Operators and Matrices
Volume 5, Number 2 (2011), 333–350

THE GENERAL SOLUTION TO A SYSTEM OF ADJOINTABLE OPERATOR EQUATIONS OVER HILBERT C^*–MODULES

QING-WEN WANG, CHANG-ZHOU DONG

(*Communicated by L. Rodman*)

Abstract. We establish necessary and sufficient conditions for the existence of solution to the system of adjointable operator equations $A_1X = D_1, XB_2 = D_2, A_3XB_3 + B_3^*X^*C_3 = D_3$ over the Hilbert C^*-modules. We also give the explicit expression of the general solution to this system when the solvability conditions are satisfied. As an application, we investigate the anti-reflexive Hermitian solution to the system of complex matrix equations $AX = B, XC = D, EXE^* = F$. The findings of this paper extend some known results in the literature.

1. Introduction

We know that investigating solutions to operator equations is a very active research topic. In 2007, Djordjević [1] considered the operator equation

$$A^*X \pm X^*A = B \tag{1.1}$$

for bounded operators on Hilbert spaces. In 2008, Cvetković-Ilić [2] gave the solvability conditions and the set of the solutions to the operator equations

$$AX + X^*C = B \tag{1.2}$$

and

$$AXB + B^*X^*A^* = C \tag{1.3}$$

for bounded operators on Hilbert spaces. Xu [3] in 2008 investigated the equation (1.3) in the general setting of the adjointable operators between the Hilbert C^*-modules. Moreover, Xu [4], Fang et al. [7] studied the system of equations

$$AX = C,\ XB = D \tag{1.4}$$

Mathematics subject classification (2010): 47A62, 46L08, 15A24, 47A52.
Keywords and phrases: Hilbert C^*-modules, operator equation, system of operator equations, Moore-Penrose inverse.
This research was supported by the grants from the Ph. D. Programs Foundation of Ministry of Education of China (20093108110001), the Scientific Research Innovation Foundation of Shanghai Municipal Education Commission (09YZ13), Singapore MoE Tier 1 Research Grant RG60/07, Shanghai Leading Academic Discipline Project (J50101), and the Innovation Funds for Graduates of Shanghai University (SHUCX101026).

Paper OaM-05-24

for adjointable operators between Hilbert C^*-modules, which generalized the main results in [5]–[6].

Hilbert C^*-module is a natural generalization of Hilbert space and C^*-algebra. Hilbert C^*-modules play important role in the theory of operator algebras, for instance, we need formulations using Hilbert C*-modules to study Morita equivalence of C*-algebras, induced representations, Multipliers, K-theory and KK-theory, index theory, Cuntz-Pimsner algebras and so on. Therefore investigating operator equations over Hilbert C^*-modules is very meaningful. Note that all of those equations (1.1)–(1.4) are special cases of the following system of adjointable operator equations

$$A_1X = D_1, XB_2 = D_2, A_3XB_3 + B_3^*X^*C_3 = D_3 \tag{1.5}$$

over the Hilbert C^*-modules, which is of interest in its own right. So far, to our knowledge, there has been little information on general solution to system (1.5) either for adjointable operator equations in the framework of Hilbert C^*-modules or for matrix equations over the complex number field. Moreover, system (1.5) has some applications, for example, using the results of system (1.5), we can investigate the anti-reflexive Hermitian solution to the system of complex matrix equations

$$AX = B, XC = D, EXE^* = F. \tag{1.6}$$

It is well-known that the reflexive and anti-reflexive matrices have many important applications in numerical analysis, information theory and linear estimate theory [8], and a large number of papers have investigated the reflexive or anti-reflexive solutions to some matrix equations [9]–[11]. We know that the anti-reflexive Hermitian solution of system (1.6) of matrix equations has not been concerned yet.

Motivated by the work mentioned above, we in this paper aim to give the solvability conditions to the system of adjointable operator equations (1.5) over the Hilbert C^*-modules, as well as present an explicit expression for the general solution to this system when the solvability conditions are satisfied.

The paper is organized as follows. In Section 2, we begin with some basic concepts and results about adjointable operators and generalized inverse of adjointable operators over the Hilbert C^*-modules. In Section 3 we present necessary and sufficient conditions for the existence of the solution to the system (1.5). When the solvability conditions are met, we also give an expression of the general solution to this system. As applications, in Section 4, we first show that some known results can be recovered from the main results of this paper, then propose the solvability conditions and the general expression of anti-reflexive Hermitian solution to the system of matrix equations (1.6). We in Section 5 give a conclusion to close this paper.

2. Preliminaries

Hilbert C^*-modules arose as generalizations of the notion Hilbert space. The basic idea was to consider modules over C^*-algebras instead of linear spaces and to allow the inner product to take values in the C^*-algebra. The structure was first used by

Kaplansky [12] in 1952. For more details of C^*-algebra and Hilbert C^*-modules, we refer the reader to [13, 14].

Let $\mathfrak{A}$ be a C^*-algebra. An inner-product $\mathfrak{A}$-module is a linear space E which is a right $\mathfrak{A}$-module (with a scalar multiplication satisfying $\lambda(xa) = x(\lambda a) = (\lambda x)a$ for $x \in E, a \in \mathfrak{A}, \lambda \in \mathbb{C}$), together with a map $E \times E \to \mathfrak{A}$, $(x,y) \to \langle x,y\rangle$ such that

(1) $\langle x, \alpha y + \beta z\rangle = \alpha \langle x,y\rangle + \beta \langle x,z\rangle$;

(2) $\langle x, ya\rangle = \langle x,y\rangle a$;

(3) $\langle x,y\rangle = \langle y,x\rangle^*$;

(4) $\langle x,x\rangle \geqslant 0$, and $\langle x,x\rangle = 0 \Leftrightarrow x = 0$ for any $x,y,z \in E$, $\alpha, \beta \in \mathbb{C}$ and $a \in \mathfrak{A}$.

An inner-product $\mathfrak{A}$-module E is called a (right) Hilbert $\mathfrak{A}$-module if it is complete with respect to the induced norm $||x|| = \langle x,x\rangle^{1/2}$.

Throughout this paper H_1 and H_2 denote two Hilbert C^*-modules, and $\mathscr{L}(H_1,H_2)$ is the set of all maps $T\colon\ H_1 \to H_2$ for which there is a map $T^*\colon\ H_2 \to H_1$ such that $\langle Tx,y\rangle = \langle x,T^*y\rangle$, for any $x \in H_1$ and $y \in H_2$. We know that any element T of $\mathscr{L}(H_1,H_2)$ is a bounded linear operator. We call $\mathscr{L}(H_1,H_2)$ the set of adjointable operators from H_1 into H_2. In case $H_1 = H_2$, $\mathscr{L}(H_1,H_1)$ which we abbreviate to $\mathscr{L}(H_1)$, is a C^*-algebra and we use the notation I to denote the identity operator. We write $\mathscr{R}(A)$ and $\mathscr{N}(A)$ for the range and null space of $A \in \mathscr{L}(H_1,H_2)$. An operator $A \in \mathscr{L}(H_1,H_2)$ is regular if there is an operator $A^- \in \mathscr{L}(H_2,H_1)$ such that $AA^-A = A$, A^- is called an inner inverse of A. It is well known that A is regular if and only if $\mathscr{R}(A)$ and $\mathscr{N}(A)$, respectively, are closed and complemented subspaces of H_2 and H_1.

The Moore-Penrose inverse of $A \in \mathscr{L}(H_1,H_2)$ is defined as the operator $A^\dagger \in \mathscr{L}(H_2,H_1)$ satisfying the Penrose equations

$$AA^\dagger A = A, A^\dagger AA^\dagger = A^\dagger, (A^\dagger A)^* = A^\dagger A, (AA^\dagger)^* = AA^\dagger.$$

For simplicity, we use L_A and R_A to stand for the projector $I - A^\dagger A$ and $I - AA^\dagger$, respectively.

An operator $A \in \mathscr{L}(H_1,H_2)$ has the (unique) Moore-Penrose inverse if and only if A has closed range, or equivalently if and only if it is regular.

By [[14], Theorem 3.2, Remark 1.1], we have the following lemma.

LEMMA 2.1. *The closeness of any one of the following sets implies the closeness of the remaining three sets* $\mathscr{R}(A), \mathscr{R}(A^*), \mathscr{R}(AA^*), \mathscr{R}(A^*A)$. *If* $\mathscr{R}(A)$ *is closed, then* $\mathscr{R}(A) = \mathscr{R}(AA^*)$, $\mathscr{R}(A^*) = \mathscr{R}(A^*A)$ *and the following orthogonal decompositions holds:*

$$H_1 = \mathscr{N}(A) \oplus \mathscr{R}(A^*), H_2 = \mathscr{R}(A) \oplus \mathscr{N}(A^*).$$

Since A is regular, it follows that A has the following matrix form:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A^*) \\ \mathscr{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{bmatrix},$$

where $A_1 : \mathscr{R}(A^*) \to \mathscr{R}(A)$ is invertible. In this case, the Moore-Penrose inverse of A has the following matrix decomposition:

$$A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(A^*) \\ \mathscr{N}(A) \end{bmatrix}.$$

For other important properties of operators and generalized inverses of operators see [15]–[18].

3. The system of adjointable operator equations (1.5)

In this section, we present necessary and sufficient conditions for the existence of the general solution of (1.5), and give an expression for the general solution to the system when the solvability conditions are met. We begin with the following lemma. The proof is analogous to operator equation [5], which is omitted here.

LEMMA 3.1. *Let* $A_1 \in \mathscr{L}(H_1,H_2)$, $B_2 \in \mathscr{L}(H_4,H_3)$ *have closed range, and let* $C_1 \in \mathscr{L}(H_3,H_2)$, $C_2 \in \mathscr{L}(H_4,H_1)$. *Then the system of adjointable operator equations*

$$A_1X = C_1, \quad XB_2 = C_2 \tag{3.1}$$

is consistent if and only if

$$R_{A_1}C_1 = 0, \quad C_2L_{B_2} = 0, \quad A_1C_2 = C_1B_2.$$

*In that case, the general solution of (*3.1*) is*

$$X = A_1^{\dagger}C_1 + L_{A_1}C_2B_2^{\dagger} + L_{A_1}YR_{B_2},$$

where $Y \in \mathscr{L}(H_3,H_1)$ *is arbitrary.*

Next lemma is due to Cvetković-Ilić [2], which can be generalized to the Hilbert C^*-modules.

LEMMA 3.2. (Corollary 3.1 in [2]) *Let* $A \in \mathscr{L}(H_1,H_2)$, $B \in \mathscr{L}(H_2,H_1)$ *and* $C \in \mathscr{L}(H_2)$. *Suppose that* B *is invertible and* $D = A^*B^{-1}$ *is regular. Then the equation (*1.3*) has a solution* $X \in \mathscr{L}(H_1)$ *if and only if* $C = C^*$, $L_DEL_D = 0$, *where* $E = (B^*)^{-1}CB^{-1}$. *In this case, the general solution of equation (*1.3*) can be expressed as*

$$X = \frac{1}{2}(D^*)^{\dagger}E + \frac{1}{2}(D^*)^{\dagger}EL_D + (Z - Z^*)D + R_DW,$$

where $Z \in \mathscr{L}(H_1)$ *and* $W \in \mathscr{L}(H_1)$ *are arbitrary.*

For the simplicity, we put

$$K_1 = (E_1^*)^{\dagger}(B^{-1})^*DB^{-1} + (E_1^*)^{\dagger}(B^{-1})^*DB^{-1}L_{E_2},$$
$$K_2 = [(B^{-1})^*DB^{-1}E_2^{\dagger} + L_{E_1}(B^{-1})^*DB^{-1}E_2^{\dagger}]^*.$$

LEMMA 3.3. *Let* $A \in \mathscr{L}(H_1,H_2)$, $B \in \mathscr{L}(H_2,H_1)$, $C \in \mathscr{L}(H_2,H_1)$ *and* $D \in \mathscr{L}(H_2)$. *Assume that* B *is invertible and* $E_1 = A^*B^{-1}$, $E_2 = CB^{-1}$ *are regular, then operator equation*

$$AXB + B^*X^*C = D \tag{3.2}$$

has a solution $X \in \mathscr{L}(H_1)$ *if*

$$AK_1B + B^*K_2^*C = 2D, \;\; AK_2B + B^*K_1^*C = 2D. \tag{3.3}$$

In this case the general solution of the equation (3.2) *can be expressed by*

$$X = \frac{1}{4}K_1 + \frac{1}{4}K_2 + (Z_1 - Z_2^*)E_2 + (Z_2 - Z_1^*)E_1 + R_{E_1}W_1 + R_{E_2}W_2, \tag{3.4}$$

where $W_1 \in \mathscr{L}(H_1), W_2 \in \mathscr{L}(H_1), Z_1 \in \mathscr{L}(H_1), Z_2 \in \mathscr{L}(H_1)$ *satisfy*

$$(Z_2 - Z_1^*)E_1 + R_{E_2}W_2 - (Z_1 - Z_2^*)E_2 - R_{E_1}W_1 = \frac{1}{4}(K_1 - K_2).$$

Proof. Suppose (3.3) is satisfied. Taking $W_1 = W_2 = Z_1 = Z_2 = 0$, we have the operator X defined by (3.4) is a solution of the operator equation (3.2).

Now assume (3.2) has a solution $\bar{X} \in \mathscr{L}(H_1)$, we want to show that it can be expressed as (3.4). Let

$$\widehat{A} = \begin{bmatrix} A & 0 \\ 0 & C^* \end{bmatrix} : H_1 \oplus H_1 \to H_2 \oplus H_2, \quad \widehat{X} = \begin{bmatrix} 0 & \bar{X} \\ \bar{X} & 0 \end{bmatrix} : H_1 \oplus H_1 \to H_1 \oplus H_1,$$

$$\widehat{B} = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} : H_2 \oplus H_2 \to H_1 \oplus H_1, \quad \widehat{D} = \begin{bmatrix} 0 & D \\ D^* & 0 \end{bmatrix} : H_2 \oplus H_2 \to H_2 \oplus H_2.$$

Then,

$$\widehat{D}^* = \begin{bmatrix} 0 & D \\ D^* & 0 \end{bmatrix} = \widehat{D}. \tag{3.5}$$

Put $E = \widehat{A}^*\widehat{B}^{-1}, F = (\widehat{B}^*)^{-1}\widehat{D}\widehat{B}^{-1}$, then

$$L_EFL_E = \begin{bmatrix} 0 & L_{E_1}(B^*)^{-1}DB^{-1}L_{E_2} \\ L_{E_2}(B^*)^{-1}D^*B^{-1}L_{E_1} & 0 \end{bmatrix} = 0. \tag{3.6}$$

By $AXB + B^*X^*C = D$,

$$\begin{aligned} \widehat{A}\widehat{X}\widehat{B} + \widehat{B}^*\widehat{X}^*\widehat{A}^* &= \begin{bmatrix} A & 0 \\ 0 & C^* \end{bmatrix} \begin{bmatrix} 0 & \bar{X} \\ \bar{X} & 0 \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} + \begin{bmatrix} B^* & 0 \\ 0 & B^* \end{bmatrix} \begin{bmatrix} 0 & \bar{X}^* \\ \bar{X}^* & 0 \end{bmatrix} \begin{bmatrix} A^* & 0 \\ 0 & C \end{bmatrix} \\ &= \begin{bmatrix} 0 & A\bar{X}B + B^*\bar{X}^*C \\ C^*\bar{X}B + B^*\bar{X}^*A^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & D \\ D^* & 0 \end{bmatrix} = \widehat{D}. \end{aligned} \tag{3.7}$$

By (3.5) and (3.6), we know the operator equation (3.7) is consistent. It follows from Lemma 3.2 that

$$\widehat{X} = \frac{1}{2}(E^*)^\dagger F + \frac{1}{2}(E^*)^\dagger FL_E + (\widehat{Z} - \widehat{Z}^*)E + R_E\widehat{W}, \tag{3.8}$$

where $\widehat{Z} \in \mathscr{L}(H_1 \oplus H_1)$ and $\widehat{W} \in \mathscr{L}(H_1 \oplus H_1)$ are arbitrary.

Let

$$\widehat{Z}=\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} : H_1\oplus H_1 \to H_1\oplus H_1, \quad \widehat{W}=\begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} : H_1\oplus H_1 \to H_1\oplus H_1.$$

Then by (3.8),

$$\widehat{X}=\begin{bmatrix} (Z_{11}-Z_{11}^*)E_1 & \frac{1}{2}K_1+(Z_{12}-Z_{21}^*)E_2+R_{E_1}W_{12} \\ \frac{1}{2}K_2+(Z_{21}-Z_{12}^*)E_1+R_{E_2}W_{21} & (Z_{22}-Z_{22}^*)E_2 \end{bmatrix},$$

implying

$$\bar{X}=\frac{1}{2}K_1+(Z_{12}-Z_{21}^*)E_2+R_{E_1}W_{12}=\frac{1}{2}K_2+(Z_{21}-Z_{12}^*)E_1+R_{E_2}W_{21},$$

$$(Z_{11}-Z_{11}^*)E_1=(Z_{22}-Z_{22}^*)E_2=0.$$

For $Z_1=Z_{12}, Z_2=Z_{21}, W_1=W_{12}, W_2=W_{21}$, we have

$$\bar{X}=\frac{1}{2}K_1+(Z_1-Z_2^*)E_2+R_{E_1}W_1=\frac{1}{2}K_2+(Z_2-Z_1^*)E_1+R_{E_2}W_2.$$

Hence, $\bar{X}$ can be expressed as

$$\bar{X}=\frac{1}{4}K_1+\frac{1}{4}K_2+(Z_1-Z_2^*)E_2+(Z_2-Z_1^*)E_1+R_{E_1}W_1+R_{E_2}W_2,$$

where $W_1\in\mathscr{L}(H_1), W_2\in\mathscr{L}(H_1), Z_1\in\mathscr{L}(H_1), Z_2\in\mathscr{L}(H_1)$ satisfy

$$(Z_2-Z_1^*)E_1+R_{E_2}W_2-(Z_1-Z_2^*)E_2-R_{E_1}W_1=\frac{1}{4}(K_1-K_2). \qquad \square$$

Now, we turn our attention to consider the system of adjointable operator equations (1.5), let $A_4=A_3L_{A_1}$, $B_4=R_{B_2}B_3$, $C_4=L_{A_1}C_3$, $E_1=A_4^*B_4^\dagger$, $E_2=C_4B_4^\dagger$, $E_3=A_{41}^*B_{41}^{-1}$, $E_4=C_{41}B_{41}^{-1}$ and

$$\begin{aligned}
D_4&=D_3-A_3(A_1^\dagger D_1+L_{A_1}D_2B_2^\dagger)B_3-B_3^*(A_1^\dagger D_1+L_{A_1}D_2B_2^\dagger)^*C_3,\\
K_1&=(E_1^*)^\dagger(B_4^\dagger)^*D_4B_4^\dagger+(E_1^*)^\dagger(B_4^\dagger)^*D_4B_4^\dagger L_{E_2},\\
K_2&=[(B_4^\dagger)^*D_4B_4^\dagger E_2^\dagger+L_{E_1}(B_4^\dagger)^*D_4B_4^\dagger E_2^\dagger]^*,\\
K_3&=(E_3^*)^\dagger(B_{41}^{-1})^*D_{41}B_{41}^{-1}+(E_3^*)^\dagger(B_{41}^{-1})^*D_{41}B_{41}^{-1}L_{E_4},\\
K_4&=[(B_{41}^{-1})^*D_{41}B_{41}^{-1}E_4^\dagger+L_{E_3}(B_{41}^{-1})^*D_{41}B_{41}^{-1}E_4^\dagger]^*.
\end{aligned}$$

We now give the main theorem of this paper as follows.

THEOREM 3.4. *Assume that* $A_1\in\mathscr{L}(H_1,H_2)$, $B_2\in\mathscr{L}(H_3,H_1)$, $A_3\in\mathscr{L}(H_1,H_4)$, $B_3\in\mathscr{L}(H_4,H_1)$, $C_3\in\mathscr{L}(H_4,H_1)$, $D_1\in\mathscr{L}(H_1,H_2)$, $D_2\in\mathscr{L}(H_3,H_1)$, $D_3\in\mathscr{L}(H_4)$, *and let* A_1, B_2, A_4, B_4, C_4, E_1, E_2 *have closed ranges such that*

$$B_4^\dagger B_4A_4=A_4B_4B_4^\dagger,\ \ B_4^\dagger B_4C_4=C_4B_4^\dagger B_4, \tag{3.9}$$

$$A_4K_1B_4 + B_4^*K_2^*C_4 = 2D_4, \;\; A_4K_2B_4 + B_4^*K_1^*C_4 = 2D_4, \tag{3.10}$$

$$L_{B_4}R_{A_4}D_4 = 0, \;\; L_{B_4}R_{C_4^*}D_4^* = 0, \tag{3.11}$$

$$R_{B_4}A_4^\dagger D_4 = R_{B_4}(C_4^*)^\dagger D_4^*, \;\; R_{B_4}L_{A_4} = R_{B_4}L_{C_4^*}. \tag{3.12}$$

*Then the system of adjointable operator equations (*1.5*) is consistent if and only if*

$$R_{A_1}D_1 = 0, \quad D_2L_{B_2} = 0, \quad A_1D_2 = D_1B_2, \tag{3.13}$$

$$L_{B_4}D_4L_{B_4} = 0. \tag{3.14}$$

*In that case, the general solution of (*1.5*) can be expressed as*

$$\begin{aligned} X = {} & A_1^\dagger D_1 + L_{A_1}D_2B_2^\dagger + L_{A_1}[\frac{1}{4}K_1 + \frac{1}{4}K_2 + B_4B_4^\dagger(Z_1 - Z_2^*)E_2 + B_4B_4^\dagger(Z_2 - Z_1^*)E_1 \\ & + B_4B_4^\dagger R_{E_1}W_1B_4B_4^\dagger + B_4B_4^\dagger R_{E_2}W_2B_4B_4^\dagger + R_{B_4}A_4^\dagger D_4B_4^\dagger + R_{B_4}L_{A_4}VB_4B_4^\dagger + UR_{B_4}]R_{B_2}, \end{aligned} \tag{3.15}$$

where $U \in \mathscr{L}(H_1), V \in \mathscr{L}(H_1)$ are arbitrary and $W_1 \in \mathscr{L}(H_1), W_2 \in \mathscr{L}(H_1)$, $Z_1 \in \mathscr{L}(H_1)$, $Z_2 \in \mathscr{L}(H_1)$ satisfy

$$\begin{aligned} & B_4B_4^\dagger(Z_2 - Z_1^*)E_1 + B_4B_4^\dagger R_{E_2}W_2B_4B_4^\dagger - B_4B_4^\dagger(Z_1 - Z_2^*)E_2 - B_4B_4^\dagger R_{E_1}W_1B_4B_4^\dagger \\ & = \frac{1}{4}(K_1 - K_2). \end{aligned}$$

Proof. Suppose that the system of adjointable operator equations (1.5) has a solution X, then X is a solution to the system of adjointable operator equations

$$A_1X = D_1, \quad XB_2 = D_2, \tag{3.16}$$

therefore (3.13) follows from Lemma 3.1. Note that X is a solution to the system of adjointable operator equations (3.16), then X can be expressed as

$$X = A_1^\dagger D_1 + L_{A_1}D_2B_2^\dagger + L_{A_1}YR_{B_2}, \tag{3.17}$$

where $Y \in \mathscr{L}(H_1)$ is arbitrary. Taking (3.17) into

$$A_3XB_3 + B_3^*X^*C_3 = D_3, \tag{3.18}$$

we have that

$$A_4YB_4 + B_4^*Y^*C_4 = D_4 \tag{3.19}$$

and (3.19) is consistent. It can be verified that

$$L_{B_4}D_4L_{B_4} = (I - B_4^\dagger B_4)(A_4XB_4 + B_4^*X^*C_4)(I - B_4^\dagger B_4) = 0.$$

Suppose (3.13) and (3.14) are satisfied. By Lemma 3.1, (3.16) is consistent. Suppose $\bar{X}$ is a general solution of (3.16). Note that

$$L_{A_1}\bar{X} = \bar{X} - A_1^\dagger A_1\bar{X} = \bar{X} - A_1^\dagger D_1.$$

So,

$$\begin{aligned}L_{A_1}\bar{X}R_{B_2} &= \bar{X} - \bar{X}B_2B_2^{\dagger} - A_1^{\dagger}D_1 + A_1^{\dagger}D_1B_2B_2^{\dagger}\\ &= \bar{X} - D_2B_2^{\dagger} - A_1^{\dagger}D_1 + A_1^{\dagger}A_1D_2B_2^{\dagger}.\end{aligned}$$

Thereby,

$$\bar{X} = A_1^{\dagger}D_1 + D_2B_2^{\dagger} - A_1^{\dagger}A_1D_2B_2^{\dagger} + L_{A_1}\bar{X}R_{B_2}. \tag{3.20}$$

Taking (3.20) into (3.18), we can get

$$A_4\bar{X}B_4 + B_4^*\bar{X}^*C_4 = D_4. \tag{3.21}$$

Using the following decompositions:

$$H_4 = \mathscr{R}(B_4^*) \oplus \mathscr{N}(B_4) \ \ and \ \ H_1 = \mathscr{R}(B_4) \oplus \mathscr{N}(B_4^*),$$

by regularity of B_4,

$$B_4 = \begin{bmatrix} B_{41} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(B_4^*) \\ \mathscr{N}(B_4) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(B_4) \\ \mathscr{N}(B_4^*) \end{bmatrix},$$

where $B_{41} : \mathscr{R}(B_4^*) \to \mathscr{R}(B_4)$ is invertible.

In this case, the Moore-Penrose inverse of B_4 has the following matrix decomposition:

$$B_4^{\dagger} = \begin{bmatrix} B_{41}^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(B_4) \\ \mathscr{N}(B_4^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(B_4^*) \\ \mathscr{N}(B_4) \end{bmatrix}.$$

Also, A_4, C_4 and D_4 have the following suitable decompositions:

$$A_4 = \begin{bmatrix} A_{41} & A_{42} \\ A_{43} & A_{44} \end{bmatrix} : \begin{bmatrix} \mathscr{R}(B_4) \\ \mathscr{N}(B_4^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(B_4^*) \\ \mathscr{N}(B_4) \end{bmatrix},$$

$$C_4 = \begin{bmatrix} C_{41} & C_{42} \\ C_{43} & C_{44} \end{bmatrix} : \begin{bmatrix} \mathscr{R}(B_4^*) \\ \mathscr{N}(B_4) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(B_4) \\ \mathscr{N}(B_4^*) \end{bmatrix},$$

$$D_4 = \begin{bmatrix} D_{41} & D_{42} \\ D_{43} & D_{44} \end{bmatrix} : \begin{bmatrix} \mathscr{R}(B_4^*) \\ \mathscr{N}(B_4) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(B_4^*) \\ \mathscr{N}(B_4) \end{bmatrix}.$$

It follows from (3.9) that $A_{42} = A_{43} = 0$ and $C_{42} = C_{43} = 0$. Therefore, the Moore-Penrose inverses of A_4 and C_4 have the following matrix decompositions:

$$A_4^{\dagger} = \begin{bmatrix} A_{41}^{\dagger} & 0 \\ 0 & A_{44}^{\dagger} \end{bmatrix} : \begin{bmatrix} \mathscr{R}(B_4^*) \\ \mathscr{N}(B_4) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(B_4) \\ \mathscr{N}(B_4^*) \end{bmatrix},$$

$$C_4^{\dagger} = \begin{bmatrix} C_{41}^{\dagger} & 0 \\ 0 & C_{44}^{\dagger} \end{bmatrix} : \begin{bmatrix} \mathscr{R}(B_4) \\ \mathscr{N}(B_4^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(B_4^*) \\ \mathscr{N}(B_4) \end{bmatrix}.$$

By computation we obtain that

$$L_{B_4}D_4L_{B_4} = \begin{bmatrix} 0 & 0 \\ 0 & D_{44} \end{bmatrix},$$

that is, $D_{44}=0$. Then for

$$\bar{X}=\begin{bmatrix}\bar{X}_1 & \bar{X}_2\\ \bar{X}_3 & \bar{X}_4\end{bmatrix}:\begin{bmatrix}\mathscr{R}(B_4)\\ \mathscr{N}(B_4^*)\end{bmatrix}\to\begin{bmatrix}\mathscr{R}(B_4)\\ \mathscr{N}(B_4^*)\end{bmatrix},$$

we can derive

$$A_4\bar{X}B_4+B_4^*\bar{X}^*C_4=\begin{bmatrix}A_{41}\bar{X}_1B_{41}+B_{41}^*\bar{X}_1^*C_{41} & B_{41}^*\bar{X}_3^*C_{44}\\ A_{44}\bar{X}_3B_{41} & 0\end{bmatrix}=\begin{bmatrix}D_{41} & D_{42}\\ D_{43} & 0\end{bmatrix},$$

which is equivalent to

$$A_{41}\bar{X}_1B_{41}+B_{41}^*\bar{X}_1^*C_{41}=D_{41}, \tag{3.22}$$

$$C_{44}^*\bar{X}_3B_{41}=D_{42}^*, \tag{3.23}$$

$$A_{44}\bar{X}_3B_{41}=D_{43}. \tag{3.24}$$

It follows from (3.10) that

$$A_{41}K_3B_{41}+B_{41}^*K_4^*C_{41}=2D_{41},\ \ A_{41}K_4B_{41}+B_{41}^*K_3^*C_{41}=2D_{41}.$$

By Lemma 3.3, the equation (3.22) is solvable. Taking the following decompositions

$$W_1=\begin{bmatrix}W_{11} & W_{12}\\ W_{13} & W_{14}\end{bmatrix}:\begin{bmatrix}\mathscr{R}(B_4)\\ \mathscr{N}(B_4^*)\end{bmatrix}\to\begin{bmatrix}\mathscr{R}(B_4)\\ \mathscr{N}(B_4^*)\end{bmatrix},$$
$$W_2=\begin{bmatrix}W_{21} & W_{22}\\ W_{23} & W_{24}\end{bmatrix}:\begin{bmatrix}\mathscr{R}(B_4)\\ \mathscr{N}(B_4^*)\end{bmatrix}\to\begin{bmatrix}\mathscr{R}(B_4)\\ \mathscr{N}(B_4^*)\end{bmatrix},$$

$$Z_1=\begin{bmatrix}Z_{11} & Z_{12}\\ Z_{13} & Z_{14}\end{bmatrix}:\begin{bmatrix}\mathscr{R}(B_4)\\ \mathscr{N}(B_4^*)\end{bmatrix}\to\begin{bmatrix}\mathscr{R}(B_4)\\ \mathscr{N}(B_4^*)\end{bmatrix},$$
$$Z_2=\begin{bmatrix}Z_{21} & Z_{22}\\ Z_{23} & Z_{24}\end{bmatrix}:\begin{bmatrix}\mathscr{R}(B_4)\\ \mathscr{N}(B_4^*)\end{bmatrix}\to\begin{bmatrix}\mathscr{R}(B_4)\\ \mathscr{N}(B_4^*)\end{bmatrix},$$

the solution of (3.22) can be expressed as

$$\bar{X}_1=\frac{1}{4}K_3+\frac{1}{4}K_4+(Z_{11}-Z_{21}^*)E_4+(Z_{21}-Z_{11}^*)E_3+R_{E_3}W_{11}+R_{E_4}W_{21},$$

where $W_{11}\in\mathscr{L}(\mathscr{R}(B_4))$, $W_{21}\in\mathscr{L}(\mathscr{R}(B_4))$, $Z_{11}\in\mathscr{L}(\mathscr{R}(B_4))$, $Z_{21}\in\mathscr{L}(\mathscr{R}(B_4))$ satisfies

$$(Z_{21}-Z_{11}^*)E_3+R_{E_4}W_{21}-(Z_{11}-Z_{21}^*)E_4-R_{E_3}W_{11}=\frac{1}{4}(K_3-K_4).$$

By (3.11),

$$A_{44}A_{44}^{\dagger}D_{43}=D_{43},\quad C_{44}^*(C_{44}^*)^{\dagger}D_{42}^*=D_{42}^*,$$

implying the equation (3.23) and (3.24) are solvable. Taking the following decompositions:

$$Y=\begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}:\begin{bmatrix} \mathscr{R}(B_4) \\ \mathscr{N}(B_4^*) \end{bmatrix}\rightarrow\begin{bmatrix} \mathscr{R}(B_4) \\ \mathscr{N}(B_4^*) \end{bmatrix},$$
$$V=\begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}:\begin{bmatrix} \mathscr{R}(B_4) \\ \mathscr{N}(B_4^*) \end{bmatrix}\rightarrow\begin{bmatrix} \mathscr{R}(B_4) \\ \mathscr{N}(B_4^*) \end{bmatrix}$$

yields that the solution of (3.23) and (3.24) can be expressed as

$$\bar{X}_3=A_{44}^{\dagger}D_{43}B_{41}^{-1}+L_{A_{44}}Y_3=(C_{44}^*)^{\dagger}D_{42}^*B_{41}^{-1}+L_{C_{44}^*}V_3,$$

where $Y_3\in\mathscr{L}(\mathscr{R}(B_4),\ \mathscr{N}(B_4^*))$ and $V_3\in\mathscr{L}(\mathscr{R}(B_4),\ \mathscr{N}(B_4^*))$ are arbitrary. By (3.12),

$$A_{44}^{\dagger}D_{43}=(C_{44}^*)^{\dagger}D_{42}^*, L_{A_{44}}=L_{C_{44}^*}.$$

Hence,

$$\bar{X}_3=A_{44}^{\dagger}D_{43}B_{41}^{-1}+L_{A_{44}}Y_3,$$

where $Y_3\in\mathscr{L}(\mathscr{R}(B_4),\ \mathscr{N}(B_4^*))$ is arbitrary.

Let

$$U=\begin{bmatrix} U_1 & X_2 \\ U_3 & X_4 \end{bmatrix}:\begin{bmatrix} \mathscr{R}(B) \\ \mathscr{N}(B^*) \end{bmatrix}\rightarrow\begin{bmatrix} \mathscr{R}(B) \\ \mathscr{N}(B^*) \end{bmatrix}.$$

Then, by computation, we can derive that $\bar{X}$ can be expressed as

$$\begin{aligned}\bar{X}=&\frac{1}{4}K_1+\frac{1}{4}K_2+B_4B_4^{\dagger}(Z_1-Z_2^*)E_2+B_4B_4^{\dagger}(Z_2-Z_1^*)E_1+B_4B_4^{\dagger}R_{E_1}W_1B_4B_4^{\dagger}\\&+B_4B_4^{\dagger}R_{E_2}W_2B_4B_4^{\dagger}+R_{B_4}A_4^{\dagger}D_4B_4^{\dagger}+R_{B_4}L_{A_4}VB_4B_4^{\dagger}+UR_{B_4},\end{aligned}\tag{3.25}$$

where $U\in\mathscr{L}(H_1)$, $V\in\mathscr{L}(H_1)$ are arbitrary and $W_1\in\mathscr{L}(H_1)$, $W_2\in\mathscr{L}(H_1)$, $Z_1\in\mathscr{L}(H_1)$, $Z_2\in\mathscr{L}(H_1)$ satisfy

$$\begin{aligned}B_4B_4^{\dagger}(Z_2-Z_1^*)E_1+B_4B_4^{\dagger}R_{E_2}W_2B_4B_4^{\dagger}-B_4B_4^{\dagger}(Z_1-Z_2^*)E_2-B_4B_4^{\dagger}R_{E_1}W_1B_4B_4^{\dagger}\\=\frac{1}{4}(K_1-K_2).\end{aligned}$$

Taking (3.25) into (3.20), we know that any solution to the system of adjointable operator equations (1.5) can be expressed as (3.15). □

4. Applications

In this section, we first consider some special cases of system (1.5) to show that some known results can be recovered from the results of this paper. Then we present the solvability conditions and an expression of the general anti-reflexive Hermitian solution to (1.6) by using the results of (1.5).

Supposing that $C_1 = D_1$, $C_2 = D_2$, $C_3 = A_3^*$ in Theorem 3.4, we can get the corresponding results to the following system of adjointable operator equations

$$A_1X = C_1, XB_2 = C_2, A_3XB_3 + B_3^*X^*A_3^* = C_3. \tag{4.1}$$

Put

$$\begin{aligned} &A_4 = A_3L_{A_1}, B_4 = R_{B_2}B_3, G = A_4^*B_4^\dagger, \\ &C_4 = C_3 - A_3(A_1^\dagger C_1 + L_{A_1}C_2B_2^\dagger)B_3 - B_3^*(A_1^\dagger C_1 + L_{A_1}C_2B_2^\dagger)^*A_3^*, \\ &K = (G^*)^\dagger(B_4^\dagger)^*C_4B_4^\dagger + (G^*)^\dagger(B_4^\dagger)^*C_4B_4^\dagger L_G. \end{aligned} \tag{4.2}$$

Then we have the following.

COROLLARY 4.1. *Let $A_1 \in \mathscr{L}(H_1,H_2)$, $B_2 \in \mathscr{L}(H_3,H_1)$, $A_3 \in \mathscr{L}(H_1,H_4)$, $B_3 \in \mathscr{L}(H_4,H_1)$, $C_1 \in \mathscr{L}(H_1,H_2)$, $C_2 \in \mathscr{L}(H_3,H_1)$, $C_3 \in \mathscr{L}(H_4)$, and let A_1, B_2, A_4, B_4, G have closed ranges such that*

$$B_4^\dagger B_4A_4 = A_4B_4B_4^\dagger, \;\; A_4KB_4 + B_4^*K^*A_4^* = 2C_4, \;\; L_{B_4}R_{A_4}C_4 = 0. \tag{4.3}$$

Then the system of adjointable operator equations (4.1) is consistent if and only if

$$R_{A_1}C_1 = 0, \quad C_2L_{B_2} = 0, \quad A_1C_2 = C_1B_2, \quad C_4 = C_4^*, \quad L_{B_4}C_4L_{B_4} = 0. \tag{4.4}$$

In that case, the general solution of (4.1) can be expressed as

$$\begin{aligned} X =&A_1^\dagger C_1 + L_{A_1}C_2B_2^\dagger + L_{A_1}[\frac{1}{2}K + B_4B_4^\dagger(Z - Z^*)G + B_4B_4^\dagger R_GWB_4B_4^\dagger + R_{B_4}A_4^\dagger C_4B_4^\dagger \\ &+ R_{B_4}L_{A_4}YB_4B_4^\dagger + UR_{B_4}]R_{B_2}, \end{aligned} \tag{4.5}$$

where $Y \in \mathscr{L}(H_1)$, $U \in \mathscr{L}(H_1)$ is arbitrary and $W \in \mathscr{L}(H_1)$, $Z \in \mathscr{L}(H_1)$ satisfy

$$B_4B_4^\dagger(Z + Z^*)G + B_4B_4^\dagger R_GWB_4B_4^\dagger = 0. \tag{4.6}$$

In Theorem 3.4, letting A_1, D_1, B_2, D_2 vanish, and $A_3 = A$, $B_3 = B$, $C_3 = C$, $D_3 = D$, we can present the solvability conditions and an expression of the general solution of the adjointable operator equation (3.2). For simplicity, we assume that K_1, K_2 are defined as

$$\begin{aligned} K_1 &= (E_1^*)^\dagger(B^\dagger)^*DB^\dagger + (E_1^*)^\dagger(B^\dagger)^*DB^\dagger L_{E_2}, \\ K_2 &= [(B^\dagger)^*DB^\dagger E_2^\dagger + L_{E_1}(B^\dagger)^*DB^\dagger E_2^\dagger]^*. \end{aligned}$$

COROLLARY 4.2. *Let $A \in \mathscr{L}(H_1,H_2)$, $B \in \mathscr{L}(H_2,H_1)$, $C \in \mathscr{L}(H_2,H_1)$ and $D \in \mathscr{L}(H_2)$. Suppose that A, B, C, $E_1 = A^*B^\dagger$, $E_2 = CB^\dagger$ are regular and*

$$B^\dagger BA = ABB^\dagger, \;\; B^\dagger BC = CB^\dagger B,$$

$$AK_1B + B^*K_2^*C = 2D, \;\; AK_2B + B^*K_1^*C = 2D,$$

$$L_BR_AD = 0, \ \ L_BR_{C^*}D^* = 0,$$

$$R_BA^{\dagger}D = R_B(C^*)^{\dagger}D^*, \quad R_BL_A = R_BL_{C^*}.$$

Then the operator equation (3.2) *has a solution* $X \in \mathscr{L}(H_1)$ *if and only if* $L_BDL_B = 0$. *In this case, the general solution of the equation* (3.2) *can be expressed by*

$$X = \frac{1}{4}K_1 + \frac{1}{4}K_2 + BB^{\dagger}(Z_1 - Z_2^*)E_2 + BB^{\dagger}(Z_2 - Z_1^*)E_1 + BB^{\dagger}R_{E_1}W_1BB^{\dagger} + BB^{\dagger}R_{E_2}W_2BB^{\dagger} + R_BA^{\dagger}DB^{\dagger} + R_BL_AYBB^{\dagger} + UR_B,$$

where $Y \in \mathscr{L}(H_1)$, $U \in \mathscr{L}(H_1)$ *are arbitrary and* $W_1 \in \mathscr{L}(H_1)$, $W_2 \in \mathscr{L}(H_1)$, $Z_1 \in \mathscr{L}(H_1)$, $Z_2 \in \mathscr{L}(H_1)$ *satisfy*

$$BB^{\dagger}(Z_2 - Z_1^*)E_1 + BB^{\dagger}R_{E_2}W_2BB^{\dagger} - BB^{\dagger}(Z_1 - Z_2^*)E_2 - BB^{\dagger}R_{E_1}W_1BB^{\dagger} = \frac{1}{4}(K_1 - K_2).$$

REMARK 4.1. Theorem 3.4, Corollary 4.1 and Corollary 4.2 are also new for finite dimension spaces.

In Theorem 3.4, suppose that A_1, D_1, B_2, D_2 vanish, and $A_3 = A$, $B_3 = B$, $C_3 = A^*$, $D_3 = C$, then we can obtain Theorem 2.1 in [3] and Theorem 3.1 in [2] as follows.

COROLLARY 4.3. *Let* $A \in \mathscr{L}(H_1,H_2)$, $B \in \mathscr{L}(H_2,H_1)$ *and* $C \in \mathscr{L}(H_2)$. *Assume that* A, B *and* $D = A^*B^{\dagger}$ *are regular and*

$$B^{\dagger}BA = ABB^{\dagger}, \quad AKB + B^*K^*A^* = 2C, \quad L_BR_AC = 0,$$

where $K = (D^*)^{\dagger}(B^{\dagger})^*CB^{\dagger} + (D^*)^{\dagger}(B^{\dagger})^*CB^{\dagger}L_D$. *Then the operator equation* (1.3) *has a solution* $X \in \mathscr{L}(H_1)$ *if and only if* $C = C^*$, $L_BCL_B = 0$. *In this case, the general solution of the equation* (1.3) *can be expressed by*

$$X = \frac{1}{2}K + BB^{\dagger}(Z - Z^*)D + BB^{\dagger}R_DWBB^{\dagger} + R_BA^{\dagger}CB^{\dagger} + R_BL_AYBB^{\dagger} + UR_B,$$

where $Y \in \mathscr{L}(H_1)$, $U \in \mathscr{L}(H_1)$ *are arbitrary and* $W \in \mathscr{L}(H_1)$, $Z \in \mathscr{L}(H_1)$ *satisfy*

$$BB^{\dagger}(Z + Z^*)D + BB^{\dagger}R_DWBB^{\dagger} = 0.$$

In Theorem 3.4, assuming that A_1, D_1, B_2, D_2 vanish and $A_3 = A$, $B_3 = I$, $C_3 = C$, $D_3 = B$, we can get Theorem 2.2 in [2].

COROLLARY 4.4. *Let* $A \in \mathscr{L}(H_1,H_2)$, $C \in \mathscr{L}(H_2,H_1)$ *and* $B \in \mathscr{L}(H_2)$ *be regular and* $K_1 = A^{\dagger}B + A^{\dagger}BL_C$, $K_2 = [BC^{\dagger} + L_{A^*}BC^{\dagger}]^*$. *Then the operator equation* (1.2) *has a solution* $X \in \mathscr{L}(H_2,H_1)$ *if and only if*

$$AK_1 + K_2^*C = 2B, \quad AK_2 + K_1^*C = 2B.$$

*In this case, the general solution of the equation (*1.2*) can be expressed by*

$$X = \frac{1}{4}K_1 + \frac{1}{4}K_2 + (Z_1 - Z_2^*)C + (Z_2 - Z_1^*)A^* + R_{A^*}W_1 + R_C W_2,$$

where $W_1 \in \mathscr{L}(H_1)$, $W_2 \in \mathscr{L}(H_1), Z_1 \in \mathscr{L}(H_1)$, $Z_2 \in \mathscr{L}(H_1)$ *satisfy*

$$(Z_2 - Z_1^*)A^* + R_C W_2 - (Z_1 - Z_2^*)C - R_{A^*}W_1 = \frac{1}{4}(K_1 - K_2).$$

In Theorem 3.4, supposing that A_1, D_1, B_2, D_2 vanish and $A_3 = A^*$, $B_3 = I$, $C_3 = A$, $D_3 = B$, we can have Theorem 2.1 in [1].

COROLLARY 4.5. *Let* $A \in \mathscr{L}(H_1, H_2)$ *is regular, and* $B \in \mathscr{L}(H_2)$*. Then the operator equation (*1.1*) has a solution* $X \in \mathscr{L}(H_1, H_2)$ *if and only if*

$$B = B^*, \quad L_A B L_A = 0.$$

*In this case, the general solution of the equation (*1.1*) can be expressed by*

$$X = \frac{1}{2}A^\dagger B + \frac{1}{2}A^\dagger B L_{A^*} + (Z - Z^*)A^* + R_{A^*}W,$$

where $W \in \mathscr{L}(H_1), Z \in \mathscr{L}(H_1)$ *satisfy*

$$(Z + Z^*)A^* + R_{A^*}W = 0.$$

In Theorem 3.4, letting A_3, B_3, C_3, D_3 vanish, we obtain the same results of the general solutions to (3.1) as [4], [5] and [6].

COROLLARY 4.6. *Let* $A_1 \in \mathscr{L}(H_1, H_2)$, $B_2 \in \mathscr{L}(H_4, H_3)$ *have closed range, and let* $C_1 \in \mathscr{L}(H_3, H_2)$, $C_2 \in \mathscr{L}(H_4, H_1)$*. Then the system of adjointable operator equations (*3.1*) is consistent if and only if*

$$R_{A_1}C_1 = 0, \quad C_2 L_{B_2} = 0, \quad A_1 C_2 = C_1 B_2.$$

*In that case, the general solution of (*3.1*) is*

$$X = A_1^\dagger C_1 + C_2 B_2^\dagger - A_1^\dagger A_1 C_2 B_2^\dagger + L_{A_1} Y R_{B_2},$$

where $Y \in \mathscr{L}(H_3, H_1)$ *is arbitrary.*

REMARK 4.2. Corollary 4.3, Corollary 4.4, Corollary 4.5 and Corollary 4.6 show that Theorem 2.1 in [3], Theorem 2.2, Theorem 3.1 in [2], Theorem 2.1 in [1] and the results of the general solutions to (3.1) in [4], [5] and [6] can be recovered from Theorem 3.4 of this paper.

Now we turn our attention to use Theorem 3.4 to investigate the anti-reflexive Hermitian solution to the system of matrix equations (1.6) in the rest of this section. Many authors have investigated the reflexive and anti-reflexive solutions to linear matrix equations. For instance, the anti-reflexive solution to the matrix equation $AX = B$ was studied in [9] and the anti-reflexive solution to the system of matrix equations (1.4) was considered in [10] over the complex number field $\mathbb{C}$. The reflexive re-nonnegative definite solution to the quaternion matrix equation $EXE^* = F$ was investigated in [11].

A matrix $A \in \mathbb{C}^{n\times n}$ is called anti-reflexive (anti-reflexive Hermitian) with respect to the nontrivial generalized reflection matrix P if $A = -PAP$ ($A^* = A, A = -PAP$), where P is the nontrivial generalized reflection matrix, i.e., $P^* = P \neq I$ and $P^2 = I$. Put

$$\mathbb{C}_{ar}^{n\times n}(P) = \{A \in \mathbb{C}^{n\times n} | A = -PAP\}.$$

$$\mathbb{HC}_{ar}^{n\times n}(P) = \{A \in \mathbb{C}^{n\times n} | A^* = A, A = -PAP\}.$$

LEMMA 4.7. (Lemma 1 in [10]) *A matrix $A \in \mathbb{C}_{ar}^{n\times n}(P)$ if and only if A can be expressed as*

$$A = U \begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix} U^*,$$

where $M \in \mathbb{C}^{r\times(n-r)}$, $N \in \mathbb{C}^{(n-r)\times r}$ and U defined as $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$, $U_1^* U_2 = 0$ is unitary.

LEMMA 4.8. (Lemma 2.7 in [11]) *Suppose that $P \in \mathbb{C}^{n\times n}$ is a nontrivial generalized reflection matrix and $K = \begin{bmatrix} I+P \\ I \end{bmatrix}$, then we have the following:*

(i) K can be reduced into $K = \begin{bmatrix} N & 0 \\ \phi & M \end{bmatrix}$, where N is a full column rank matrix of size $n \times r$ and $r = rank(I+P)$, by applying a sequence of elementary column operations on K.

(ii) Perform the Gram-Schmidt process to the columns of N and M, suppose that the corresponding orthonormal matrices are U_1 and U_2.

(iii) Put $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$, then

$$P = U \begin{bmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{bmatrix} U^*.$$

REMARK 4.3. Lemma 4.8 gives a practical method to represent the unitary matrix U in Lemma 4.7.

By Lemma 4.7, we have the following.

LEMMA 4.9. *A matrix $A \in \mathbb{HC}_{ar}^{n\times n}(P)$ if and only if A can be expressed as*

$$A = U \begin{bmatrix} 0 & M \\ M^* & 0 \end{bmatrix} U^*,$$

where $M \in \mathbb{C}^{r\times(n-r)}$ and U is defined as $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$, $U_1^* U_2 = 0$ is unitary.

Now we consider the anti-reflexive Hermitian solution to the system (1.6), where $A, B \in \mathbb{C}^{(m_1+m_3)\times(n_1+n_2)}$, $C, D \in \mathbb{C}^{(n_1+n_2)\times(m_2+m_4)}$, $E \in \mathbb{C}^{m_5\times(n_1+n_2)}$, $F \in \mathbb{C}^{m_5\times m_5}$ are known and $X \in \mathbb{HC}_{ar}^{(n_1+n_2)\times(n_1+n_2)}(P)$ unknown. By Lemma 4.9, we can assume that

$$X = U \begin{bmatrix} 0 & X_1 \\ X_1^* & 0 \end{bmatrix} U^*, \tag{4.7}$$

where $X_1 \in \mathbb{C}^{n_1\times n_2}$. Suppose that

$$AU = \begin{bmatrix} A_{s1} & A_{s2} \end{bmatrix}, BU = \begin{bmatrix} B_{s1} & B_{s2} \end{bmatrix}, \tag{4.8}$$

$$U^*C = \begin{bmatrix} C_{s1} \\ C_{s2} \end{bmatrix}, U^*D = \begin{bmatrix} D_{s1} \\ D_{s2} \end{bmatrix}, \tag{4.9}$$

$$EU = \begin{bmatrix} A_3 & B_3^* \end{bmatrix}, F = C_3, \tag{4.10}$$

where $A_{s1}, B_{s1} \in \mathbb{C}^{m_1\times n_1}$, C_{s1}, $D_{s1} \in \mathbb{C}^{n_1\times m_2}$, A_{s2}, $B_{s2} \in \mathbb{C}^{m_3\times n_2}$, C_{s2}, $D_{s2} \in \mathbb{C}^{n_2\times m_4}$, $A_3 \in \mathbb{C}^{m_5\times n_1}$, $B_3 \in \mathbb{C}^{m_5\times n_2}$, $C_3 \in \mathbb{C}^{m_5\times m_5}$. Let

$$\begin{bmatrix} A_{s1} \\ C_{s1}^* \end{bmatrix} = A_1, \begin{bmatrix} B_{s2} \\ D_{s2}^* \end{bmatrix} = C_1, \tag{4.11}$$

$$\begin{bmatrix} A_{s2}^* & C_{s2} \end{bmatrix} = B_2, \begin{bmatrix} B_{s1}^* & D_{s1} \end{bmatrix} = C_2. \tag{4.12}$$

Then the system of matrix equation (1.6) has anti-reflexive Hermitian solution if and only if the system of matrix equation (4.1) is consistent. By Theorem 3.4, we have the following theorem.

THEOREM 4.10. *Let (4.3) hold. Then System (1.6) has an anti-reflexive Hermitian solution $X \in \mathbb{HC}_{ar}^{(n_1+n_2)\times(n_1+n_2)}(P)$ if and only if the equalities in (4.4) hold. In this case, the general anti-reflexive Hermitian solution to (1.6) can be expressed as (4.5).*

We now give an algorithm for finding the anti-reflexive Hermitian solution to system (1.6), and present a numerical example to illustrate our results. Base on Remark 3, Lemma 4.9 and Theorem 4.9, we propose the following algorithm for solving the anti-reflexive Hermitian solution to system (1.6).

ALGORITHM 4.1. (1) Input $A, B \in \mathbb{C}^{(m_1+m_3)\times(n_1+n_2)}$, $C, D \in \mathbb{C}^{(n_1+n_2)\times(m_2+m_4)}$, $E \in \mathbb{C}^{m_5\times(n_1+n_2)}$, $F \in \mathbb{C}^{m_5\times m_5}$ and the nontrivial generalized reflection matrix $P \in \mathbb{C}^{(n_1+n_2)\times(n_1+n_2)}$.

(2) Compute r and U by the way of Lemma 4.8.

(3) Compute $A_1, B_2, A_3, B_3, C_1, C_2, C_3$ by (4.8)–(4.12).

(4) Check whether (4.3) and (4.4) hold or not. If all hold, then go into the following.

(5) Compute X_1 by (4.5).

(6) Compute X by (4.7).

EXAMPLE 4.1. Given a generalized reflection matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and the parameter matrices of system (1.6)

$$A = \begin{bmatrix} 1-i & 2 & i & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2+i & 2+4i & 5+5i \end{bmatrix},$$

$$C = \begin{bmatrix} 0 \\ 5.35 \\ 2 \\ i \end{bmatrix}, \quad D = \begin{bmatrix} -5 \\ 2+4i \\ 5.35-10.7i \\ 0 \end{bmatrix},$$

$$E = \begin{bmatrix} 2 & 1 & 1 & 7i \\ 0 & 0 & i & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 1.42 & 0.02-0.01i \\ 0.02+0.01i & 0 \end{bmatrix}.$$

By Lemma 4.8, we obtain $r = 2$ and

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

According to (4.8)–(4.12), we derived

$$A_1 = \begin{bmatrix} 1-i & i \\ 0 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 & 5.35 \\ 0 & i \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 1 \\ 0 & i \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 & 0 \\ -7i & 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 2+i & 5+5i \\ 2-4i & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & -5 \\ 2-4i & 5.35-10.7i \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 1.42 & 0.02-0.01i \\ 0.02+0.01i & 0 \end{bmatrix}.$$

By computation, (4.3) and (4.4) holds. Using Theorem 4.10, the general solution of system (4.1) is

$$X_1 = S_1 + S_2[S_3 + S_4(Z - Z^*)S_5 + S_6WS_4 + S_7 + S_8YS_4 + US_9]S_{10},$$

where Y, U are arbitrary, Z, W satisfy

$$S_4(Z - Z^*)S_5 + S_6WS_4 = 0,$$

and

$$S_1 = \begin{bmatrix} 0 & 2+3i \\ 1-2i & 0 \end{bmatrix}, \; S_2 = \begin{bmatrix} 0.22-0.17i & -0.11+0.056i \\ 0.047+0.014i & -0.22+0.07i \end{bmatrix},$$

$$S_3 = \begin{bmatrix} -4-0.17i & 0.02i \\ 3.2-0.78i & -0.003-0.014i \end{bmatrix}, \quad S_4 = \begin{bmatrix} 1 & -0.004i \\ 0.004i & 0 \end{bmatrix},$$

$$S_5 = \begin{bmatrix} 0.074+0.05i & 0 \\ -0.07-0.03i & 0 \end{bmatrix}, \quad S_6 = \begin{bmatrix} 0.4+0.002i & 0.48+0.09i \\ 0.002i & 0.002i \end{bmatrix},$$

$$S_7 = \begin{bmatrix} 0.012-0.004i & 0 \\ 0.089-2.79i & 0.012+0.004i \end{bmatrix}, \quad S_8 = \begin{bmatrix} 0 & 0.002i \\ 0.001+0.05i & 0.44+0.05i \end{bmatrix},$$

$$S_9 = \begin{bmatrix} 0 & 0.004i \\ -0.004i & 1 \end{bmatrix}, \quad S_{10} = \begin{bmatrix} 0.33 & 0.89i \\ 0.03i & 0 \end{bmatrix}.$$

Then by (4.7), we obtain the general anti-reflexive Hermitian solution $X \in \mathbb{HC}_{ar}^{4\times 4}(P)$ to system (1.6), which can be expressed as the following

$$X = \begin{bmatrix} 0 & X_1 \\ X_1^* & 0 \end{bmatrix}.$$

5. Conclusion

In this paper, we have investigated the system of adjointable operator equations (1.5) over the Hilbert C^*-modules, we have presented necessary and sufficient conditions for the existence and the expression of the general solution to the system (1.5). Some special cases of system (1.5) have been considered in Section 4 to show that some known results can be recovered from the results of this paper. As an application, we have proposed the solvability conditions and the general expression of anti-reflexive Hermitian solution to the system of matrix equations (1.6) over $\mathbb{C}$. Moreover, we have given an algorithm for finding the anti-reflexive solution to system (1.6) and presented a numerical example to illustrate our results.

Acknowledgements. The authors would like to thank an anonymous referee and Professor Leiba Rodman for their helpful suggestions and constructive comments on earlier versions of this paper.

REFERENCES

[1] D. S. DJORDJEVIĆ, *Explicit solution of the operator equation $A^*X + X^*A = B$*, J. Comput. Appl. Math., **200** (2007), 701–704.
[2] D. S. CVETKOVIĆ-ILIĆ, *The solutions of some operator equations*, J. Korean Math. Soc., **45(5)** (2008), 1417–1425.
[3] Q. XU, L. SHENG, Y. GU, *The solutions to some operator equations*, Linear Algebra Appl., **429** (2008), 1997–2024.
[4] Q. XU, *Common Hermitian and positive solutions to the adjointable operator equations $AX = C$, $XB = D$*, Linear Algebra Appl., **429** (2008), 1–11.
[5] A. DAJIĆ, J. J. KOLIHA, *Positive solutions to the equations $AX = C$ and $XB = D$ for Hilbert space operators*, J. Math. Anal. Appl., **333** (2007), 567–576.
[6] A. DAJIĆ, J. J. KOLIHA, *Equations $ax = c$ and $xb = d$ in rings and rings with involution with applications to Hilbert space operators*, Linear Algebra Appl., **429** (2008), 1779–1809.
[7] X. FANG, J. YU, H. YAO, *Solutions to operator equations on Hilbert C^*-modules*, Linear Algebra Appl., **431** (2009), 2142–2153.
[8] H. C. CHEN, *Generalized reflexive matrices: special properties and applications*, SIAM J. Matrix Appl., **19** (1998), 140–153.

[9] Z. Y. PENG, X. Y. HU, *The reflexive and anti-reflexive solutions of matrix equation* $AX = B$, , Linear Algebra Appl., **375** (2003), 147–155.
[10] H. X. CHANG, Q. W. WANG, *Reflexive solutions to a system of matrix equations*, Journal of Shanghai University (English edition), **11** (2007), 355–358.
[11] Q. W. WANG, F. ZHANG, *The reflexive re-nonnegative definite solution to a quaternion matrix equation*, Electron. J. Linear Algebra, **17** (2008), 88–101.
[12] I. KAPLANSKY, *Modules over operator algebras*, Amer J. Math., **75** (1953), 839–858.
[13] N. E. WEGGE-OLSEN, *K-theory and C^*-algebras–a friendly approach*, Oxford University Press, Oxford, England, 1993.
[14] E. C. LANCE, *Hilbert C^*-modules-A Toolkit for Operator Algebraists,* Cambridge University Press, 1995.
[15] S. L. CAMPBELL, C. D. MEYER JR., *Generalized inverses of linear transformations,* Dover Publication, Inc., New York, 1991.
[16] J. C. HOU, J. L. CUI, *Theorem of linear mapping on operator algebras,* Science Press, BeiJing, 2002.
[17] A. BEN-ISRAEL, T. N. E. GREVILLE, *Generalized inverse: theory and applications,* Second Ed., Springer, 2003.
[18] G. R. WANG, *Generalized Inverse: Theory and Computations,* Science Press, BeiJing, 2004.

(Received August 5, 2010)

Qing-Wen Wang
Department of Mathematics
Shanghai University
Shanghai 200444
P.R. China
e-mail: wqw858@yahoo.com.cn

Chang-Zhou Dong
Department of Mathematics
Shanghai University
Shanghai 200444
P.R.China