbb{D}.$$

The reproducing kernel at $\lambda \in \mathbb{D}$ for the Hardy space is the the Szego kernel $K_\lambda := (1-\overline{\lambda}z)^{-1}$. $S$ denotes the shift operator $f \mapsto zf$ on $H^2$. Its adjoint (the backward shift) is the operator

$$S^* f = \frac{f - f(0)}{z}.$$

A Toeplitz operator is the compression of a multiplication operator on $L^2(\mathbb{T})$ to $H^2$. In other words, given $\Phi \in L^2(\mathbb{T})$ (called the symbol of the operator), $T_\Phi = PM_\Phi$ is the operator that sends $f$ to $P(\Phi f)$ for all $f \in H^2$. This operator is bounded if and only if $\Phi \in L^\infty(\mathbb{T})$, and the mapping $\Phi \to T_\Phi$ from $L^\infty$ to the space of bounded operators on $H^2$ is linear and one-to-one. In the case that $\Phi \in H^\infty$, the Toeplitz operator $T_\Phi$ is just the multiplication operator $M_\Phi$. In [2], Brown and Halmos describe the algebraic properties of Toeplitz operators. Among other things, they found necessary and sufficient conditions for the product of two Toeplitz operators to itself be a Toeplitz operator, namely that either the first operator's symbol is antiholomorphic or the second

*Mathematics subject classification* (2010): Primary 47L80; Secondary 47A05, 47B35.
*Keywords and phrases*: Toeplitz operator, truncated Toeplitz operator, model space, reproducing kernel, complex symmetric operator, operator algebra, maximal algebra, conjugation, bounded operator interpolation.


Paper OaM-05-22

operator's symbol is holomorphic. In either case, the symbol of the product is the product of the symbols (i.e. $T_\Phi T_\Psi = T_{\Phi\Psi}$).

More recently, Sarason [11] found analogues to several of Brown and Halmos's results for truncated Toeplitz operators on the model spaces $H^2 \ominus uH^2$, where $u$ is some non-constant inner function. The model spaces are the backward-shift invariant subspaces of $H^2$ (that they are backward shift invariant follows easily from the fact that $uH^2$ is clearly shift invariant). Let $K_u^2$ denote the space $H^2 \ominus uH^2$ from here forward. Let $P_u = P - M_u P M_{\overline{u}}$ denote the projection from $L^2$ to $K_u^2$.

Given $\Phi \in L^2(\mathbb{T})$ we then define the truncated Toeplitz operator (TTO) $A_\Phi$ to be the operator that sends $f$ to $P_u(\Phi f)$ for all $f \in K_u^2$. $A_\Phi$ is well-defined on the set of bounded functions in $K_u^2$, which is dense in $K_u^2$ and which we denote $K_u^\infty$. We let $\mathscr{T}_u$ denote the set of truncated Toeplitz operators which extend to be bounded on all of $K_u^2$.

Truncated Toeplitz operators have many of the same properties as ordinary Toeplitz operators (for example, $A_\Phi^* = A_{\overline{\Phi}}$) but there are also striking differences. For example, there are bounded truncated Toeplitz operators with unbounded symbols [1] (though any truncated Toeplitz operator with a bounded symbol is itself bounded). Additionally, symbols are not unique: the same operator can be generated from more than one symbol, and we say that $\Psi$ is a symbol for $A_\Phi$ if $A_\Phi = A_\Psi$. Given two functions $\Psi$ and $\Phi$, we write $\Psi \stackrel{A}{\equiv} \Phi$ to mean that $A_\Psi = A_\Phi$.

The truncated Toeplitz operators in $\mathscr{T}_u$ do not form an algebra. There are, however, weakly closed algebras contained in $\mathscr{T}_u$. The goal of this paper is to describe the maximal algebras contained in $\mathscr{T}_u$, where by maximal we mean that any weakly closed algebra in $\mathscr{T}_u$ is contained within one of these maximal algebras.

In what follows, for functions $f, g$ in $L^2(\mathbb{T})$, $\langle f, g\rangle = \int_{\mathbb{T}} f\overline{g}\, dm$, $\|f\| = \sqrt{\langle f, f\rangle}$ and $f \otimes g$ is the rank one operator that maps $h$ to $f\langle h, g\rangle$. Further, if $A$ is an operator on a Hilbert space, $[A]'$ denotes the commutant of $A$.

## 2. Background

In this section we lay out basic facts about operators in $\mathscr{T}_u$ and model spaces. Let $u$ be a non-trivial inner function. $K_u^2$ is then a reproducing kernel Hilbert space with reproducing kernels $K_\lambda^u := P_u K_\lambda = \frac{1-\overline{u(\lambda)}u}{1-\overline{\lambda}z}$ for $\lambda \in \mathbb{D}$. Note that $K_\lambda^u$ is bounded for all $\lambda$, and hence in $K_u^\infty$.

The function $u$ is said to have an angular derivative in the sense of Caratheodory (ADC) at the point $\zeta \in \mathbb{T}$ if $u$ has a nontangential limit $u(\zeta)$ of unit modulus at $\zeta$ and $u'$ has a nontangential limit $u'(\zeta)$ at $\zeta$. It is known that $u$ has an ADC at $\zeta$ if and only if every function in $K_u^2$ has a nontangential limit at $\zeta$ [10]. Thus there exists a reproducing kernel function $K_\zeta^u$ such that $\left\langle f, K_\zeta^u \right\rangle = f(\zeta)$. Specifically, $K_\zeta^u$ is the limit of $K_\lambda^u$ as $\lambda$ approaches $\zeta$ nontangentially in the disc and so $K_\zeta^u = \frac{1-\overline{u(\zeta)}u}{1-\overline{\zeta}z}$. If $u$ is a finite Blaschke product, both $u$ and $u'$ are holomorphic in a domain which compactly contains $\mathbb{D}$ and so these boundary reproducing kernels are defined for every unimodular $\zeta$.

Truncated Toeplitz operators have a symmetry property called $C$-symmetry. This concept is due to Garcia and Putinar [6, 7, 8]. Given a $\mathbb{C}$-Hilbert space $\mathscr{H}$ and an antilinear isometric involution $C$ on $\mathscr{H}$, we say that a bounded operator $T$ is a $C$-symmetric operator (CSO) if $T^* = CTC$. Here by isometric we mean that $\langle Cf, Cg \rangle = \langle g, f \rangle$.

In $L^2(\mathbb{T})$, the operator $Cf = u\overline{zf}$ is a conjugation which bijectively maps $uH^2$ to $\overline{zH^2}$ and $K_u^2$ to itself. By restricting ourselves to $K_u^2$, $C$ can be thought of as a conjugation on $K_u^2$. From here on, $C$ always refers to this operator. We will sometimes write $\widetilde{f}$ for $Cf$ for sake of readability. The conjugate reproducing kernel is $\widetilde{K_\lambda^u}(z) = \frac{u(z)-u(\lambda)}{z-\lambda}$ for $z \neq \lambda$ and $\widetilde{K_\lambda^u}(\lambda) = u'(\lambda)$ and has the property that for $f \in K_u^2$, $\widetilde{f}(\lambda) = \left\langle \widetilde{K_\lambda^u}, f \right\rangle$.

Consider the operator $CA_\Phi C$, where $\Phi \in L^2(\mathbb{T})$ and $A_\Phi \in \mathscr{T}_u$. If $f, g \in K_u^2$, then

$$\begin{aligned} \langle CA_\Phi Cf, g \rangle &= \langle Cg, A_\Phi Cf \rangle \\ &= \left\langle u\overline{zg}, \Phi u\overline{zf} \right\rangle \\ &= \left\langle \overline{\Phi} f, g \right\rangle \\ &= \left\langle (A_\Phi)^* f, g \right\rangle \end{aligned}$$

and so we see that operators in $\mathscr{T}_u$ are $C$-symmetric.

Two CSOs commute if and only if their product is $C$-symmetric.

PROPOSITION 2.1. *Let $A_1$ and $A_2$ be $C$-symmetric. Then $A_1A_2$ is $C$-symmetric if and only if $A_1$ and $A_2$ commute.*

*Proof.* Say $A_1A_2$ is $C$-symmetric. Then

$$A_1A_2 = CA_2^*A_1^*C = CA_2^*CCA_1^*C = A_2A_1.$$

On the other hand, if $A_1$ and $A_2$ commute, then so do their adjoints, and so

$$CA_1A_2C = A_1^*A_2^* = A_2^*A_1^*. \quad \square$$

The operator $S_u = P_uS = A_z$ is critical to what follows. Since $K_u^2$ is invariant under $S^*$ we see that $S_u^* = S^*$. Let $f \in K_u^2$ such that $f(0) = 0$, i.e. $f \perp K_0^u$. Then $S^*f = f/z$. On the other hand, $S^*K_0^u = (1 - \overline{u(0)}u - 1 + |u(0)|^2)/z = -\overline{u(0)}\widetilde{K_0^u}$. $S_u$ is $C$-symmetric, and so $S_u$ is characterized by the following equations: $S_uf = zf$ for $f \perp \widetilde{K_0^u}$, and $S_u\widetilde{K_0^u} = -u(0)K_0^u$.

The symbols of TTOs are a more complex issue than the symbols of Toeplitz operators. Sarason proved the following results in [11] as Theorem 3.1 and Theorem 4.1 respecitively.

PROPOSITION 2.2. *If $\Phi \in L^2(\mathbb{T})$ then $A_\Phi = 0$ if and only if $\Phi \in uH^2 + \overline{uH^2}$.*

PROPOSITION 2.3. *$A$ is in $\mathscr{T}_u$ iff $A - S_u A S_u^* = \Phi \otimes K_0^u + K_0^u \otimes \Psi$ for some $\Phi, \Psi \in K_u^2$, in which case $A = A_{\Phi + \overline{\Psi}}$.*

Thus we have a way of finding a symbol for a TTO, but TTOs do not have unique symbols.

The following is a necessary and sufficient condition for a TTO with symbol in $K_u^2 + \overline{K_u^2}$ to equal zero.

PROPOSITION 2.4. *Let $\varphi_1, \varphi_2 \in K_u^2$. Then $A_{\varphi_1 + \overline{\varphi_2}} = 0$ if and only if $\varphi_1 = cK_0^u$ and $\varphi_2 = -\overline{c}K_0^u$ for some $c \in \mathbb{C}$.*

*Proof.* Let $\varphi_1 = cK_0^u$ and $\varphi_2 = -\overline{c}K_0^u$. Then

$$A_{\varphi_1 + \overline{\varphi_2}} = A_{cK_0^u - c\overline{K_0^u}} = A_{cu(z)\overline{u(0)} - c\overline{u(z)}u(0)}$$

so $A_{\varphi_1 + \overline{\varphi_2}} = 0$.

Now suppose $A_{\varphi_1 + \overline{\varphi_2}} = 0$. Then $A - S_u A S_u^* = 0 = \varphi_1 \otimes K_0^u + K_0^u \otimes \varphi_2$, so $\varphi_1 = cK_0^u$ for some $c \in \mathbb{C}$. Hence $cK_0^u \otimes K_0^u + K_0^u \otimes \varphi_2 = 0$ and so $\varphi_2 = -\overline{c}K_0^u$ as required. $\square$

Since $I = A_{K_0^u}$ we can compute the identities

$$I - S_u S_u^* = K_0^u \otimes K_0^u \tag{2.1}$$

and

$$I - S_u^* S_u = \widetilde{K_0^u} \otimes \widetilde{K_0^u} \tag{2.2}$$

from which it follows that

$$S_u \widetilde{S_u \widetilde{\varphi}} = S_u S_u^* \varphi = \varphi - \varphi(0) K_0^u \tag{2.3}$$

for all $\varphi \in K_u^2$.

The following identities are Lemma 2.2 of [11].

PROPOSITION 2.5.

(1) *If $\lambda \in \mathbb{D}$,*

$$S_u^* K_\lambda^u = \overline{\lambda} K_\lambda^u - \overline{u(\lambda)} \widetilde{K_0^u}$$

*and*

$$S_u \widetilde{K_\lambda^u} = \lambda \widetilde{K_\lambda^u} - u(\lambda) K_0^u.$$

(2) *If $\lambda \in \mathbb{D}$ is nonzero,*

$$S_u K_\lambda^u = \frac{1}{\overline{\lambda}} \left( K_\lambda^u - K_0^u \right)$$

*and*

$$S_u^* \widetilde{K_\lambda^u} = \frac{1}{\lambda} \left( \widetilde{K_\lambda^u} - \widetilde{K_0^u} \right).$$

(3) *These equalities all hold for $\lambda \in \mathbb{T}$ such that $u$ has an ADC at $\lambda$.*

## 3. Generalized Shifts

We now define the generalized compressed shift operator. Our definition follows Sarason's definition in Section 14 of [11].

DEFINITION 3.1. Let $\alpha \in \overline{\mathbb{D}}$. Then $S_u^\alpha = S_u + \frac{\alpha}{1-\alpha\overline{u(0)}} K_0^u \otimes \widetilde{K_0^u}$.

Again, we can think about the generalized shift as follows. If $f \in K_u^2$ and $f \perp \widetilde{K_0^u}$, then $S_u^\alpha f = zf$. On the other hand,

$$\begin{aligned}
S_u^\alpha \widetilde{K_0^u} &= S_u \widetilde{K_0^u} + \frac{\alpha \left\langle \widetilde{K_0^u}, \widetilde{K_0^u} \right\rangle}{1-\alpha\overline{u(0)}} K_0^u \\
&= -u(0) K_0^u + \frac{\alpha(1-|u(0)|^2)}{1-\alpha\overline{u(0)}} K_0^u \\
&= \frac{\alpha - u(0)}{1-\alpha\overline{u(0)}} K_0^u.
\end{aligned}$$

The corollary to Theorem 10.1 in [11] states that if a bounded operator $A$ on $K_u^2$ is in $[S_u^\alpha]'$ then $A$ is in $\mathscr{T}_u$. The following proof gives us the symbol of any operator in $[S_u^\alpha]'$.

PROPOSITION 3.2. *Let $\alpha \in \overline{\mathbb{D}}$. If $A$ is a bounded operator that commutes with $S_u^\alpha$ then $A$ is in $\mathscr{T}_u$ and has a symbol $\varphi + \alpha \overline{S_u \widetilde{\varphi}}$ where $\varphi = A K_0^u (1-\alpha\overline{u(0)})^{-1}$.*

*Proof.* First note that

$$AS_u^\alpha = AS_u + \frac{\alpha}{1-\alpha\overline{u(0)}} (AK_0^u) \otimes \widetilde{K_0^u} \tag{3.1}$$

and

$$\begin{aligned}
S_u^\alpha A &= S_u A + \frac{\alpha}{1-\alpha\overline{u(0)}} K_0^u \otimes \left( A^* \widetilde{K_0^u} \right) \\
&= S_u A + \frac{\alpha}{1-\alpha\overline{u(0)}} K_0^u \otimes \left( \widetilde{AK_0^u} \right).
\end{aligned} \tag{3.2}$$

If $A$ and $S_u^\alpha$ commute then we can use Equations (3.1) and (3.2) to see that

$$\begin{aligned}
S_u A &= S_u^\alpha A - \frac{\alpha}{1-\alpha\overline{u(0)}} K_0^u \otimes \left( \widetilde{AK_0^u} \right) \\
&= AS_u^\alpha - \frac{\alpha}{1-\alpha\overline{u(0)}} K_0^u \otimes \left( \widetilde{AK_0^u} \right) \\
&= AS_u + \frac{\alpha}{1-\alpha\overline{u(0)}} (AK_0^u) \otimes \widetilde{K_0^u} - \frac{\alpha}{1-\alpha\overline{u(0)}} K_0^u \otimes \left( \widetilde{AK_0^u} \right).
\end{aligned}$$

It follows that

$$\begin{aligned} A - S_u A S_u^* &= A - A S_u S_u^* - \frac{\alpha}{1-\alpha\overline{u(0)}} A K_0^u \otimes S_u \widetilde{K_0^u} + \frac{\alpha}{1-\alpha\overline{u(0)}} K_0^u \otimes S_u \widetilde{AK_0^u} \\ &= AK_0^u \otimes K_0^u + \frac{\overline{u(0)}\alpha}{1-\alpha\overline{u(0)}} AK_0^u \otimes K_0^u + \frac{\alpha}{1-\alpha\overline{u(0)}} K_0^u \otimes S_u \widetilde{AK_0^u} \\ &= \frac{AK_0^u}{1-\alpha\overline{u(0)}} \otimes K_0^u + K_0^u \otimes \overline{\alpha} S_u C\left(\frac{AK_0^u}{1-\alpha\overline{u(0)}}\right). \end{aligned}$$

The conclusion then follows from Proposition 2.3. □

COROLLARY 1. *Let $A$ be a bounded opeator that commutes with $S_u^{\alpha *}$, for $\alpha \in \overline{\mathbb{D}}$. Then $A$ is in $\mathscr{T}_u$ and has a symbol of the form $\overline{\alpha}\psi + \overline{S_u\widetilde{\psi}} + c$ for $\psi \in K_u^2$ and $c \in \mathbb{C}$.*

*Proof.* $A^*$ commutes with $S_u^\alpha$ and therefore has symbol $\varphi + \alpha\overline{S_u\widetilde{\varphi}}$ where $\varphi = A^* K_0^u(1-\alpha\overline{u(0)})$ by the previous proposition. Therefore $A$ has symbol $\overline{\alpha} S_u\widetilde{\varphi} + \overline{\varphi}$. Define $\psi = S_u\widetilde{\varphi}$. Then by Equation 2.3 $S_u\widetilde{\psi} = S_u\widetilde{S_u\widetilde{\varphi}} = \varphi - \varphi(0)K_0^u$ and $\overline{\alpha}\psi + \overline{S_u\widetilde{\psi}} + \overline{\varphi(0)}$ is a symbol for $A$. □

Suppose $A_\Phi$ and $A_\Psi$ are in $\mathscr{T}_u$ and both commute with $S_u^\alpha$ for some $\alpha \in \overline{\mathbb{D}}$. Then their product $A_\Phi A_\Psi$ also commutes with $S_u^\alpha$, and is therefore also in $\mathscr{T}_u$. So we know of two cases when the product of two operators in $\mathscr{T}_u$ is itself in $\mathscr{T}_u$ — when both operators commute with some $S_u^\alpha$ or $S_u^{\alpha *}$, or when one of the operators is $A_c = cI$ for some $c \in \mathbb{C}$. We will show in Section 5 that these are the only cases where the product of two operators in $\mathscr{T}_u$ is itself in $\mathscr{T}_u$.

## 4. TTOs of type $\alpha$

If $A_\Phi$ is in $\mathscr{T}_u$ and commutes with $S_u^\alpha$, then $A_{\Phi+c}$ also commutes with $S_u^\alpha$ for all $c \in \mathbb{C}$. If $\alpha \in \overline{\mathbb{D}} \setminus \{0\}$, then $\overline{\alpha}^{-1} \in \mathbb{C} \setminus \mathbb{D}$, and by the corollary to Proposition 3.2 any operator in $\mathscr{T}_u$ which commutes with $S_u^{\alpha *}$ has a symbol of the form $\psi + \overline{\alpha}^{-1}\overline{S_u\widetilde{\psi}} + c$ with $\psi \in K_u^2$ and $c \in \mathbb{C}$. We therefore make the following definition.

DEFINITION 4.1. An operator $A \in \mathscr{T}_u$ is said to be a TTO of type $\alpha$ for $\alpha \in \mathbb{C}$ if $A$ has a symbol of the form $\varphi + \alpha\overline{S_u\widetilde{\varphi}} + c$, where $\varphi \in K_u^2$ and $c \in \mathbb{C}$. Note that an operator in $\mathscr{T}_u$ is of type $0$ if and only if it has a holomorphic symbol. We say an operator in $\mathscr{T}_u$ is of type $\infty$ if it has an antiholomorphic symbol.

PROPOSITION 4.2. *Let $A := A_{\varphi_1 + \overline{\varphi_2}}$ be in $\mathscr{T}_u$, where $\varphi_i \in K_u^2$.*

(1) *If $\alpha \in \mathbb{C}$, then $A$ is of type $\alpha$ if and only if $\overline{\alpha} S_u\widetilde{\varphi_1} - \varphi_2 \in \mathbb{C}K_0^u$.*

(2) *$A$ is of type $\infty$ if and only if $\varphi_1 \in \mathbb{C}K_0^u$ if and only if $S_u\widetilde{\varphi_1} \in \mathbb{C}K_0^u$.*

*Proof.*

(1) Let $A_{\varphi_1+\overline{\varphi_2}}$ be of type $\alpha$. Then by Proposition 3.2 and its corollary there is some $\varphi \in K_u^2$ and $c \in \mathbb{C}$ such that $A_{\varphi_1+\overline{\varphi_2}} = A_{\varphi+cK_0^u+\alpha\overline{S_u\widetilde{\varphi}}}$, or, equivalently

$$A_{\varphi_1-\varphi-cK_0^u+\overline{\varphi_2}-\alpha\overline{S_u\widetilde{\varphi}}} = 0$$

By Proposition 2.4 we have that $\varphi_1-\varphi \in \mathbb{C}K_0^u$ and that $\varphi_2-\overline{\alpha}S_u\widetilde{\varphi} \in \mathbb{C}K_0^u$. So then by Proposition 2.5 we have that $S_u\widetilde{\varphi_1}-S_u\widetilde{\varphi} \in \mathbb{C}K_0^u$ and so $\overline{\alpha}S_u\widetilde{\varphi_1}-\varphi_2 = \overline{\alpha}S_u\widetilde{\varphi_1}-\overline{\alpha}S_u\widetilde{\varphi}-\varphi_2+\overline{\alpha}S_u\widetilde{\varphi} \in \mathbb{C}K_0^u$.

Now suppose that $\overline{\alpha}S_u\widetilde{\varphi_1}-\varphi_2 \in \mathbb{C}K_0^u$. Then $\varphi_2 = \overline{\alpha}S_u\widetilde{\varphi_1}+cK_0^u$ for some $c \in \mathbb{C}$ and thus $A_{\varphi_1+\overline{\varphi_2}} = A_{\varphi_1+\alpha\overline{S_u\widetilde{\varphi_1}}+\overline{cK_0^u}}$ is of type $\alpha$.

(2) $A$ is of type $\infty$ if and only if $\varphi_1+\overline{\varphi_2} \stackrel{A}{\equiv} \overline{\psi}$ for some $\psi \in K_u^2$, which is true if and only if $\varphi_1 = P_u(\overline{\psi-\varphi_2}) \stackrel{A}{\equiv} \overline{\psi(0)-\varphi_2(0)}$ which is true if and only if $\varphi_1 \in \mathbb{C}K_0^u$. If $\varphi_1 = cK_0^u$ then $S_u\widetilde{\varphi_1} = -\overline{c}u(0)K_0^u$ by Proposition 2.5. On the other hand, if $S_u\widetilde{\varphi_1} = cK_0^u$ then

$$\begin{aligned}
\varphi_1 &= (S_uS_u^*-K_0^u\otimes K_0^u)\varphi_1 \\
&= S_u\widetilde{S_u\widetilde{\varphi_1}}-\varphi_1(0)K_0^u \\
&= S_u\widetilde{cK_0^u}-\varphi_1(0)K_0^u \\
&= -\overline{c}u(0)K_0^u-\varphi_1(0)K_0^u \\
&\in \mathbb{C}K_0^u \quad \square
\end{aligned}$$

PROPOSITION 4.3. *Any TTO of type $\alpha \in \mathbb{C}$ has a symbol of the form $\varphi_0+\alpha\overline{S_u\widetilde{\varphi_0}}+cK_0^u$ where $\varphi_0(0)=0$ and $c \in \mathbb{C}$, and any TTO of antiholomorphic type has a symbol of the form $\overline{\varphi_0}+cK_0^u$ where $\varphi_0(0)=0$.*

*Proof.* To prove the first statement, let $A$ be of type $\alpha \in \mathbb{C}$ and let $\varphi+\alpha\overline{S_u\widetilde{\varphi}}+cK_0^u$ be a symbol of $A$, where $\varphi \in K_u^2$ and $c \in \mathbb{C}$. Define $\varphi_0 = \varphi - \frac{\langle\varphi,K_0^u\rangle}{\langle K_0^u,K_0^u\rangle}K_0^u$. Then $\varphi_0 \perp K_0^u$, or in other words, $\varphi_0(0)=0$. Then since by Proposition 2.5 $S_u\widetilde{K_0^u} = -u(0)K_0^u$ we have that

$$\begin{aligned}
\varphi+\alpha\overline{S_u\widetilde{\varphi}}+cK_0^u &\stackrel{A}{\equiv} \varphi_0+\frac{\langle\varphi,K_0^u\rangle}{\langle K_0^u,K_0^u\rangle}K_0^u+\alpha\overline{S_u\widetilde{\varphi_0}}+\alpha\frac{\overline{\langle\varphi,K_0^u\rangle}}{\langle K_0^u,K_0^u\rangle}\overline{K_0^u}+cK_0^u \\
&\stackrel{A}{\equiv} \varphi_0+\alpha\overline{S_u\widetilde{\varphi_0}}+c_1K_0^u
\end{aligned}$$

where $c_1 \in \mathbb{C}$.

To prove the second statement, consider $A = A_{\overline{\varphi}}$ and let $\varphi_0 = \varphi - \frac{\langle\varphi,K_0^u\rangle}{\langle K_0^u,K_0^u\rangle}K_0^u$. Then $\overline{\varphi} \stackrel{A}{\equiv} \overline{\varphi_0}+\frac{\langle K_0^u,\varphi\rangle}{\langle K_0^u,K_0^u\rangle}K_0^u$. $\square$

Let $\alpha \in \mathbb{C} \setminus \{0\}$. Then if $A = A_{\varphi_1 + \overline{\varphi_2}}$ is of type $\alpha$, its adjoint is $A^* = A_{\psi_1 + \overline{\psi_2}}$ where $\psi_1 = \varphi_2$ and $\psi_2 = \varphi_1$. By Proposition 4.2 it follows that

$$\overline{\alpha} S_u \psi_2 - \psi_1 \in \mathbb{C} K_0^u.$$

It follows by Proposition 2.5 that

$$\begin{aligned} S_u C(\overline{\alpha} S_u \widetilde{\psi_2} - \psi_1) &= \alpha S_u S_u^* \psi_2 - S_u \widetilde{\psi_1} \\ &= \alpha \psi_2 - S_u \widetilde{\psi_1} + \alpha \langle \psi_2, K_0^u \rangle K_0^u \\ &\in \mathbb{C} K_0^u. \end{aligned}$$

The second equation follows from Equation 2.3. Hence we have that $\alpha^{-1} S_u \widetilde{\psi_1} - \psi_2 \in \mathbb{C} K_0^u$ and so it follows that $A^*$ is of type $\overline{\alpha^{-1}}$. In the case that $A$ is of type 0, $A$ has a holomorphic symbol, and so its adjoint $A^*$ has an antiholomorphic symbol, and is therefore of type $\infty$. Thus we can state the following duality relationship.

PROPOSITION 4.4. *An operator in $\mathscr{T}_u$ is of type $\alpha \in \mathbb{C}^*$ if and only if its adjoint is of type $\overline{\alpha^{-1}}$ using the convention that $0^{-1} = \infty$ and $\infty^{-1} = 0$.*

The operator $cI = A_{cK_0^u} = A_{\overline{cK_0^u}}$ is, by the above definition, of type $\alpha$ for every $\alpha \in \mathbb{C}^*$. This is the only way that an operator in $\mathscr{T}_u$ can be of more than one type. Specifically, this means that any $A \in \mathscr{T}_u$ is either of no type, one type, or every type.

PROPOSITION 4.5. *Let $A \in \mathscr{T}_u$ be of type $\alpha$ and of type $\beta$, where $\alpha \neq \beta$. Then $A = cI$ for some $c \in \mathbb{C}$.*

*Proof.* If $\alpha = 0$ and $\beta = \infty$, then there are $\varphi, \psi \in K_u^2$ such that $A = A_\varphi = A_{\overline{\psi}}$ and so $A_\varphi - S_u A_\varphi S_u^* = \varphi \otimes K_0^u$ and $A_{\overline{\psi}} - S_u A_{\overline{\psi}} S_u^* = K_0^u \otimes \psi$ by Proposition 2.3. Thus $\varphi \otimes K_0^u = K_0^u \otimes \psi$ and $\varphi = cK_0^u$ for some $c \in \mathbb{C}$, and so $A = cI$.

Now suppose that at least one of $\alpha$ and $\beta$ is in $\mathbb{C} \setminus \{0\}$. By looking at $A^*$ if needed we can assume without loss of generality that neither $\alpha$ or $\beta$ is $\infty$. By Proposition 4.3 there are $\varphi, \psi \in K_u^2$ and $c, d \in \mathbb{C}$ such that $\varphi(0) = \psi(0) = 0$ and both $\varphi + \alpha \overline{S_u \widetilde{\varphi}} + c$ and $\psi + \beta \overline{S_u \widetilde{\psi}} + d$ are symbols for $A$. It follows that

$$\begin{aligned} A - S_u A S_u^* &= \varphi \otimes K_0^u + c K_0^u \otimes K_0^u + \alpha K_0^u \otimes S_u \widetilde{\varphi} \\ &= \psi \otimes K_0^u + d K_0^u \otimes K_0^u + \beta K_0^u \otimes S_u \widetilde{\psi}. \end{aligned}$$

By rearranging terms we see that $\varphi - \psi \in \mathbb{C} K_0^u$. Since $\varphi, \psi \perp K_0^u$ it follows that $\varphi = \psi$ and

$$(c-d) K_0^u \otimes K_0^u = (\beta - \alpha) K_0^u \otimes S_u \widetilde{\varphi}.$$

Therefore $S_u \widetilde{\varphi} = \frac{c-d}{\beta - \alpha} K_0^u$ but since

$$\langle S_u \widetilde{\varphi}, K_0^u \rangle = \left\langle \widetilde{K_0^u}, S_u^* \varphi \right\rangle = \left\langle S_u \widetilde{K_0^u}, \varphi \right\rangle = \langle -u(0) K_0^u, \varphi \rangle = 0$$

we get that $c = d$ and $S_u \widetilde{\varphi} = 0$.

Finally we calculate $\varphi = (I - K_0^u \otimes K_0^u)\varphi = S_u \widetilde{S_u \widetilde{\varphi}} = 0$ and get that $A = A_c = cI$. $\square$

For the rest of this section fix $\alpha \in \overline{\mathbb{D}}$. By Proposition 3.2 if an operator $A \in \mathscr{T}_u$ is in $[S_u^\alpha]'$ then it is of type $\alpha$. We spend the remainder of this section proving that every TTO of type $\alpha$ is in $[S_u^\alpha]'$. Specifically, we will show that the product of two TTOs of type $\alpha$ is itself in $\mathscr{T}_u$. Therefore any two TTOs of type $\alpha$ commute and so any TTO of type $\alpha$ commutes with $S_u^\alpha$. Therefore for $\alpha \in \overline{\mathbb{D}}$, $[S_u^\alpha]'$ is precisely the TTOs of type $\alpha$, and therefore $[S_u^{\alpha *}]'$ is precisely the TTOs of type $\overline{\alpha}^{-1}$ with the convention that $\frac{1}{0} = \infty$.

First we prove a lemma that will prove useful here and later.

LEMMA 4.6. *Let* $\Phi = \varphi_1 + \overline{\varphi_2}$ *and* $\Psi = \psi_1 + \overline{\psi_2}$ *where* $\varphi_i, \psi_i \in K_u^2$ *such that* $A_\Phi, A_\Psi \in \mathscr{T}_u$. *Then* $A_\Phi A_\Psi$ *is in* $\mathscr{T}_u$ *if and only if*

$$\varphi_1 \otimes \psi_2 - (S_u \widetilde{\varphi_2}) \otimes (S_u \widetilde{\psi_1}) = \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0$$

*for some* $\Phi_0, \Psi_0 \in K_u^2$.

*Proof.* In what follows, $\Phi_0$ and $\Psi_0$ represent functions in $K_u^2$ that can be different from use to use. By Proposition 2.3, $A_\Phi A_\Psi \in \mathscr{T}_u$ if and only if $A_\Phi A_\Psi - S_u A_\Phi A_\Psi S_u^* = \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0$. It suffices to show that $A_\Phi A_\Psi - S_u A_\Phi A_\Psi S_u^* = \varphi_1 \otimes \psi_2 - (S_u \widetilde{\varphi_2}) \otimes (S_u \widetilde{\psi_1}) + \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0$. Recall Equation 2.2, which states that $I = S_u^* S_u + \widetilde{K_0^u} \otimes \widetilde{K_0^u}$. Therefore

$$\begin{aligned} S_u A_\Phi A_\Psi S_u^* &= S_u A_\Phi (S_u^* S_u + \widetilde{K_0^u} \otimes \widetilde{K_0^u}) A_\Psi S_u^* \\ &= S_u A_\Phi S_u^* S_u A_\Psi S_u^* + \left(S_u A_\Phi \widetilde{K_0^u}\right) \otimes \left(S_u A_{\overline{\Psi}} \widetilde{K_0^u}\right). \end{aligned} \tag{4.1}$$

Since $A_\Phi \widetilde{K_0^u} = P_u\left[(\varphi_1 + \overline{\varphi_2})\left(\overline{z}(u - u(0))\right)\right]$ we have

$$\begin{aligned} S_u A_\Phi \widetilde{K_0^u} &= S_u \left(\widetilde{\varphi_2} + \varphi_1(0)\widetilde{K_0^u} - u(0) S_u^* \varphi_1\right) \\ &= S_u \widetilde{\varphi_2} - u(0)\varphi_1(0) K_0^u - u(0) S_u S_u^* \varphi_1 \\ &= S_u \widetilde{\varphi_2} - u(0)\varphi_1(0) K_0^u - u(0)\varphi_1 + u(0)\left(K_0^u \otimes K_0^u\right)\varphi_1 \\ &= S_u \widetilde{\varphi_2} - u(0)\varphi_1(0) K_0^u - u(0)\varphi_1 + u(0)\varphi_1(0) K_0^u \\ &= S_u \widetilde{\varphi_2} - u(0)\varphi_1 \end{aligned}$$

so the second term of (4.1) is

$$\begin{aligned} \left(S_u A_\Phi \widetilde{K_0^u}\right) \otimes \left(S_u A_{\overline{\Psi}} \widetilde{K_0^u}\right) &= \left(S_u \widetilde{\varphi_2} - u(0)\varphi_1\right) \otimes \left(S_u \widetilde{\psi_1} - u(0)\psi_2\right) \\ &= S_u \widetilde{\varphi_2} \otimes S_u \widetilde{\psi_1} - u(0)\left[\varphi_1 \otimes S_u \widetilde{\psi_1}\right] \\ &\quad - \overline{u(0)}\left[S_u \widetilde{\varphi_2} \otimes \psi_2\right] + |u(0)|^2 \left[\varphi_1 \otimes \psi_2\right]. \end{aligned}$$

By Proposition 2.3 we have that $S_uA_\Phi S_u^* = A_\Phi - \varphi_1 \otimes K_0^u - K_0^u \otimes \varphi_2$, and so the first term of (4.1) is

$$\begin{aligned} S_uA_\Phi S_u^* S_uA_\Psi S_u^* &= (A_\Phi - \varphi_1 \otimes K_0^u - K_0^u \otimes \varphi_2)(A_\Psi - \psi_1 \otimes K_0^u - K_0^u \otimes \psi_2) \\ &= A_\Phi A_\Psi - \Phi_0 \otimes K_0^u - (A_\Phi K_0^u) \otimes \psi_2 \\ &\quad - \varphi_1 \otimes \left(A_{\overline{\Psi}} K_0^u\right) + \left(1 - |u(0)|^2\right) \varphi_1 \otimes \psi_2 - K_0^u \otimes \Psi_0 \\ &= A_\Phi A_\Psi + \Phi_0 \otimes K_0^u - K_0^u \otimes \Psi_0 - \left(1 + |u(0)|^2\right) \varphi_1 \otimes \psi_2 \\ &\quad + \overline{u(0)} \left(S_u \widetilde{\varphi_2} \otimes \psi_2\right) + u(0) \left(\varphi_1 \otimes S_u \widetilde{\psi_1}\right). \end{aligned}$$

By combining the expanded terms together, we get

$$S_uA_\Phi A_\Psi S_u^* = S_u\widetilde{\varphi_2} \otimes S_u\widetilde{\psi_1} - \varphi_1 \otimes \psi_2 + \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0 + A_\Phi A_\Psi$$

and the result follows. $\square$

THEOREM 4.7. *Let $\alpha \in \overline{\mathbb{D}}$, and let $A$ be a bounded operator on $K_u^2$. Then $A$ is a TTO of type $\alpha$ if and only if $A$ is in $[S_u^\alpha]'$.*

*Proof.* Proposition 3.2 proves that everything in $[S_u^\alpha]'$ is of type $\alpha$, so assume $A$ is of type $\alpha$. We will prove that $AS_u^\alpha$ is in $\mathscr{T}_u$, and hence $C$-symmetic, and so $AS_u^\alpha = S_u^\alpha A$ by Proposition 2.1.

$S_u^\alpha$ commutes with itself, and therefore is of type $\alpha$. By Definition 3.1

$$S_u^\alpha K_0^u = S_uK_0^u + \frac{\alpha\overline{u'(0)}}{1-\alpha\overline{u(0)}} K_0^u.$$

So by Proposition 3.2

$$(1-\alpha\overline{u(0)})^{-1}(S_uK_0^u + \alpha\overline{S_u\widetilde{S_uK_0^u}} + \frac{\alpha\overline{u'(0)}}{1-\alpha\overline{u(0)}}(K_0^u + \alpha\overline{S_u\widetilde{K_0^u}}))$$

is a symbol for $S_u^\alpha$. By Proposition 2.5

$$K_0^u + \alpha\overline{S_u\widetilde{K_0^u}} \overset{A}{\equiv} (1-\alpha\overline{u(0)})$$

and so it follows that

$$(1-\alpha\overline{u(0)})^{-1}(S_uK_0^u + \alpha\overline{S_u\widetilde{S_uK_0^u}} + \alpha\overline{u'(0)}K_0^u) \tag{4.2}$$

is also a symbol for $S_u^\alpha$.

Suppose $A$ is of type $\alpha$. Then we may without loss of generality assume that $\varphi + \alpha\overline{S_u\widetilde{\varphi}}$ is a symbol for $A$ where $\varphi$ is in $K_u^2$. Applying Lemma 4.6 we see that $AS_u^\alpha$ is in $\mathscr{T}_u$ if and only if there exist $\Phi, \Psi \in K_u^2$ such that

$$\varphi \otimes \left(\overline{\alpha} S_u\widetilde{S_uK_0^u}\right) - \left(S_u\widetilde{\overline{\alpha} S_u\widetilde{\varphi}}\right) \otimes S_u\widetilde{S_uK_0^u} = \Phi \otimes K_0^u + K_0^u \otimes \Psi$$

Factoring $\alpha$ out of the left-hand side, we get

$$\begin{aligned}\varphi \otimes \left(S_u\widetilde{S_u K_0^u}\right) - \left(S_u\widetilde{S_u\widetilde{\varphi}}\right) \otimes S_u\widetilde{S_u K_0^u} &= ((I - S_u S_u^*)\,\varphi) \otimes S_u\widetilde{S_u K_0^u} \\ &= \varphi(0)K_0^u \otimes S_u\widetilde{S_u K_0^u}\end{aligned}$$

The conclusion follows. □

## 5. Algebras of TTOs

The results of the previous section show that the TTOs of type $\alpha$ form a weakly closed commutative algebra for any $\alpha \in \mathbb{C}^*$, which we denote $\mathscr{B}^\alpha$. In this section we will show that these algebras are maximal — any algebra in $\mathscr{T}_u$ is a subalgebra of at least one $\mathscr{B}^\alpha$.

We begin by showing that if $A_\Phi$ is of type $\alpha$, $A_\Psi \in \mathscr{T}_u$, and their product is in $\mathscr{T}_u$, then either $A_\Phi$ is a multiple of $I$, or $A_\Psi$ is of type $\alpha$ as well.

LEMMA 5.1. *Let $A_\Phi, A_\Phi \in \mathscr{T}_u$ such that $A_\Phi A_\Psi \in \mathscr{T}_u$ and let $\alpha \in \mathbb{C}^*$. If one of the operators in the product is of type $\alpha$, then either it is a constant multiple of the identity operator, or the other is of type $\alpha$ as well.*

*Proof.* Since $A_\Phi A_\Psi$ is in $\mathscr{T}_u$, it is a CSO, and so $A_\Phi A_\Psi = A_\Psi A_\Phi$ by Proposition 2.1. Thus we assume without loss of generality that $A_\Phi$ is of type $\alpha$. Additionally $A_\Phi A_\Psi$ is in $\mathscr{T}_u$ if and only if its adjoint $CA_\Phi A_\Psi C = A_{\overline{\Phi}} A_{\overline{\Psi}}$ is as well, where $A_{\overline{\Phi}}$ is of type $\overline{\alpha}^{-1}$, so we assume without loss of generality that $A_\Phi$ is of type $\alpha \in \overline{\mathbb{D}}$. So $\Phi \overset{A}{\equiv} \varphi_0 + \alpha\overline{S_u\widetilde{\varphi_0}} + cK_0^u$ and $\Psi \overset{A}{\equiv} \psi_1 + \overline{\psi_2}$ for some $\varphi_0, \psi_1, \psi_2 \in K_u^2$, where by Proposition 4.3 we may assume that $\varphi_0(0) = 0$, $c \in \mathbb{C}$. By Lemma 4.6, there exists $\Phi_0, \Psi_0 \in K_u^2$ such that

$$\begin{aligned}\Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0 &= (\varphi_0 + cK_0^u) \otimes \psi_2 - \left(S_u\left(\widetilde{\overline{\alpha}S_u\widetilde{\varphi_0}}\right)\right) \otimes \left(S_u\widetilde{\psi_1}\right) \\ &= \varphi_0 \otimes \psi_2 + cK_0^u \otimes \psi_2 - \varphi_0 \otimes \left(\overline{\alpha}S_u\widetilde{\psi_1}\right) \\ &= \varphi_0 \otimes \left(\psi_2 - \overline{\alpha}S_u\widetilde{\psi_1}\right) + cK_0^u \otimes \psi_2\end{aligned}$$

So $\varphi_0 \otimes \left(\psi_2 - \overline{\alpha}S_u\widetilde{\psi_1}\right) = \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_1$ for some $\Psi_1 \in K_u^2$. So either $\Phi_0$ and $K_0^u$ are linearly dependent or $\Psi_1$ and $K_0^u$ are. If $\Phi_0$ and $K_0^u$ are linearly dependent, then $\Phi_0 = c_1 K_0^u$ which means $\varphi_0 = c_2 K_0^u$, but this and $\varphi_0(0) = 0$ then imply that $c_2 = 0$, and so $\varphi_0 = 0$ and $A_\Phi = cI$. Otherwise, $\Psi_1 = c_3 K_0^u$ and so $\psi_2 - \overline{\alpha}S_u\widetilde{\psi_1} = c_4 K_0^u$, which means $A_\Psi$ is of type $\alpha$ by Proposition 4.2. □

We now prove the main theorem of this section.

THEOREM 5.2. *Let $\Phi, \Psi \in L^2(\mathbb{T})$ such that $A_\Phi, A_\Psi \in \mathscr{T}_u$. Then $A_\Phi A_\Psi \in \mathscr{T}_u$ if and only if one of two (not mutually exclusive) cases holds:*

*Trivial case: Either $A_\Phi$ or $A_\Psi$ is equal to $cI$ for some $c \in \mathbb{C}$.*

*Non-trivial case: $A_\Phi$ and $A_\Psi$ are both of type $\alpha$ for some $\alpha \in \mathbb{C}^*$, in which case their product is of type $\alpha$ as well.*

*Proof.* The sufficiency of either case follows from earlier discussion, so we prove their necessity. In what follows we will use the fact that if $\Phi$ and $\Psi$ are functions such that $A_\Phi A_\Psi \in \mathscr{T}_u$, then for any complex constants $c_1, c_2$ $A_{\Phi+c_1}A_{\Psi+c_2} \in \mathscr{T}_u$.

Suppose $A_\Phi A_\Psi \in \mathscr{T}_u$. By Lemma 5.1 it suffices to show that one of $A_\Phi$ and $A_\Psi$ is of type $\alpha$ for some $\alpha \in \mathbb{C}^*$.

There exists $\varphi_i, \psi_i \in K_u^2$ such that we may assume without loss of generality that $\Phi = \varphi_1 + \overline{\varphi_2}$ and that $\Psi = \psi_1 + \overline{\psi_2}$. Then it follows by Lemma 4.6 that

$$\varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1}) = \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0$$

holds for some $\Phi_0, \Psi_0$ in $K_u^2$. If at least one of $\Phi_0$ and $\Psi_0$ is non-zero, but one of them is in $\mathbb{C}K_0^u$, then the right-hand side of this equation is a rank one operator $f \otimes g$. Thus we consider the following three cases.

(1) $\varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1}) = 0$

(2) $\varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1}) = f \otimes g;\ f, g \in K_u^2$

(3) $\varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1}) = \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0;\ \Phi_0, \Psi_0 \neq cK_0^u$

In what follows, $c$ and $c_i$ represent complex constants that may change from paragraph to paragraph.

*Case 1*: We have $\varphi_1 \otimes \psi_2 = (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1})$, which means that $\psi_2$ and $S_u\widetilde{\psi_1}$ are linearly dependent. Both $\psi_2$ and $S_u\widetilde{\psi_1}$ are non-zero, so $\psi_2 = \overline{\alpha} S_u\widetilde{\psi_1}$ for $\alpha \neq 0$ and it follows from Proposition 4.2 that $A_\Psi$ is of type $\alpha$.

*Case 2*: We have $\varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1}) = f \otimes g;\ f, g \in K_u^2$. So either $\varphi_1$ and $S_u\widetilde{\varphi_2}$ are linearly dependent or $S_u\widetilde{\psi_1}$ and $\psi_2$ are. In the latter case, we again get that $A_\Psi$ is of type $\alpha$ for some $\alpha \neq 0$. Assume instead that $\varphi_1 = c_1 S_u\widetilde{\varphi_2}$ for $c_1 \neq 0$. Then by Equation 2.3 $c_2 S_u\widetilde{\varphi_1} = S_u\widetilde{S_u\widetilde{\varphi_2}} = \varphi_2 - \langle \varphi_2, K_0^u \rangle K_0^u$, and so $\varphi_2 - c_2 S_u\widetilde{\varphi_1} \in \mathbb{C}K_0^u$ and therefore by Proposition 4.2 $A_\Phi$ is of type $\alpha = \overline{c_2}$.

*Case 3*: We have $\varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1}) = \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0; \Phi_0, \Psi_0 \neq cK_0^u$. There exists $f \in K_u^2$ such that $f(0) = 0$ and $\langle f, \Phi_0 \rangle = 1$. Then we have

$$\begin{aligned} K_0^u &= (\Psi_0 \otimes K_0^u + K_0^u \otimes \Phi_0) f \\ &= (\psi_2 \otimes \varphi_1) f - \left(S_u\widetilde{\psi_1} \otimes S_u\widetilde{\varphi_2}\right) f \\ &= \psi_2 \langle f, \varphi_1 \rangle - S_u\widetilde{\psi_1} \left\langle f, S_u\widetilde{\varphi_2} \right\rangle \end{aligned}$$

If $\langle f, \varphi_1 \rangle = 0$, then $cK_0^u = S_u\widetilde{\psi_1}$, and so by Proposition 4.2 $A_\Psi$ is of type $\infty$. Similarly, if $\left\langle f, S_u\widetilde{\varphi_2} \right\rangle = 0$, then $cK_0^u = \psi_2$ and $A_\Psi$ is of type $0$. So we can assume that $\psi_2 = \overline{\alpha} S_u\widetilde{\psi_1} + cK_0^u$ for some $\alpha \neq 0$. Thus $A_\Psi$ is of type $\alpha$ by Proposition 4.2. □

EXAMPLE 5.3. Theorem 5.1 of [11] classifies all the rank one operators in $\mathscr{T}_u$ and finds symbols for them. Specifically, for $\lambda \in \mathbb{D}$ $\widetilde{K_\lambda^u} \otimes K_\lambda^u$ is in $\mathscr{T}_u$ and has with symbol $u/(z-\lambda)$, and if $u$ has an ADC at $\zeta \in \mathbb{T}$ then $K_\zeta^u \otimes K_\zeta^u$ is in $\mathscr{T}_u$ and has symbol $K_\zeta^u + \overline{K_0^u}[\zeta] - 1$. We will show that all of them are of type $\alpha$ for some $\alpha \in \mathbb{C}^*$, and compute $\alpha$.

Let $\lambda \in \mathbb{D}$ and consider $A = \widetilde{K_\lambda^u} \otimes K_\lambda^u$, with symbol $u/(z-\lambda)$. Since $\widetilde{K_\lambda^u}(\lambda) = u'(\lambda)$, $\left(\widetilde{K_\lambda^u} \otimes K_\lambda^u\right)^2 = u'(\lambda)\widetilde{K_\lambda^u} \otimes K_\lambda^u$ so it follows that $\widetilde{K_\lambda^u} \otimes K_\lambda^u$ is of type $\alpha$ for some $\alpha \in \mathbb{C}^*$. Since

$$\begin{aligned} u/(z-\lambda) &\overset{A}{\equiv} \widetilde{K_\lambda^u} + u(\lambda)/(z-\lambda) \\ &\overset{A}{\equiv} \widetilde{K_\lambda^u} + u(\lambda)\overline{zK_\lambda} \\ &\overset{A}{\equiv} \widetilde{K_\lambda^u} + u(\lambda)\overline{S_u K_\lambda^u} \end{aligned}$$

$A$ is of type $u(\lambda)$.

Now instead suppose that $\zeta \in \mathbb{T}$ such that $u$ has an ADC at $\zeta$, and consider $A = K_\zeta^u \otimes K_\zeta^u$ which has symbol $K_\zeta^u + \overline{K_\zeta^u} - 1$. Again it is clear that $A^2$ is a scalar multiple of $A$ and hence $A$ is of type $\alpha$ for some $\alpha$. Since $A$ is self-adjoint, it follows that $\alpha$ is unimodular. We compute

$$\begin{aligned} \widetilde{K_\zeta^u} &= \frac{u - u(\zeta)}{z - \zeta} \\ &= \frac{u(\zeta)\left(1 - \overline{u(\zeta)}u\right)}{\zeta\left(1 - \overline{\zeta}z\right)} \\ &= \overline{\zeta}u(\zeta)K_\zeta^u \end{aligned}$$

so

$$S_u\widetilde{K_\zeta^u} = \zeta\widetilde{K_\zeta^u} - u(\zeta)K_0^u = u(\zeta)\left(K_\zeta^u - K_0^u\right)$$

Thus $K_\zeta^u - 1 \overset{A}{\equiv} \overline{u(\zeta)S_u\widetilde{K_\zeta^u}}$ and so $K_\zeta^u + u(\zeta)\overline{S_u\widetilde{K_\zeta^u}}$ is a symbol for $A$, which is therefore of type $u(\zeta)$.

Theorem 5.2 has the following consequence which is an analogue of Corollary 2 in [2].

THEOREM 5.4. *Let $A \in \mathscr{T}_u$ be invertible. Then $A^{-1} \in \mathscr{T}_u$ if and only if $A$ is of type $\alpha$ for some $\alpha \in \mathbb{C}^*$. If $A^{-1} \in \mathscr{T}_u$, then $A$ and $A^{-1}$ are of the same type*

*Proof.* If $A^{-1} \in \mathscr{T}_u$, then both $A$ and $A^{-1}$ are of type $\alpha$ for some $\alpha \in \mathbb{C}^*$ by Theorem 5.2 since their product is $I = A_{K_0^u}$. If $A$ is of type $\alpha$, either $|\alpha| \leqslant 1$ or $A^*$ is of type $\beta = 1/\overline{\alpha} \leqslant 1$. In the first case, we have that $AS_u^\alpha = S_u^\alpha A$, so $A^{-1}S_u^\alpha = A^{-1}S_u^\alpha AA^{-1} = A^{-1}AS_u^\alpha A^{-1} = S_u^\alpha A^{-1}$ and $A^{-1}$ is a TTO of type $\alpha$. In the second case, we have that $A^*$ is an invertible TTO of type $\beta$ where $|\beta| \leqslant 1$, so its inverse is a TTO of type $\beta$ as well. By taking adjoints again, the result follows. $\square$

$\mathbb{C}I$ is a subalgebra of $\mathscr{B}^\alpha$ for every $\alpha$, and the intersection of $\mathscr{B}^\alpha$ and $\mathscr{B}^\beta$ is either $\mathscr{B}^\alpha$ or $\mathbb{C}I$ depending on whether $\alpha = \beta$ or not. We now consider an arbitrary algebra $\mathscr{A}$ contained in $\mathscr{T}_u$ and its relationship to $\mathscr{B}^\alpha$.

THEOREM 5.5. *Let $\mathscr{A}$ be an algebra contained in $\mathscr{T}_u$. Then there exists an $\alpha \in \mathbb{C}^*$ such that $\mathscr{A}$ is a subalgebra of $\mathscr{B}^\alpha$.*

*Proof.* Suppose every $A$ in $\mathscr{A}$ is of the form $cI$, for $c \in \mathbb{C}$. Then $I \in \mathscr{A}$ and so $\mathscr{A} = \mathbb{C}I$ which is a subalgebra of every $\mathscr{B}^\alpha$.

Suppose then that there is $A \in \mathscr{A}$ not of the form $cI$. $A^2 \in \mathscr{T}_u$ so by Theorem 5.2 $A$ is of type $\alpha$ for some unique $\alpha$. If $B \in \mathscr{A}$ then $AB \in \mathscr{T}_u$ and so since $A \neq cI$ it follows from Theorem 5.2 that $B$ is of type $\alpha$ as well, and therefore every operator in $\mathscr{A}$ is of type $\alpha$, and so it is a subalgebra of $\mathscr{B}^\alpha$ $\square$

## 6. Properties of $\mathscr{B}^\alpha$

Due to the duality between $\mathscr{B}^\alpha$ and $\mathscr{B}^{(\overline{\alpha}^{-1})}$ via taking adjoints, in order to study these algebras we can look at the cases where $\alpha \in \overline{\mathbb{D}}$. These algebras can then be divided into two different groups, $\alpha \in \mathbb{D}$ and $\alpha \in \mathbb{T}$. Different techniques are needed to deal with each of these cases. We discuss what the product of two TTOs of type $\alpha$ is, and expand on Theorem 5.4 by finding necessary and sufficient conditions for a TTO of type $\alpha$ to be invertible, based on its symbol.

### 6.1. $\alpha \in \mathbb{D}$

In this subsection, assume $\alpha \in \mathbb{D}$.

Sarason's Commutant Lifting Theorem [9] states that if $A$ is a bounded operator that commutes with $S_u$, then there exists a function $\varphi \in H^\infty$ such that $\|A\| = \|\varphi\|_\infty$ and $A = A_\varphi$. The goal of this subsection is to find a Commutant Lifting Theorem for $[S_u^\alpha]'$.

Let $u_\alpha = \frac{u-\alpha}{1-\overline{\alpha}u}$ for $\alpha \in \mathbb{D}$. In what follows, we will be dealing with operators in both $\mathscr{T}_u$ and $\mathscr{T}_{u_\alpha}$. Let $A_\Phi^u$ refer to an operator in $\mathscr{T}_u$ and $A_\Phi^{u_\alpha}$ an operator in $\mathscr{T}_{u_\alpha}$.

$T_\alpha = M_{(1-|\alpha|^2)^{-1/2}(1-\overline{\alpha}u)}$ is an unitary map from $K_{u_\alpha}^2$ onto $K_u^2$ called a Crofoot transform [5]. Note that $T_\alpha^{-1} = M_{(1-|\alpha|^2)^{1/2}(1-\overline{\alpha}u)^{-1}}$. Sarason [11] showed that $S_u^\alpha = A^u_{z/(1-\alpha\overline{u})}$ and that $T_\alpha^{-1} S_u^\alpha T_\alpha = A_z^{u_\alpha}$, the compressed shift on $K_{u_\alpha}^2$. Thus there is a unitary equivalence between $\mathscr{B}^\alpha$ on $K_u^2$ and $\mathscr{B}^0$ on $K_{u_\alpha}^2$. The following propositions describe the operators of the form $A^u_{\varphi/(1-\alpha\overline{u})}$ for $\varphi \in H^2$, which are in fact the operators in $\mathscr{B}^\alpha$.

PROPOSITION 6.1.

(1) *For $\varphi \in K_u^2$ and $\alpha \in \mathbb{D}$, $A^u_{\varphi/(1-\alpha\overline{u})} = A^u_{\varphi(1+\alpha\overline{u})} = A^u_{\varphi - \alpha\overline{S_u\widetilde{\varphi}}}$.*

(2) *If $\varphi \in H^2$, then $A^u_{\overline{\varphi}/(1-\alpha\overline{u})} = A^u_{\overline{\varphi}}$. Specifically, $A^u_{(1-\alpha\overline{u})^{-1}} = I$.*

(3) $S_u^\alpha = A^u_{z/(1-\alpha\overline{u})}$.

*Proof.*

(1) Since
$$\frac{1}{1-\alpha\overline{u}} = \sum_{n=0}^{\infty} (\alpha\overline{u})^n$$
we can compute
$$\frac{\varphi}{1-\alpha\overline{u}} = \sum_{n=0}^{\infty} \varphi(\alpha\overline{u})^n$$
But since $\overline{u}\varphi \in \overline{zH^2}$ it follows that $\sum_{n=0}^{\infty} \varphi(\alpha\overline{u})^n \overset{A}{\equiv} \varphi(1+\alpha\overline{u})$ and so $A^u_{\varphi/(1-\alpha\overline{u})} = A^u_{\varphi(1+\alpha\overline{u})}$. The second equality then holds because
$$\overline{S_u\widetilde{\varphi}} = \overline{\widetilde{S_u^*\varphi}} = \overline{u}z\frac{\varphi-\varphi(0)}{z} \overset{A}{\equiv} \varphi\overline{u}.$$

(2) $\overline{\varphi}/(1-\alpha\overline{u}) \overset{A}{\equiv} \overline{\varphi} + \alpha\overline{u\varphi}/(1-\alpha\overline{u}) \overset{A}{\equiv} \overline{\varphi}$ by Proposition 2.2, since $\overline{u\varphi}/(1-\alpha\overline{u}) \in \overline{uH^2}$.

(3) Equation (4.2) and part $(1)$ of this proof imply that $S_u^\alpha$ has symbol
$$\frac{1}{1-\alpha\overline{u(0)}}\left(\frac{S_uK_0^u}{1-\alpha\overline{u}} + \alpha\overline{u'(0)}\right)$$
so it suffices to show that
$$\frac{z(1-\alpha\overline{u(0)})}{1-\alpha\overline{u}} \overset{A}{\equiv} \frac{S_uK_0^u}{1-\alpha\overline{u}} + \alpha\overline{u'(0)}.$$
Since $z = S_uK_0^u + uP(\overline{u}z)$,
$$\begin{aligned}\frac{z}{1-\alpha\overline{u}} &\overset{A}{\equiv} \frac{S_uK_0^u}{1-\alpha\overline{u}} + \frac{uP(\overline{u}z)}{1-\alpha\overline{u}}\\ &\overset{A}{\equiv} \frac{S_uK_0^u}{1-\alpha\overline{u}} + \frac{\alpha P(\overline{u}z)}{1-\alpha\overline{u}}.\end{aligned}$$
Since $\widetilde{K_0^u} = (u-u(0))\overline{z}$, $P(\overline{u}z) = \overline{\widetilde{K_0^u}(0)} + \overline{u(0)}z = \overline{u'(0)} + \overline{u(0)}z$,
$$\begin{aligned}\frac{z(1-\alpha\overline{u(0)})}{1-\alpha\overline{u}} &\overset{A}{\equiv} \frac{z}{1-\alpha\overline{u}} - \frac{\alpha\overline{u(0)}z}{1-\alpha\overline{u}}\\ &\overset{A}{\equiv} \frac{S_uK_0^u}{1-\alpha\overline{u}} + \frac{\alpha\overline{u'(0)}}{1-\alpha\overline{u}} + \frac{\alpha\overline{u(0)}z}{1-\alpha\overline{u}} - \frac{\alpha\overline{u(0)}z}{1-\alpha\overline{u}}\\ &\overset{A}{\equiv} \frac{S_uK_0^u}{1-\alpha\overline{u}} + \alpha\overline{u'(0)}. \qquad \square\end{aligned}$$

LEMMA 6.2. *Let $\varphi \in H^2$ and $\alpha \in \mathbb{D}$. Then $T_\alpha A^{u_\alpha}_\varphi T_\alpha^{-1} = A^u_{\varphi/(1-\alpha\overline{u})}$ and $T_\alpha A^{u_\alpha}_{\overline{\varphi}} T_\alpha^{-1} = A^u_{\overline{\varphi}/(1-\overline{\alpha}u)}$. Therefore $A^{u_\alpha}_\varphi$ and $A^u_{\varphi/(1-\alpha\overline{u})}$ (respectively $A^{u_\alpha}_{\overline{\varphi}}$ and $A^u_{\overline{\varphi}/(1-\overline{\alpha}u)}$) have the same norm, and if $\psi \in H^2$, then $A^u_{\varphi/(1-\alpha\overline{u})} = A^u_{\psi/(1-\alpha\overline{u})}$ (respectively $A^u_{\overline{\varphi}/(1-\overline{\alpha}u)} = A^u_{\overline{\psi}/(1-\overline{\alpha}u)}$) if and only if $u_\alpha | (\varphi - \psi)$.*

*Proof.* It suffices to show that the equalities hold on $K_u^\infty$, so let $f \in K_u^\infty$. Then

$$A^u_{\varphi/(1-\alpha\overline{u})} f = P_u\left(\frac{f\varphi}{1-\alpha\overline{u}}\right) = P\left(\frac{f\varphi}{1-\alpha\overline{u}}\right) - uP\left(\frac{\overline{u}f\varphi}{1-\alpha\overline{u}}\right)$$

On the other hand,

$$\begin{aligned} T_\alpha A^{u_\alpha}_\varphi T_\alpha^{-1} f &= (1-\overline{\alpha}u) P_{u_\alpha}\left(\frac{f\varphi}{1-\overline{\alpha}u}\right) \\ &= (1-\overline{\alpha}u)\left[\frac{f\varphi}{1-\overline{\alpha}u} - u_\alpha P\left(\frac{\overline{u_\alpha} f\varphi}{1-\overline{\alpha}u}\right)\right] \\ &= f\varphi - (u-\alpha)P\left(\frac{\overline{u}f\varphi}{1-\alpha\overline{u}}\right) \\ &= f\varphi + P\left(\frac{\alpha\overline{u}f\varphi}{1-\alpha\overline{u}}\right) - uP\left(\frac{\overline{u}f\varphi}{1-\alpha\overline{u}}\right) \\ &= P\left(\frac{f\varphi}{1-\alpha\overline{u}}\right) - uP\left(\frac{\overline{u}f\varphi}{1-\alpha\overline{u}}\right) \end{aligned}$$

Since $T_\alpha$ is unitary, it follows that $A^u_{\varphi/(1-\alpha\overline{u})} = A^u_{\psi/(1-\alpha\overline{u})}$ if and only if $A^{u_\alpha}_\varphi = A^{u_\alpha}_\psi$, but by Proposition 2.2 the latter is true if and only if $u_\alpha | \varphi - \psi$.

Since $T_\alpha$ is unitary, we have

$$\begin{aligned} A^u_{\overline{\varphi}/(1-\overline{\alpha}u)} &= \left(A^u_{\varphi/(1-\alpha\overline{u})}\right)^* \\ &= \left(T_\alpha A^{u_\alpha}_\varphi T_\alpha^{-1}\right)^* \\ &= T_\alpha A^{u_\alpha}_{\overline{\varphi}} T_\alpha^{-1} \end{aligned}$$

proving the result for the adjoints. □

THEOREM 6.3. *Let $A$ be an bounded operator on $K_u^2$ and let $\alpha \in \mathbb{D}$. Then $A$ is of type $\alpha$ if and only if there is a function $\varphi \in H^2$ such that $A = A^u_{\varphi/(1-\alpha\overline{u})}$. If $A$ is of type $\alpha$ then there is a function $\psi \in H^\infty$ such that $\|\psi\|_\infty = \|A\|$ and $A = A^u_{\psi/(1-\alpha\overline{u})}$ and therefore every operator of type $\alpha$ has a bounded symbol. Further, if $\varphi, \psi$ are in $H^\infty$ then $A^u_{\varphi/(1-\alpha\overline{u})} A^u_{\psi/(1-\alpha\overline{u})} = A^u_{\varphi\psi/(1-\alpha\overline{u})}$.*

*Proof.* Let $B = T_\alpha^{-1} A T_\alpha$. Then

$$AA^u_{z/(1-\alpha\overline{u})} = A^u_{z/(1-\alpha\overline{u})} A$$

if and only if

$$BA_z^{u_\alpha} = T_\alpha^{-1} A A^u_{z/(1-\alpha\overline{u})} T_\alpha = T_\alpha^{-1} A^u_{z/(1-\alpha\overline{u})} A T_\alpha = A_z^{u_\alpha} B$$

But this is true if and only if $B = A_\varphi^{u_\alpha}$ for some $\varphi \in H^2$ which is true if and only if $A = A^u_{\varphi/(1-\alpha\overline{u})}$ for some $\varphi \in H^2$, hence the first claim holds. By the Commutant Lifting Theorem, there is a function $\psi \in H^\infty$ such that $A_\varphi^{u_\alpha} = A_\psi^{u_\alpha}$ and $\|A_\varphi^{u_\alpha}\| = \|\psi\|_\infty$. By Lemma 6.2 it follows that $A = A^u_{\psi/(1-\alpha\overline{u})}$. Since $T_\alpha$ is unitary, $\|A\| = \|\psi\|_\infty$.

To prove the last claim, we compute

$$A^u_{\varphi/(1-\alpha\overline{u})} A^u_{\psi/(1-\alpha\overline{u})} = T_\alpha^{-1} A_\varphi^{u_\alpha} A_\psi^{u_\alpha} T_\alpha = T_\alpha^{-1} A_{\varphi\psi}^{u_\alpha} T_\alpha = A^u_{\varphi\psi/(1-\alpha\overline{u})} \qquad \square$$

Just as $A_\varphi^u = \varphi(S_u)$ for $\varphi \in H^\infty$, we get that $A^u_{\varphi/(1-\alpha\overline{u})} = \varphi(S_u^\alpha)$ for $\varphi \in H^\infty$.

Note that $\lambda$ is in the spectrum of $A_\varphi^u$ if and only if $\inf_{z\in\mathbb{D}}(|u(z)| + |\varphi(z) - \lambda|) = 0$ [3].

PROPOSITION 6.4. *Let $\alpha \in \mathbb{D}$ and let $\varphi \in H^\infty$. Then $A^u_{\varphi/(1-\alpha\overline{u})}$ is invertible if and only if* $\inf_{z\in\mathbb{D}}(|u_\alpha(z)| + |\varphi(z)|) > 0$

*Proof.* $A^u_{\varphi/(1-\alpha\overline{u})}$ is invertible if and only if $A_\varphi^{u_\alpha}$ is invertible, which is true if and only if $\inf_{z\in\mathbb{D}}(|u_\alpha(z)| + |\varphi(z)|) > 0$. $\square$

**6.2.** $\alpha \in \mathbb{T}$

The case of $|\alpha| = 1$ is indirectly dealt with in [11, 1] and we collect those results here. There are TTOs of unimodular type without a bounded symbol under certain conditions. Specifically, in [1] it is shown that there exists $u$ an inner function with an ADC at $\zeta \in \mathbb{T}$ such that $K_\zeta^u \otimes K_\zeta^u \in \mathscr{T}_u$ does not have a bounded symbol.

Example 5.3 shows that $K_\zeta^u \otimes K_\zeta^u$ is of type $u(\zeta)$, and hence it is an example of a TTO of unimodular type without a bounded symbol.

If, however, we weaken what we mean by "bounded symbol" we can find a bounded symbol for any TTO of unimodular type. Specifically, we change the measure with respect to which we take the sup norm of a function.

Let $\alpha$ be unimodular, and fixed for the rest of this section. An operator is of type $\alpha$ if and only if it commutes with $S_u^\alpha$, which is in this case a unitary operator known as a Clark unitary operator, and is unitarily equivalent to $M_z$ on the space $L^2(\mathbb{T}, \mu_\alpha)$ where $\mu_\alpha$ is the Clark measure associated with $S_u^\alpha$ [4]. $[M_z]'$ is the space of multiplication operators induced by $L^\infty(\mu_\alpha)$ and so by using the unitary equivalence, every operator of type $\alpha$ is equal to $\Phi(S_u^\alpha)$ where $\Phi \in L^\infty(\mu_\alpha)$. In this sense we can think about $\Phi$ as a "bounded symbol" for the operator. This gives us a symbol calculus of sorts for operators of type $\alpha$: given $\Phi, \Psi$ bounded $\mu_\alpha$-almost everywhere, the product of $M_\Phi$ and $M_\Psi$ is $M_{\Phi\Psi}$ where $\Phi\Psi$ is itself bounded $\mu_\alpha$-almost everywhere. Hence $\Phi(S_u^\alpha)\Psi(S_u^\alpha) = \Phi\Psi(S_u^\alpha)$. It follows that a TTO of type $\alpha$ is invertible if and only if it is of the form $\Phi(S_u^\alpha)$, where $|\Phi| \geqslant \delta > 0$ $\mu_\alpha$-almost everywhere.

We can use this symbol calculus to precisely describe the unitary operators in $\mathscr{T}_u$ on a given model space.

PROPOSITION 6.5. *Let $A \in \mathscr{T}_u$. Then $A$ is unitary if and only if it is equal to $\Phi(S_u^\alpha)$ for some $\alpha \in \mathbb{T}$ and some $\Phi \in L^\infty(\mathbb{T}, \mu_\alpha)$ such that $|\Phi| = 1$ $\mu_\alpha$-almost everywhere. Specifically, any unitary operator in $\mathscr{T}_u$ is of unimodular type, and commutes with the Clark unitary operator of the same type.*

*Proof.* If $A$ is unitary then $AA^* = I$, which means that $A$ and $A^*$ must both be of the same type $\alpha \in \mathbb{C}^*$. Thus $\alpha = \overline{\alpha}^{-1}$ which implies that $\alpha$ is of unimodular type. So $A = \Phi(S_u^\alpha)$ for some $\Phi \in L^\infty(\mathbb{T}, \mu_\alpha)$. Then $I = AA^* = \Phi(S_u^\alpha)\overline{\Phi}(S_u^\alpha) = |\Phi|^2(S_u^\alpha)$ which implies that $|\Phi| = 1$ $\mu_\alpha-$almost everywhere. The other direction is obvious. $\square$

## REFERENCES


[1] A. BARANOV, I. CHALENDAR, E. FRICAIN, J. MASHREGHI, AND D. TIMOTIN, *Bounded symbols and reproducing kernel thesis for truncated Toeplitz operators*, arXiv:0909.0131v1, 2009.

[2] A. BROWN AND P. R. HALMOS, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math., **213** (1963-1964), 89–102.

[3] J. A. CIMA, A. L. MATHESON, AND W. T. ROSS, *The Cauchy Transform*, Mathematical Surveys and Monographs, 125, American Mathematical Society, Providence, RI, 1998.

[4] D. N. CLARK, *One dimensional perturbations of restricted shifts*, J. Analyse Math., **25** (1972), 169–191.

[5] R. B. CROFOOT,

M ultipliers between invariant subspaces of the backward shift, Pacific J. Math., **166**, 2 (1994), 225–246.

[6] S. R. GARCIA, *Conjugation and Clark operators*, Contemporary Mathematics, **393** (2006), 67–111.

[7] S. R. GARCIA AND M. PUTINAR, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc., **358**, 3 (2006), 1285–1315.

[8] S. R. GARCIA AND M. PUTINAR, *Complex symmetric operators and applications ii*, Trans. Amer. Math. Soc., **359**, 8 (2007), 3913–3931.

[9] D. SARASON, *Generalized interpolation in $H^\infty$*, Trans. Amer. Math. Soc, **127** (1967), 179–203.

[10] D. SARASON, *Sub-Hardy Hilbert spaces in the unit disc*, John Wiley and Sons, Inc., New York, 1994.

[11] D. SARASON, *Algebraic properties of truncated Toeplitz operators*, Operators and Matrices, **1**, 4 (2007), 491–526.


(Received June 18, 2010)

*N. A. Sedlock*
*Framingham State University*
*Department of Mathematics*
*100 State Street, P.O. Box 9101*
*Framingham, MA 01701-9101*
*U.S.A.*
*e-mail:* nsedlock@framingham.edu