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ON 2×2 OPERATOR MATRICES

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Abstract. In this paper, we show that some 2×2 operator matrices have scalar extensions. In particular, we focus on some 2-hyponormal operators and their generalizations. As a corollary, we get that such operator matrices have nontrivial invariant subspaces if their spectra have nonempty interiors in the complex plane.

1. Introduction

Let $\mathscr{H}$ be a separable complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. If $T\in\mathscr{L}(\mathscr{H})$, we write $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, and $\sigma_e(T)$ for the spectrum, the point spectrum, the approximate point spectrum, and the essential spectrum of T, respectively.

An operator $T\in\mathscr{L}(\mathscr{H})$ is said to be p*-hyponormal* if $(T^*T)^p \geqslant (TT^*)^p$ for $0<p<\infty$. Especially, if T is 1-hyponormal (resp. $\frac{1}{2}$-hyponormal), then it is called *hyponormal* (resp. *semi-hyponormal*). An operator $A\in\mathscr{L}(\oplus_1^n\mathscr{H})$ is said to be an n*-hyponormal* operator if

$$A=\begin{pmatrix} T_{1,1} & T_{1,2} & \cdots & T_{1,n} \\ T_{2,1} & T_{2,2} & \cdots & T_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ T_{n,1} & T_{n,2} & \cdots & T_{n,n} \end{pmatrix}$$

where $\{T_{i,j}\}$ are mutually commuting hyponormal operators on $\mathscr{H}$.

An arbitrary operator $T\in\mathscr{L}(\mathscr{H})$ has a unique polar decomposition $T=U|T|$, where $|T|=(T^*T)^{\frac{1}{2}}$ and U is the appropriate partial isometry satisfying $kerU = ker|T| = kerT$ and $kerU^*=kerT^*$. Associated with T is a related operator $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, called the *Aluthge transform* of T, and denoted throughout this paper by $\widehat{T}$. For an arbitrary operator $T\in\mathscr{L}(\mathscr{H})$, the sequence $\{\widehat{T}^{(n)}\}$ of Aluthge iterates of T is defined by $\widehat{T}^{(0)}=T$ and $\widehat{T}^{(n+1)}=\widehat{\widehat{T}^{(n)}}$ for every positive integer n (see [1], [8], and [9] for more details).

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An operator $T \in \mathscr{L}(\mathscr{H})$ is called *scalar* of order m if it possesses a spectral distribution of order m, i.e. if there is a continuous unital homomorphism of topological algebras

$$\Phi : C_0^m(\mathbb{C}) \to \mathscr{L}(\mathscr{H})$$

such that $\Phi(z) = T$, where as usual z stands for the identical function on $\mathbb{C}$ and $C_0^m(\mathbb{C})$ for the space of all compactly supported functions continuously differentiable of order m, $0 \leqslant m \leqslant \infty$. An operator is *subscalar* of order m if it is similar to the restriction of a scalar operator of order m to an invariant subspace.

In 1984, M. Putinar showed in [17] that every hyponormal operator is subscalar of order 2. In 1987, his theorem was used to show that hyponormal operators with thick spectra have a nontrivial invariant subspace, which was a result due to S. Brown (see [2]). In 1995, one author of this paper proved in [11] that every upper triangular n-hyponormal operator is subscalar, and in the same paper he raised an open question about the subscalarity of 2-hyponormal operators. As an effort to solve this question, we obtain partial solutions of the question and more generalized results.

2. Preliminaries

An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have the *single-valued extension property at* z_0 if for every neighborhood D of z_0 and any analytic function $f : D \to \mathscr{H}$, with $(T-z)f(z) \equiv 0$, it results $f(z) \equiv 0$. An operator $T \in \mathscr{L}(\mathscr{H})$ having the single-valued extension property at every z in the complex plane $\mathbb{C}$ is said to have the *single-valued extension property* (or SVEP). For $T \in \mathscr{L}(\mathscr{H})$ and $x \in \mathscr{H}$, the set $\rho_T(x)$ is defined to consist of elements z_0 in $\mathbb{C}$ such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0, with values in $\mathscr{H}$, which verifies $(T-z)f(z) \equiv x$, and it is called *the local resolvent set* of T at x. We denote the complement of $\rho_T(x)$ by $\sigma_T(x)$, called *the local spectrum* of T at x, and define *the local spectral subspace* of T, $H_T(F) = \{x \in \mathscr{H} : \sigma_T(x) \subset F\}$ for each subset F of $\mathbb{C}$. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have *the property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to \mathscr{H}$ of $\mathscr{H}$-valued analytic functions such that $(T-z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, then $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have *Dunford's property* (C) if $H_T(F)$ is closed for each closed subset F of $\mathbb{C}$. It is well known by [13] that

$$\text{Property } (\beta) \Rightarrow \text{Dunford's property } (C) \Rightarrow \text{SVEP}.$$

Let z be the coordinate in the complex plane $\mathbb{C}$ and $d\mu(z)$ the planar Lebesgue measure. Consider a bounded (connected) open subset U of $\mathbb{C}$. We shall denote by $L^2(U,\mathscr{H})$ the Hilbert space of measurable functions $f : U \to \mathscr{H}$, such that

$$\|f\|_{2,U} = \left(\int_U \|f(z)\|^2 d\mu(z)\right)^{\frac{1}{2}} < \infty.$$

The space of functions $f \in L^2(U,\mathscr{H})$ that are analytic in U is denoted by

$$A^2(U,\mathscr{H}) = L^2(U,\mathscr{H}) \cap \mathscr{O}(U,\mathscr{H})$$

where $\mathscr{O}(U,\mathscr{H})$ denotes the Fr$\acute{e}$chet space of $\mathscr{H}$-valued analytic functions on U with respect to uniform topology. $A^2(U,\mathscr{H})$ is called the Bergman space for U. Note that $A^2(U,\mathscr{H})$ is a Hilbert space.

Now, let us define a special Sobolev type space. For a fixed non-negative integer m, the vector-valued Sobolev space $W^m(U,\mathscr{H})$ with respect to $\bar{\partial}$ and of order m will be the space of those functions $f\in L^2(U,\mathscr{H})$ whose derivatives $\bar{\partial}f,\cdots,\bar{\partial}^m f$ in the sense of distributions still belong to $L^2(U,\mathscr{H})$. Endowed with the norm

$$\|f\|^2_{W^m}=\sum_{i=0}^{m}\|\bar{\partial}^i f\|^2_{2,U},$$

$W^m(U,\mathscr{H})$ becomes a Hilbert space contained continuously in $L^2(U,\mathscr{H})$.

We can easily show that the linear operator M of multiplication by z on $W^m(U,\mathscr{H})$ is continuous and it has a spectral distribution Φ of order m defined by the following relation; for $\varphi\in C_0^m(\mathbb{C})$ and $f\in W^m(U,\mathscr{H})$, $\Phi(\varphi)f=\varphi f$. Hence M is a scalar operator of order m.

3. 2-hyponormal operators

In this section, we will show that some 2-hyponormal operators have scalar extensions. For this, we begin with the following lemmas.

LEMMA 3.1. Let $T\in\mathscr{L}(\mathscr{H})$ be a hyponormal operator and let D be a bounded disk in $\mathbb{C}$. If $\{f_n\}$ is any sequence in $W^m(D,\mathscr{H})$ $(m\geqslant 2)$ such that

$$\lim_{n\to\infty}\|(T-z)\bar{\partial}^i f_n\|_{2,D}=0$$

for $i=0,1,2,\cdots,m$, then

$$\lim_{n\to\infty}\|\bar{\partial}^i f_n\|_{2,D_0}=0$$

for $i=0,1,2,\cdots,m-2$, where D_0 is a disk with $D_0\subsetneq D$ and P denotes the orthogonal projection of $L^2(D,\mathscr{H})$ onto $A^2(D,\mathscr{H})$.

Proof. Since T is hyponormal, by [17] there exists a constant C_D such that

$$\|(I-P)\bar{\partial}^i f_n\|_{2,D}\leqslant C_D\big(\|(T-z)\bar{\partial}^{i+1}f_n\|_{2,D}+\|(T-z)\bar{\partial}^{i+2}f_n\|_{2,D}\big)\tag{1}$$

for $i=0,1,2,\cdots,m-2$. From (1), we have

$$\lim_{n\to\infty}\|(I-P)\bar{\partial}^i f_n\|_{2,D}=0\tag{2}$$

for $i=0,1,2,\cdots,m-2$. So, it holds that

$$\lim_{n\to\infty}\|(T-z)P\bar{\partial}^i f_n\|_{2,D}=0\tag{3}$$

for $i=0,1,2,\cdots,m-2$. Since T has the property (β), from (3) we have

$$\lim_{n\to\infty}\|P\overline{\partial}^i f_n\|_{2,D_0}=0 \tag{4}$$

for $i=0,1,2,\cdots,m-2$, where D_0 denotes a disk with $D_0 \subsetneqq D$. From (2) and (4), we get that

$$\lim_{n\to\infty}\|\overline{\partial}^i f_n\|_{2,D_0}=0$$

for $i=0,1,2,\cdots,m-2$. $\square$

LEMMA 3.2. Let $T\in\mathscr{L}(\mathscr{H})$ and let D be a bounded disk in $\mathbb{C}$ containing $\sigma(T)$. Suppose that $f_n\in W^m(D,\mathscr{H})$ and $h_n\in\mathscr{H}$ are sequences such that

$$\lim_{n\to\infty}\|(T-z)Pf_n+1\otimes h_n\|_{2,D}=0$$

where P is the orthogonal projection of $L^2(D,\mathscr{H})$ onto $A^2(D,\mathscr{H})$ and $1\otimes h$ denotes the constant function sending any $z\in D$ to $h\in\mathscr{H}$. Then $\lim_{n\to\infty}\|h_n\|=0$.

Proof. Let Γ be a curve in D surrounding $\sigma(T)$. Then

$$\lim_{n\to\infty}\|Pf_n(z)+(T-z)^{-1}(1\otimes h_n)(z)\|=0$$

uniformly for all $z\in\Gamma$. Applying the Riesz-Dunford functional calculus, we obtain that

$$\begin{aligned}0&=\lim_{n\to\infty}\left\|\frac{1}{2\pi i}\int_\Gamma Pf_n(z)\,dz+\frac{1}{2\pi i}\int_\Gamma (T-z)^{-1}(1\otimes h_n)(z)\,dz\right\|\\&=\lim_{n\to\infty}\left\|\frac{1}{2\pi i}\int_\Gamma Pf_n(z)\,dz+h_n\right\|.\end{aligned}$$

But $\frac{1}{2\pi i}\int_\Gamma Pf_n(z)\,dz=0$ by the Cauchy's theorem. Hence $\lim_{n\to\infty}\|h_n\|=0$. $\square$

Recall that an operator $T\in\mathscr{L}(\mathscr{H})$ is said to be *nilpotent* of order k if $T^k=0$ for some positive integer k.

LEMMA 3.3. Let $A=\begin{pmatrix}T_1 & T_2\\ T_3 & T_4\end{pmatrix}$ be a 2-hyponormal operator defined on $\mathscr{H}\oplus\mathscr{H}$. For a bounded disk D in $\mathbb{C}$ containing $\sigma(A)$ and a positive integer m, define the map $V_m:\mathscr{H}\oplus\mathscr{H}\to H(D)$ by

$$V_m h=\widetilde{1\otimes h}(\equiv 1\otimes h+\overline{(A-z)\oplus_1^2 W^m(D,\mathscr{H})})$$

where $1\otimes h$ denotes the constant function sending any $z\in D$ to $h\in\mathscr{H}\oplus\mathscr{H}$ and $H(D):=\oplus_1^2W^m(D,\mathscr{H})/\overline{(A-z)\oplus_1^2W^m(D,\mathscr{H})}$. Then the following statements hold.

(a) If either T_2 or T_3 is nilpotent, then V_4 is one-to-one and has closed range.

(b) If T_4 is nilpotent and $T_2-T_3=\pm T_1$, then V_6 is one-to-one and has closed range.

(c) If T_1 is nilpotent and $T_2-T_3=\pm T_4$, then V_6 is one-to-one and has closed range.

(d) If $T_j=\gamma_j T_1$ for $j=2,3,4$ and $1-\gamma_4=\pm(\gamma_2-\gamma_3)$ where $\gamma_j\in\mathbb{C}$ for $j=2,3,4$, then V_6 is one-to-one and has closed range.

(e) If $T_2T_3=0$, then V_8 is one-to-one and has closed range.

(f) If T_1+T_4 is hyponormal and $\det(A):=T_1T_4-T_2T_3=0$, then V_8 is one-to-one and has closed range.

Proof. Since every operator both hyponormal and nilpotent is the zero operator, the proof of (a) follows from [11].

In order to show the others, let $h_n=(h_n^1,h_n^2)^t\in\mathscr{H}\oplus\mathscr{H}$ and $f_n=(f_n^1,f_n^2)^t\in\oplus_1^2 W^m(D,\mathscr{H})$ be sequences such that

$$\lim_{n\to\infty}\|(A-z)f_n+1\otimes h_n\|_{\oplus_1^2 W^m}=0. \tag{5}$$

Then from (5) we have

$$\begin{cases}\lim_{n\to\infty}\|(T_1-z)f_n^1+T_2f_n^2+1\otimes h_n^1\|_{W^m}=0\\ \lim_{n\to\infty}\|T_3f_n^1+(T_4-z)f_n^2+1\otimes h_n^2\|_{W^m}=0.\end{cases} \tag{6}$$

By the definition of the norm for the Sobolev space, (6) implies that

$$\begin{cases}\lim_{n\to\infty}\|(T_1-z)\bar{\partial}^i f_n^1+T_2\bar{\partial}^i f_n^2\|_{2,D}=0\\ \lim_{n\to\infty}\|T_3\bar{\partial}^i f_n^1+(T_4-z)\bar{\partial}^i f_n^2\|_{2,D}=0\end{cases} \tag{7}$$

for $i=1,2,\cdots,m$.

(b) Set $m=6$ and note that $T_4=0$ because T_4 is hyponormal and nilpotent. By (7), we get that

$$\lim_{n\to\infty}\|\{(T_1\pm T_3)-z\}\bar{\partial}^i f_n^1+(T_2\mp z)\bar{\partial}^i f_n^2\|_{2,D}=0 \tag{8}$$

for $i=1,2,\cdots,6$. Since $T_2-T_3=\pm T_1$, from (8) we have

$$\lim_{n\to\infty}\|(T_2\mp z)(\bar{\partial}^i f_n^1\pm\bar{\partial}^i f_n^2)\|_{2,D}=0 \tag{9}$$

for $i=1,2,\cdots,6$. Since T_2 is hyponormal, we obtain from Lemma 3.1 and (9) that

$$\lim_{n\to\infty}\|\bar{\partial}^i f_n^1\pm\bar{\partial}^i f_n^2\|_{2,D_1}=0 \tag{10}$$

for $i=1,2,3,4$, where $\sigma(A)\subsetneq D_1\subsetneq D$ (note that the one-to-one correspondence $z\mapsto -z$ on $\mathbb{C}$ may be necessary for the case when $T_2-T_3=-T_1$). In addition,

$$\|(T_3\pm z)\bar{\partial}^i f_n^1\|_{2,D_1}\leqslant\|T_3\bar{\partial}^i f_n^1-z\bar{\partial}^i f_n^2\|_{2,D_1}+\|z(\bar{\partial}^i f_n^1\pm\bar{\partial}^i f_n^2)\|_{2,D_1}$$

for $i=1,2,3,4$, which implies together with (7) and (10) that

$$\lim_{n\to\infty}\|(T_3\pm z)\overline{\partial}^i f_n^1\|_{2,D_1}=0 \tag{11}$$

for $i=1,2,3,4$. Since T_3 is hyponormal, by Lemma 3.1 and (11) we have

$$\lim_{n\to\infty}\|\overline{\partial}^i f_n^1\|_{2,D_2}=0 \tag{12}$$

for $i=1,2$, where $\sigma(A)\subsetneqq D_2\subsetneqq D_1$. Due to (10) and (12),

$$\lim_{n\to\infty}\|\overline{\partial}^i f_n^2\|_{2,D_2}=0$$

for $i=1,2$. Hence, it follows that

$$\lim_{n\to\infty}\|z\overline{\partial}^i f_n^1\|_{2,D_2}=\lim_{n\to\infty}\|z\overline{\partial}^i f_n^2\|_{2,D_2}=0.$$

By applying [17], we get that

$$\lim_{n\to\infty}\|(I-P)f_n^1\|_{2,D_2}=\lim_{n\to\infty}\|(I-P)f_n^2\|_{2,D_2}=0 \tag{13}$$

where P denotes the orthogonal projection of $L^2(D_2,\mathscr{H})$ onto $A^2(D_2,\mathscr{H})$. (5) and (13) imply that

$$\lim_{n\to\infty}\|(A-z)Pf_n+1\otimes h_n\|_{2,D_2}=0 \tag{14}$$

where $Pf_n:=\binom{Pf_n^1}{Pf_n^2}$. Therefore, $\lim_{n\to\infty}\|h_n\|=0$ from Lemma 3.2. Thus V_6 is one-to-one and has closed range.

(c) We can show (c) by the same method as in the proof of (b).

(d) Put $m=6$. Since $1-\gamma_4=\pm(\gamma_2-\gamma_3)$, from (7) we get that

$$\lim_{n\to\infty}\|\{(1\pm\gamma_3)T_1-z\}(\overline{\partial}^i f_n^1\pm\overline{\partial}^i f_n^2)\|_{2,D}=0 \tag{15}$$

for $i=1,2,\cdots,6$. Because $(1\pm\gamma_3)T_1$ is hyponormal, (15) and Lemma 3.1 imply that

$$\lim_{n\to\infty}\|\overline{\partial}^i f_n^1\pm\overline{\partial}^i f_n^2\|_{2,D_1}=0 \tag{16}$$

for $i=1,2,3,4$, where $\sigma(A)\subsetneqq D_1\subsetneqq D$. Since

$$\begin{aligned}\|\{(\gamma_3\mp\gamma_4)T_1\pm z\}\overline{\partial}^i f_n^1\|_{2,D_1}\leqslant{}&\|\gamma_3T_1\overline{\partial}^i f_n^1+(\gamma_4T_1-z)\overline{\partial}^i f_n^2\|_{2,D_1}\\&+\|(\gamma_4T_1-z)(\overline{\partial}^i f_n^1\pm\overline{\partial}^i f_n^2)\|_{2,D_1}\end{aligned}$$

for $i=1,2,3,4$, the equations (7) and (16) induce that

$$\lim_{n\to\infty}\|\{(\gamma_3\mp\gamma_4)T_1\pm z\}\overline{\partial}^i f_n^1\|_{2,D_1}=0 \tag{17}$$

for $i = 1,2,3,4$. Since $(\gamma_3 \mp \gamma_4)T_1$ is hyponormal, we obtain from (17) and Lemma 3.1 that

$$\lim_{n\to\infty} \|\overline{\partial}^i f_n^1\|_{2,D_2} = 0 \tag{18}$$

for $i = 1,2$, where $\sigma(A) \subsetneqq D_2 \subsetneqq D_1$. Due to (16) and (18),

$$\lim_{n\to\infty} \|\overline{\partial}^i f_n^2\|_{2,D_2} = 0$$

for $i = 1,2$. Hence, by the same process as (13) and (14), V_6 is one-to-one and has closed range.

(e) Set $m = 8$. Since $T_2T_3 = 0$, multiplying the second equation of (7) by T_2, we get that

$$\lim_{n\to\infty} \|(T_4 - z)T_2\overline{\partial}^i f_n^2\|_{2,D} = 0 \tag{19}$$

for $i = 1,2,\cdots,8$. Since T_4 is hyponormal, we obtain from (19) and Lemma 3.1 that

$$\lim_{n\to\infty} \|T_2\overline{\partial}^i f_n^2\|_{2,D_1} = 0 \tag{20}$$

for $i = 1,2,\cdots,6$, where $\sigma(A) \subsetneqq D_1 \subsetneqq D$. By the first equation of (7) and (20), we get that

$$\lim_{n\to\infty} \|(T_1 - z)\overline{\partial}^i f_n^1\|_{2,D_1} = 0 \tag{21}$$

for $i = 1,2,\cdots,6$. Thus, by the hyponormality of T_1, (21) and Lemma 3.1 imply that

$$\lim_{n\to\infty} \|\overline{\partial}^i f_n^1\|_{2,D_2} = 0 \tag{22}$$

for $i = 1,2,3,4$, where $\sigma(A) \subsetneqq D_2 \subsetneqq D_1$. From the second equation of (7) and (22), it holds that

$$\lim_{n\to\infty} \|(T_4 - z)\overline{\partial}^i f_n^2\|_{2,D_2} = 0 \tag{23}$$

for $i = 1,2,3,4$. Since T_4 is hyponormal, (23) and Lemma 3.1 result in the equation,

$$\lim_{n\to\infty} \|\overline{\partial}^i f_n^2\|_{2,D_3} = 0$$

for $i = 1,2$, where $\sigma(A) \subsetneqq D_3 \subsetneqq D_2$. Hence, by the same process as (13) and (14), we can conclude that V_8 is one-to-one and has closed range.

(f) Set $m = 8$. By (7), we obtain that

$$\begin{cases} \lim_{n\to\infty} \|(T_1T_3 - zT_3)\overline{\partial}^i f_n^1 + T_2T_3\overline{\partial}^i f_n^2\|_{2,D} = 0 \\ \lim_{n\to\infty} \|T_1T_3\overline{\partial}^i f_n^1 + (T_1T_4 - zT_1)\overline{\partial}^i f_n^2\|_{2,D} = 0 \end{cases} \tag{24}$$

for $i = 1,2,\cdots,8$. Since $\det(A) = T_1T_4 - T_2T_3 = 0$, (24) implies that

$$\lim_{n\to\infty} \|z(T_1\overline{\partial}^i f_n^2 - T_3\overline{\partial}^i f_n^1)\|_{2,D} = 0 \tag{25}$$

$i = 1, 2, \cdots, 8$. Since the zero operator is hyponormal, by (25) and Lemma 3.1 we can have

$$\lim_{n\to\infty} \|T_1 \overline{\partial}^i f_n^2 - T_3 \overline{\partial}^i f_n^1\|_{2,D_1} = 0 \tag{26}$$

for $i = 1, 2, \cdots, 6$, where $\sigma(A) \subsetneqq D_1 \subsetneqq D$. From (26) and the second equation of (7), we get that

$$\lim_{n\to\infty} \|(T_1 + T_4 - z)\overline{\partial}^i f_n^2\|_{2,D_1} = 0 \tag{27}$$

for $i = 1, 2, \cdots, 6$. Since $T_1 + T_4$ is hyponormal, it holds by (27) and Lemma 3.1 that

$$\lim_{n\to\infty} \|\overline{\partial}^i f_n^2\|_{2,D_2} = 0 \tag{28}$$

for $i = 1, 2, 3, 4$, where $\sigma(A) \subsetneqq D_2 \subsetneqq D_1$. Thus it can be obtained from (28) and the first equation of (7) that

$$\lim_{n\to\infty} \|(T_1 - z)\overline{\partial}^i f_n^1\|_{2,D_2} = 0 \tag{29}$$

for $i = 1, 2, 3, 4$. Because T_1 is hyponormal, by (29) and Lemma 3.1 we can conclude that

$$\lim_{n\to\infty} \|\overline{\partial}^i f_n^1\|_{2,D_3} = 0 \tag{30}$$

for $i = 1, 2$, where $\sigma(A) \subsetneqq D_3 \subsetneqq D_2$. So, as in the proof of (b), we obtain from (28) and (30) that

$$\lim_{n\to\infty} \|(I - P)f_n^1\|_{2,D_3} = \lim_{n\to\infty} \|(I - P)f_n^2\|_{2,D_3} = 0$$

where P denotes the orthogonal projection of $L^2(D_3, \mathscr{H})$ onto $A^2(D_3, \mathscr{H})$. Hence, by the same process as (13) and (14), V_8 is one-to-one and has closed range. $\square$

Now we are ready to prove that some 2-hyponormal operators have scalar extensions.

THEOREM 3.4. Let $A = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \in \mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ be a 2-hyponormal operator. If $\{T_i\}_{i=1}^4$ satisfy one of the conditions in Lemma 3.3, then A is a subscalar operator of order m where $m = 4$ in the case of (a), $m = 6$ in the cases of from (b) to (d), and $m = 8$ in the cases of (e) and (f) in Lemma 3.3.

Proof. Let D be an arbitrary bounded open disk in $\mathbb{C}$ that contains $\sigma(A)$ and consider the quotient space

$$H(D) = \oplus_1^2 W^m(D, \mathscr{H}) / \overline{(A - z)\oplus_1^2 W^m(D, \mathscr{H})}$$

endowed with the Hilbert space norm, where $m = 4$ in the case of (a), $m = 6$ in the cases of from (b) to (d), and $m = 8$ in the cases of (e) and (f) in Lemma 3.3. The class of a vector f or an operator S on $H(D)$ will be denoted by $\widetilde{f}$, respectively $\widetilde{S}$. Let M be the operator of multiplication by z on $\oplus_1^2 W^m(D, \mathscr{H})$. Then M is a scalar operator of order m and has a spectral distribution Φ. Since the range of $A - z$ is invariant under M, $\widetilde{M}$ can be well-defined. Moreover, consider the spectral distribution $\Phi : C_0^m(\mathbb{C}) \to \mathscr{L}(\oplus_1^2 W^m(D,\mathscr{H}))$ defined by the following relation; for $\varphi \in C_0^m(\mathbb{C})$ and $f \in \oplus_1^2 W^m(D,\mathscr{H})$, $\Phi(\varphi)f = \varphi f$. Then the spectral distribution Φ of M commutes with $A-z$, and so $\widetilde{M}$ is still a scalar operator of order m with $\widetilde{\Phi}$ as a spectral distribution. Consider the operator $V_m : \mathscr{H}\oplus\mathscr{H} \to H(D)$ given by $V_m h = \widetilde{1\otimes h}$ with the same notation of Lemma 3.3, and denote the range of V_m by $\operatorname{ran}(V_m)$. Since

$$V_m Ah = \widetilde{1\otimes Ah} = \widetilde{z\otimes h} = \widetilde{M}(\widetilde{1\otimes h}) = \widetilde{M}V_m h$$

for all $h \in \mathscr{H}\oplus\mathscr{H}$, $V_m A = \widetilde{M}V_m$. In particular, $\operatorname{ran}(V_m)$ is invariant under $\widetilde{M}$. Furthermore, $\operatorname{ran}(V_m)$ is closed by Lemma 3.3, and hence $\operatorname{ran}(V_m)$ is a closed invariant subspace of the scalar operator $\widetilde{M}$. Since A is similar to the restriction $\widetilde{M}|_{\operatorname{ran}(V_m)}$ and $\widetilde{M}$ is a scalar operator of order m, A is a subscalar operator of order m. $\square$

4. Generalizations of 2-hyponormal operators

In this section, we consider the following question in the sense of the completion problem; given a 2×2 operator matrix A with main diagonal of p-hyponormal operators, when is A subscalar? We give some solutions for this question (see Theorem 4.2). The following lemma is the key step to prove that such operator matrices are subscalar.

LEMMA 4.1. Let A be an operator matrix on $\mathscr{H}\oplus\mathscr{H}$ such that $A = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ where T_i are mutually commuting, and T_1 and T_4 are p-hyponormal. For a bounded disk D containing $\sigma(A)$, define the map $V_m : \mathscr{H}\oplus\mathscr{H} \to H(D)$ as in Lemma 3.3. If either T_2 or T_3 is nilpotent of order k, then V_{12k+8} is one-to-one and has closed range.

Proof. We may assume that T_2 is nilpotent of order k (the proof for which T_3 is nilpotent of order k is similar). It suffices to consider only the case of $0 < p < \frac{1}{2}$. Let $h_n = (h_n^1, h_n^2)^t \in \mathscr{H}\oplus\mathscr{H}$ and $f_n = (f_n^1, f_n^2)^t \in \oplus_1^2 W^{12k+8}(D,\mathscr{H})$ be sequences such that

$$\lim_{n\to\infty} \|(A-z)f_n + 1\otimes h_n\|_{\oplus_1^2 W^{12k+8}} = 0. \tag{31}$$

By (31), we get that

$$\begin{cases} \lim_{n\to\infty} \|(T_1-z)f_n^1 + T_2 f_n^2 + 1\otimes h_n^1\|_{W^{12k+8}} = 0 \\ \lim_{n\to\infty} \|T_3 f_n^1 + (T_4-z)f_n^2 + 1\otimes h_n^2\|_{W^{12k+8}} = 0. \end{cases} \tag{32}$$

By the definition of the norm for the Sobolev space, (32) implies that

$$\begin{cases} \lim_{n\to\infty} \|(T_1-z)\bar{\partial}^i f_n^1 + T_2 \bar{\partial}^i f_n^2\|_{2,D} = 0 \\ \lim_{n\to\infty} \|T_3 \bar{\partial}^i f_n^1 + (T_4-z)\bar{\partial}^i f_n^2\|_{2,D} = 0 \end{cases} \tag{33}$$

for $i = 1,2,\cdots,12k+8$.

CLAIM. It holds for every $j=0,1,2,\cdots,k$ that

$$\lim_{n\to\infty}\|T_2^{k-j}\overline{\partial}^i f_n^2\|_{2,D_j}=0 \tag{34}$$

for $i=1,2,\cdots,12(k-j)+8$, where $\sigma(A)\subsetneqq D_k\subsetneqq D_{k-1}\subsetneqq\cdots\subsetneqq D_1\subsetneqq D_0=D$.

To prove the claim, we will apply the induction on j. Since $T_2{}^k=0$, (34) holds obviously when $j=0$. Suppose that the claim is true for $j=r<k$. Then

$$\lim_{n\to\infty}\|T_2^{k-r}\overline{\partial}^i f_n^2\|_{2,D_r}=0 \tag{35}$$

for $i=1,2,\cdots,12(k-r)+8$. By (33) and (35), we get that

$$\lim_{n\to\infty}\|(T_1-z)T_2^{k-r-1}\overline{\partial}^i f_n^1\|_{2,D_r}=0 \tag{36}$$

for $i=1,2,\cdots,12(k-r)+8$. Let $T_1=U_1|T_1|$ and $\widehat{T_1}=V|\widehat{T_1}|$ be the polar decompositions of T_1 and $\widehat{T_1}$, respectively. Since $\widehat{S}|S|^{\frac{1}{2}}=|S|^{\frac{1}{2}}S$ holds for every operator $S\in\mathscr{L}(\mathscr{H})$, we obtain from (36) that

$$\begin{cases}\lim_{n\to\infty}\|(\widehat{T_1}-z)|T_1|^{\frac{1}{2}}T_2^{k-r-1}\overline{\partial}^i f_n^1\|_{2,D_r}=0\\ \lim_{n\to\infty}\|(\widehat{T_1}^{(2)}-z)|\widehat{T_1}|^{\frac{1}{2}}|T_1|^{\frac{1}{2}}T_2^{k-r-1}\overline{\partial}^i f_n^1\|_{2,D_r}=0\end{cases} \tag{37}$$

for $i=1,2,\cdots,12(k-r)+8$. Since T_1 is p-hyponormal, $\widehat{T_1}^{(2)}$ is hyponormal by [1] or [8]. It follows from (37) and Lemma 3.1 that

$$\lim_{n\to\infty}\||\widehat{T_1}|^{\frac{1}{2}}|T_1|^{\frac{1}{2}}T_2^{k-r-1}\overline{\partial}^i f_n^1\|_{2,D_{r,1}}=0 \tag{38}$$

for $i=1,2,\cdots,12(k-r)+6$, where $\sigma(A)\subsetneqq D_{r,1}\subsetneqq D_r$. Since $T_1=U_1|T_1|$ and $\widehat{T_1}=V|\widehat{T_1}|$, from (37) and (38) we have

$$\lim_{n\to\infty}\|z|T_1|^{\frac{1}{2}}T_2^{k-r-1}\overline{\partial}^i f_n^1\|_{2,D_{r,1}}=0 \tag{39}$$

for $i=1,2,\cdots,12(k-r)+6$. Applying Lemma 3.1 with $T=(0)$, we obtain from (39) that

$$\lim_{n\to\infty}\||T_1|^{\frac{1}{2}}T_2^{k-r-1}\overline{\partial}^i f_n^1\|_{2,D_{r,2}}=0$$

for $i=1,2,\cdots,12(k-r)+4$, where $\sigma(A)\subsetneqq D_{r,2}\subsetneqq D_{r,1}$, which induces that

$$\lim_{n\to\infty}\|T_1T_2^{k-r-1}\overline{\partial}^i f_n^1\|_{2,D_{r,2}}=0 \tag{40}$$

for $i=1,2,\cdots,12(k-r)+4$. By (36) and (40), we get that

$$\lim_{n\to\infty}\|zT_2^{k-r-1}\overline{\partial}^i f_n^1\|_{2,D_{r,2}}=0 \tag{41}$$

for $i=1,2,\cdots,12(k-r)+4$. Again applying Lemma 3.1 with $T=(0)$, then we can conclude from (41) that

$$\lim_{n\to\infty}\|T_2^{k-r-1}\bar{\partial}^i f_n^1\|_{2,D_{r,3}}=0 \tag{42}$$

for $i=1,2,\cdots,12(k-r)+2$, where $\sigma(A)\subsetneqq D_{r,3}\subsetneqq D_{r,2}$. From (42) and (33), we have

$$\lim_{n\to\infty}\|(T_4-z)T_2^{k-r-1}\bar{\partial}^i f_n^2\|_{2,D_{r,3}}=0 \tag{43}$$

for $i=1,2,\cdots,12(k-r)+2$. Since T_4 is p-hyponormal, by the same method as the procedure from (36) to (42) we get that

$$\lim_{n\to\infty}\|T_2^{k-r-1}\bar{\partial}^i f_n^2\|_{2,D_{r+1}}=0 \tag{44}$$

for $i=1,2,\cdots,12(k-r-1)+8$, where $\sigma(A)\subsetneqq D_{r+1}\subsetneqq D_{r,3}$. Hence we complete the proof of our claim.

By the claim with $j=k$, we get that

$$\lim_{n\to\infty}\|\bar{\partial}^i f_n^2\|_{2,D_k}=0 \tag{45}$$

for $i=1,2,\cdots,8$. Combining (45) with (33), we obtain that

$$\lim_{n\to\infty}\|(T_1-z)\bar{\partial}^i f_n^1\|_{2,D_k}=0$$

for $i=1,2,\cdots,8$. Since T_1 is p-hyponormal, by the same method as the procedure from (36) to (42) we can show that

$$\lim_{n\to\infty}\|\bar{\partial}^i f_n^1\|_{2,D_{k,1}}=0 \tag{46}$$

for $i=1,2$, where $\sigma(A)\subsetneqq D_{k,1}\subsetneqq D_k$. (45) and (46) imply that

$$\lim_{n\to\infty}\|z\bar{\partial}^i f_n^1\|_{2,D_{k,1}}=\lim_{n\to\infty}\|z\bar{\partial}^i f_n^2\|_{2,D_{k,1}}=0$$

for $i=1,2$. Thus it follows from [17] that

$$\lim_{n\to\infty}\|(I-P)f_n^1\|_{2,D_{k,1}}=\lim_{n\to\infty}\|(I-P)f_n^2\|_{2,D_{k,1}}=0 \tag{47}$$

where P denotes the orthogonal projection of $L^2(D_{k,1},\mathscr{H})$ onto $A^2(D_{k,1},\mathscr{H})$. Set $Pf_n:=\binom{Pf_n^1}{Pf_n^2}$. Combining (47) with (31), we have

$$\lim_{n\to\infty}\|(A-z)Pf_n(z)+1\otimes h_n\|_{2,D_{k,1}}=0,$$

which induces by Lemma 3.2 that $\lim_{n\to\infty}\|h_n\|=0$, and so V_{12k+8} is one-to-one and has closed range. $\square$

THEOREM 4.2. Let $A = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ be an operator matrix with the same hypotheses as Lemma 4.1. Then A is a subscalar operator of order $12k+8$.

Proof. Let D be an arbitrary bounded open disk in $\mathbb{C}$ that contains $\sigma(A)$ and consider the quotient space

$$H(D) = \oplus_1^2 W^{12k+8}(D, \mathcal{H}) / \overline{(A-z)\oplus_1^2 W^{12k+8}(D, \mathcal{H})}$$

endowed with the Hilbert space norm. The class of a vector f or an operator S on $H(D)$ will be denoted by $\widetilde{f}$, respectively $\widetilde{S}$. Let M be the operator of multiplication by z on $\oplus_1^2 W^{12k+8}(D, \mathcal{H})$. Then M is a scalar operator of order $12k+8$ and has a spectral distribution Φ. Moreover, $\widetilde{M}$ is a scalar operator of order $12k+8$ with $\widetilde{\Phi}$ as a spectral distribution. Consider the operator $V_{12k+8} : \mathcal{H} \oplus \mathcal{H} \to H(D)$ given by $V_{12k+8} h = \widetilde{1 \otimes h}$ with the same notations as Lemma 4.1, and denote the range of V_{12k+8} by $\operatorname{ran}(V_{12k+8})$. Since $V_{12k+8} A = \widetilde{M} V_{12k+8}$, $\operatorname{ran}(V_{12k+8})$ is invariant under $\widetilde{M}$. Hence, by Lemma 4.1, $\operatorname{ran}(V_{12k+8})$ is a closed invariant subspace of the scalar operator $\widetilde{M}$. Since A is similar to the restriction $\widetilde{M}|_{\operatorname{ran}(V_{12k+8})}$ and $\widetilde{M}$ is a scalar operator of order $12k+8$, A is a subscalar operator of order $12k+8$. $\square$

5. Some applications

In this section we give some applications of our main theorems. In particular, the following corollary gives a partial solution for the invariant subspace problem.

COROLLARY 5.1. Let A be an operator matrix on $\mathcal{H} \oplus \mathcal{H}$ having one of the forms in Theorem 3.4 or Theorem 4.2. If $\sigma(A)$ has nonempty interior in $\mathbb{C}$, then A has a nontrivial invariant subspace.

Proof. The proof follows from Theorem 3.4 or Theorem 4.2 and [5]. $\square$

Before giving the next corollary, we recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *power regular* if $\lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}}$ exists for every $x \in \mathcal{H}$.

COROLLARY 5.2. Let A be an operator matrix on $\mathcal{H} \oplus \mathcal{H}$ with the same assumptions as in Theorem 3.4 or Theorem 4.2. Then

(a) A has the property (β), Dunford's property (C), and the single-valued extension property.

(b) A is power regular.

Proof. (a) From section 2, it suffices to prove that A has the property (β). Since the property (β) is transmitted from an operator to its restrictions to closed invariant subspaces, we are reduced by Theorem 3.4 or Theorem 4.2 to the case of a scalar operator order m, where m is taken for each of the cases. Since every scalar operator has the property (β) (see [17]), A has the property (β).

(b) From Theorem 3.4 or Theorem 4.2, A is similar to the restriction of a scalar operator to one of its invariant subspaces. Since a scalar operator is power regular and the restrictions of power regular operators to their invariant subspaces are still power regular, A is also power regular. $\square$

Recall that an $X \in \mathscr{L}(\mathscr{H},\mathscr{K})$ is called a *quasiaffinity* if it has trivial kernel and dense range. An operator $S \in \mathscr{L}(\mathscr{H})$ is said to be a *quasiaffine transform* of an operator $T \in \mathscr{L}(\mathscr{K})$ if there is a quasiaffinity $X \in \mathscr{L}(\mathscr{H},\mathscr{K})$ such that $XS = TX$. Furthermore, operators $S \in \mathscr{L}(\mathscr{H})$ and $T \in \mathscr{L}(\mathscr{K})$ are *quasisimilar* if there are quasiaffinities $X \in \mathscr{L}(\mathscr{H},\mathscr{K})$ and $Y \in \mathscr{L}(\mathscr{K},\mathscr{H})$ such that $XS = TX$ and $SY = YT$.

COROLLARY 5.3. Let A and B be operator matrices on $\mathscr{H}\oplus\mathscr{H}$ with the same assumptions as in Theorem 3.4 or Theorem 4.2. If A and B are quasisimilar, then $\sigma(A) = \sigma(B)$ and $\sigma_e(A) = \sigma_e(B)$.

Proof. Since A and B satisfy the property (β) from Corollary 5.2, the proof follows from [19]. $\square$

THEOREM 5.4. If A is an operator matrix on $\mathscr{H}\oplus\mathscr{H}$ with the same notations as in Theorem 3.4 or Theorem 4.2, then the equality $\sigma_{\widetilde{M}}(V_m h) = \sigma_A(h)$ holds for each $h \in \mathscr{H}\oplus\mathscr{H}$ where m is the appropriately chosen integer as in Theorem 3.4 or Theorem 4.2.

Proof. Let $h \in \mathscr{H}\oplus\mathscr{H}$ be given. If $\lambda_0 \in \rho_A(h)$, then there is an $\mathscr{H}\oplus\mathscr{H}$-valued analytic function g defined on a neighborhood U of λ_0 such that $(A-\lambda)g(\lambda) = h$ for all $\lambda \in U$. Then

$$(\widetilde{M}-\lambda)V_m g(\lambda) = V_m(A-\lambda)g(\lambda) = V_m h$$

for all $\lambda \in U$. Hence $\lambda_0 \in \rho_{\widetilde{M}}(V_m h)$. That is, $\sigma_{\widetilde{M}}(V_m h) \subset \sigma_A(h)$.

On the other hand, suppose $\lambda_0 \in \rho_{\widetilde{M}}(V_m h)$. Then there exists an $H(D)$-valued analytic function $\widetilde{f}$ on some neighborhood U of λ_0 such that $(\widetilde{M}-\lambda)\widetilde{f}(\lambda) = V_m h$ for all $\lambda \in U$, where $H(D) = \oplus_1^2 W^m(D,\mathscr{H})/\overline{(A-z)\oplus_1^2 W^m(D,\mathscr{H})}$. Let $f \in \mathscr{O}(U,\oplus_1^2 W^m(D,\mathscr{H}))$ be a holomorphic lifting of $\widetilde{f}$ and let $f(\lambda,z) = (f(\lambda))(z)$ for $\lambda \in U$ and $z \in D$. Fix $\zeta \in U$. Then for $z \in D$,

$$h - (z-\zeta)f(\zeta,z) \in \overline{(A-z)\oplus_1^2 W^m(D,\mathscr{H})}.$$

Note that from Grothendieck theorem in [13],

$$\mathscr{O}(U,\oplus_1^2 W^m(D,\mathscr{H})) = \mathscr{O}(U)\hat{\otimes}(\oplus_1^2 W^m(D,\mathscr{H}))$$

where $\mathscr{O}(U)$ denotes the Fr$\acute{e}$chet space of all complex-valued analytic functions on U (i.e. $\mathscr{O}(U) := \mathscr{O}(U,\mathbb{C})$) and $\hat{\otimes}$ is the complete topological tensor product (see [13] for more details). Since the dense range property of a Hilbert space operator is preserved by the topological tensor product with the nuclear space $\mathscr{O}(U)$, there exists a sequence $\{g_n\} \subset \mathscr{O}(U,\oplus_1^2 W^m(D,\mathscr{H}))$ satisfying that

$$\lim_{n\to\infty}\big(h-(z-\zeta)f(\zeta,z)-(A-z)g_n(\zeta,z)\big) = 0 \tag{48}$$

with respect to Fr$\acute{e}$chet space topology of the space $\mathscr{O}(U,\oplus_1^2 W^m(D,\mathscr{H}))$. Let U_0 be a neighborhood of λ_0, relatively compact in U. Let $\mathfrak{r}$ be the unique continuous linear extension

$$\mathfrak{r}:\mathscr{O}(U)\hat{\otimes}(\oplus_1^2 W^m(D,\mathscr{H}))\to\oplus_1^2 W^m(U_0,\mathscr{H})$$

of the map $u\otimes v\to(u\cdot v)|_{U_0}$ where $u\in\mathscr{O}(U)$ and $v\in\oplus_1^2 W^m(D,\mathscr{H})$. Then

$$\mathfrak{r}\big(h-(z-\zeta)f(\zeta,z)-(A-z)g_n(\zeta,z)\big)=h-(A-z)f_n(z) \tag{49}$$

where $f_n(z):=g_n(z,z)$ for $z\in U_0$. Hence from the equations (48) and (49), we have

$$\lim_{n\to\infty}\|h-(A-z)f_n\|_{\oplus_1^2 W^m(U_0,\mathscr{H})}=0.$$

From the applications of the proof in Lemma 3.3 or Lemma 4.1, we obtain that

$$\lim_{n\to\infty}\|(I-P)f_n\|_{2,U_1}=0$$

where U_1 is an open neighborhood of λ_0 with $U_1\subsetneqq U_0$, and so

$$\lim_{n\to\infty}\|h-(A-z)Pf_n\|_{2,U_1}=0.$$

Thus $h\in\overline{(A-z)\oplus_1^2\mathscr{O}(U_2,\mathscr{H})}$ where U_2 is an open neighborhood of λ_0 with $U_2\subsetneqq U_1$. Since A has the property (β) from Corollary 5.2, $A-z$ should have closed range on $\oplus_1^2\mathscr{O}(U_2,\mathscr{H})$. Hence $h\in(A-z)\oplus_1^2\mathscr{O}(U_2,\mathscr{H})$, i.e., $\lambda_0\in\rho_A(h)$. □

COROLLARY 5.5. If A is an operator matrix on $\mathscr{H}\oplus\mathscr{H}$ with the same notations as in Theorem 3.4 or Theorem 4.2, then $\sigma(A)=\sigma(\widetilde{M})$.

Proof. Since $\sigma_A(h)=\sigma_{\widetilde{M}}(V_m h)$ for all $h\in\mathscr{H}\oplus\mathscr{H}$ by Theorem 5.4, where m is the appropriately chosen integer as in Theorem 3.4 or Theorem 4.2, $\sigma_A(h)\subset\sigma(\widetilde{M})$ for all $h\in\mathscr{H}\oplus\mathscr{H}$. Hence $\bigcup\{\sigma_A(h):h\in\mathscr{H}\oplus\mathscr{H}\}\subset\sigma(\widetilde{M})$. Since A has the single valued extension property by Corollary 5.2, $\sigma(A)=\bigcup\{\sigma_A(h):h\in\mathscr{H}\}\subset\sigma(\widetilde{M})$.

Conversely, note that if $U\subset\mathbb{C}$ is any bounded open set containing $\sigma(A)$ and M is the multiplication operator by z on $\oplus_1^2 W^m(U,\mathscr{H})$, then $\sigma(\widetilde{M})\subset\sigma(M)\subset\overline{U}$ holds. From this property, if $\lambda\in\rho(A)$, then we can choose an bounded open set D so that $\widetilde{M}-\lambda$ is invertible. Since this algebraic property is independent of the choice of D, we get $\sigma(\widetilde{M})\subset\sigma(A)$. □

COROLLARY 5.6. Let A be an operator matrix on $\mathscr{H}\oplus\mathscr{H}$ with the same notations as in Theorem 3.4 or Theorem 4.2. If A is quasinilpotent, then it is nilpotent.

Proof. If $\sigma(A)=\{0\}$, then $\widetilde{M}$ is nilpotent from [3], say with order k. Since $V_mA=\widetilde{M}V_m$ and V_m is one-to-one, $A^k=0$. □

A closed subspace of $\mathscr{H}$ is said to be *hyperinvariant* for T if it is invariant under every operator in the commutant $\{T\}'$ of T. An operator $T\in\mathscr{L}(\mathscr{H})$ is *decomposable* provided that, for each open cover $\{U,V\}$ of $\mathbb{C}$, there exist closed T-invariant subspaces Y, Z of $\mathscr{H}$ such that $\mathscr{H}=Y+Z$, $\sigma(T|_Y)\subset U$, and $\sigma(T|_Z)\subset V$.

THEOREM 5.7. Let A be an operator matrix on $\mathscr{H}\oplus\mathscr{H}$ having one of the forms in Theorem 3.4 or Theorem 4.2 and let $A\neq zI$ for all $z\in\mathbb{C}$. If S is a decomposable quasiaffine transform of A, then A has a nontrivial hyperinvariant subspace.

Proof. If S is a decomposable quasiaffine transform of A, there exists a quasiaffinity X such that $XS=AX$ where S is decomposable. If A has no nontrivial hyperinvariant subspace, we may assume that $\sigma_p(A)=\emptyset$ and $H_A(F)=\{0\}$ for each closed set F proper in $\sigma(A)$ by Lemma 3.6.1 of [14]. Let $\{U,V\}$ be an open cover of $\mathbb{C}$ with $\sigma(A)\backslash\overline{U}\neq\emptyset$ and $\sigma(A)\backslash\overline{V}\neq\emptyset$. If $x\in H_S(\overline{U})$, then $\sigma_S(x)\subset\overline{U}$. So there exists an analytic $\mathscr{H}\oplus\mathscr{H}$-valued function f defined on $\mathbb{C}\setminus\overline{U}$ such that $(S-z)f(z)\equiv x$ for all $z\in\mathbb{C}\setminus\overline{U}$. Hence $(A-z)Xf(z)=X(S-z)f(z)=Xx$ for all $z\in\mathbb{C}\setminus\overline{U}$. Thus $\mathbb{C}\setminus\overline{U}\subset\rho_A(Xx)$, which implies that $Xx\in H_A(\overline{U})$, i.e., $XH_S(\overline{U})\subset H_A(\overline{U})$. Similarly, $XH_S(\overline{V})\subset H_A(\overline{V})$. Then since S is decomposable,

$$X\mathscr{H}=XH_S(\overline{U})+XH_S(\overline{V})\subset H_A(\overline{U})+H_A(\overline{V})=\{0\}.$$

But this is a contradiction. So A has a nontrivial hyperinvariant subspace. $\square$

6. Further results

In this section, we consider some properties of 2×2 operator matrices. First we will consider some spectral properties of 2×2 operator matrices.

PROPOSITION 6.1. Let $A=\begin{pmatrix} T_1 & T_2\\ T_3 & T_4\end{pmatrix}$ be an operator matrix defined on $\mathscr{H}\oplus\mathscr{H}$, where T_j are mutually commuting operators on $\mathscr{H}$ for $j=1,2,3,4$.

(a) If $T_2T_3=0$, then $\sigma_p(A)\subset\sigma_p(T_1)\cup\sigma_p(T_4)$, $\sigma_{ap}(A)\subset\sigma_{ap}(T_1)\cup\sigma_{ap}(T_4)$ and $\sigma(A)\subset\sigma(T_1)\cup\sigma(T_4)$. In this case, $\sigma_p(A)=\sigma_p(T_1)\cup\sigma_p(T_4)$ when $0\notin\sigma_p(T_2)\cup\sigma_p(T_3)$, and $\sigma_{ap}(A)=\sigma_{ap}(T_1)\cup\sigma_{ap}(T_4)$ when $0\notin\sigma_{ap}(T_2)\cup\sigma_{ap}(T_3)$.

(b) If $\det(A):=T_1T_4-T_2T_3=0$, then $\sigma_p(A)\setminus\{0\}\subset\sigma_p(T_1)\cup\sigma_p(T_1+T_4)$, $\sigma_{ap}(A)\setminus\{0\}\subset\sigma_{ap}(T_1)\cup\sigma_{ap}(T_1+T_4)$, and $\sigma(A)\setminus\{0\}=\sigma(T_1+T_4)\setminus\{0\}$.

Proof. (a) Let $T_2T_3=0$. If $\lambda\in\sigma_{ap}(A)$, then there exists a sequence $\{x_n^1\oplus x_n^2\}$ of unit vectors in $\mathscr{H}\oplus\mathscr{H}$ such that

$$\lim_{n\to\infty}\|(A-\lambda)(x_n^1\oplus x_n^2)\|=0.$$

From this, we have

$$\begin{cases}\lim_{n\to\infty}\|(T_1-\lambda)x_n^1+T_2x_n^2\|=0\\ \lim_{n\to\infty}\|T_3x_n^1+(T_4-\lambda)x_n^2\|=0.\end{cases}\tag{50}$$

Since $T_2T_3=0$, it follows from (50) that

$$\lim_{n\to\infty}\|(T_1-\lambda)T_3x_n^1\|=0.\tag{51}$$

If $\lim_{n\to\infty} \|T_3 x_n^1\| \neq 0$, then $\lambda \in \sigma_{ap}(T_1)$. Otherwise, it holds by (50) that

$$\lim_{n\to\infty} \|(T_4-\lambda)x_n^2\| = 0.$$

If $\lim_{n\to\infty} \|x_n^2\| \neq 0$, then $\lambda \in \sigma_{ap}(T_4)$. Suppose that $\lim_{n\to\infty} \|x_n^2\| = 0$. Since $\|x_n^1\|^2 + \|x_n^2\|^2 = 1$ for all n, $\lim_{n\to\infty} \|x_n^1\| \neq 0$. In addition $\lim_{n\to\infty} \|(T_1-\lambda)x_n^1\| = 0$, which implies $\lambda \in \sigma_{ap}(T_1)$. Hence we can conclude that $\sigma_{ap}(A) \subset \sigma_{ap}(T_1) \cup \sigma_{ap}(T_4)$. Similarly, we can show that $\sigma_p(A) \subset \sigma_p(T_1) \cup \sigma_p(T_4)$. For the last inclusion, let $\lambda \in \sigma(A)$. Then $(T_1-\lambda)(T_4-\lambda)$ is not invertible by [7]. Thus, at least one of $T_1-\lambda$ and $T_4-\lambda$ is not invertible, and so $\sigma(A) \subset \sigma(T_1) \cup \sigma(T_4)$.

Now suppose $0 \notin \sigma_{ap}(T_2) \cup \sigma_{ap}(T_3)$. If $\lambda \in \sigma_{ap}(T_1)$, then there is a sequence $\{x_n\}$ of unit vectors in $\mathscr{H}$ such that $\lim_{n\to\infty} \|(T_1-\lambda)x_n\| = 0$. Since $T_2T_3 = 0$, we have

$$\lim_{n\to\infty} \left\|(A-\lambda)\begin{pmatrix} T_2x_n \\ 0 \end{pmatrix}\right\| = \lim_{n\to\infty} \left\|\begin{pmatrix} T_2(T_1-\lambda)x_n \\ T_2T_3x_n \end{pmatrix}\right\| = 0.$$

Since $0 \notin \sigma_{ap}(T_2)$, it must hold that $\lim_{n\to\infty} \|T_2x_n\| \neq 0$, and hence $\lambda \in \sigma_{ap}(A)$. Similarly, if $\lambda \in \sigma_{ap}(T_4)$, then we can derive $\lambda \in \sigma_{ap}(A)$ by using the assumption $0 \notin \sigma_{ap}(T_3)$. Therefore, $\sigma_{ap}(A) = \sigma_{ap}(T_1) \cup \sigma_{ap}(T_4)$. By the same way, if $0 \notin \sigma_p(T_2) \cup \sigma_p(T_3)$, then we get that $\sigma_p(A) = \sigma_p(T_1) \cup \sigma_p(T_4)$.

(b) We will first show that $\sigma_{ap}(A) \setminus \{0\} \subset \sigma_{ap}(T_1) \cup \sigma_{ap}(T_1+T_4)$. If $\lambda \in \sigma_{ap}(A) \setminus \{0\}$, then we can choose a sequence $\{x_n^1 \oplus x_n^2\}$ of unit vectors in $\mathscr{H} \oplus \mathscr{H}$ such that

$$\lim_{n\to\infty} \|(A-\lambda)(x_n^1 \oplus x_n^2)\| = 0.$$

This induces that

$$\begin{cases} \lim_{n\to\infty} \|(T_1-\lambda)x_n^1 + T_2x_n^2\| = 0 \\ \lim_{n\to\infty} \|T_3x_n^1 + (T_4-\lambda)x_n^2\| = 0. \end{cases} \tag{52}$$

By (52), we get that

$$\begin{cases} \lim_{n\to\infty} \|(T_1T_3-\lambda T_3)x_n^1 + T_2T_3x_n^2\| = 0 \\ \lim_{n\to\infty} \|T_1T_3x_n^1 + (T_1T_4-\lambda T_1)x_n^2\| = 0. \end{cases} \tag{53}$$

Since $T_1T_4 = T_2T_3$ and $\lambda \neq 0$, we obtain from (53) that

$$\lim_{n\to\infty} \|T_1x_n^2 - T_3x_n^1\| = 0.$$

Combining this with (52), we have

$$\lim_{n\to\infty} \|(T_1+T_4-\lambda)x_n^2\| = 0.$$

If $\lim_{n\to\infty} \|x_n^2\| \neq 0$, then $\lambda \in \sigma_{ap}(T_1+T_4)$. If $\lim_{n\to\infty} \|x_n^2\| = 0$, then it follows that $\lim_{n\to\infty} \|x_n^1\| \neq 0$ and $\lim_{n\to\infty} \|(T_1-\lambda)x_n^1\| = 0$. Therefore, $\lambda \in \sigma_{ap}(T_1)$. Similarly, we can prove the case of the point spectrum.

Finally, it remains to show that $\sigma(A) \setminus \{0\} = \sigma(T_1+T_4) \setminus \{0\}$. Let $\lambda \in \mathbb{C} \setminus \{0\}$. From [7], $\lambda \in \sigma(A)$ is equivalent to the statement that $(T_1-\lambda)(T_4-\lambda) - T_2T_3$ is not invertible; that is, $T_1+T_4-\lambda$ is not invertible, because $T_1T_4 - T_2T_3 = 0$ and $\lambda \neq 0$. Hence $\sigma(A) \setminus \{0\} = \sigma(T_1+T_4) \setminus \{0\}$. $\square$

PROPOSITION 6.2. Let $A=\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ be an operator matrix defined on $\mathscr{H}\oplus\mathscr{H}$, where T_j are mutually commuting operators on $\mathscr{H}$ for $j=1,2,3,4$. If T_3 is nilpotent of order k, then $\sigma_{T_4}(T_3^{k-1}y)\subset\sigma_A(x\oplus y)$ for any $x\oplus y\in\mathscr{H}\oplus\mathscr{H}$. If, in addition, T_2 is nilpotent of order m, then $\sigma_{T_1}(T_2^{m-1}x)\cup\sigma_{T_4}(T_3^{k-1}y)\subset\sigma_A(x\oplus y)$ for any $x\oplus y\in\mathscr{H}\oplus\mathscr{H}$.

Proof. Let $z_0\in\rho_A(x\oplus y)$. Then there exist analytic functions $f(z)$ and $g(z)$ on some neighborhood U of z_0 on which

$$(A-z)(f(z)\oplus g(z))\equiv x\oplus y.$$

This implies that

$$\begin{cases}(T_1-z)f(z)+T_2g(z)=x\\ T_3f(z)+(T_4-z)g(z)=y\end{cases}\tag{54}$$

for all $z\in U$. Since $T_3^k=0$, we get from (54) that $(T_4-z)T_3^{k-1}g(z)=T_3^{k-1}y$, and so $z_0\in\rho_{T_4}(T_3^{k-1}y)$. Hence, $\sigma_{T_4}(T_3^{k-1}y)\subset\sigma_A(x\oplus y)$. Similarly, if T_2 is nilpotent of order m, $\sigma_{T_1}(T_2^{m-1}x)\subset\sigma_A(x\oplus y)$. Hence $\sigma_{T_1}(T_2^{m-1}x)\cup\sigma_{T_4}(T_3^{k-1}y)\subset\sigma_A(x\oplus y)$. □

COROLLARY 6.3. Let $A=\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ be an operator matrix defined on $\mathscr{H}\oplus\mathscr{H}$, where T_j are mutually commuting operators on $\mathscr{H}$ for $j=1,2,3,4$. If T_2 and T_3 are nilpotent of order m and k, respectively, then $(T_2^{m-1}\oplus T_3^{k-1})H_A(F)\subset H_{T_1\oplus T_4}(F)$ for any subset F in $\mathbb{C}$.

Proof. If $x\oplus y\in H_A(F)$, then $\sigma_A(x\oplus y)\subset F$. First we will claim that $\sigma_{T_1}(T_2^{m-1}x)\cup\sigma_{T_4}(T_3^{k-1}y)=\sigma_{T_1\oplus T_4}(T_2^{m-1}x\oplus T_3^{k-1}y)$. Suppose that there are $\mathscr{H}$-valued analytic functions f_1 and f_2 on some open set U in $\mathbb{C}$ such that

$$(T_1\oplus T_4-z)(f_1(z)\oplus f_2(z))=T_2^{m-1}x\oplus T_3^{k-1}y$$

for all $z\in U$. This is equivalent to the following; for all $z\in U$

$$\begin{cases}(T_1-z)f_1(z)=T_2^{m-1}x \text{ and}\\ (T_4-z)f_2(z)=T_3^{k-1}y.\end{cases}$$

Hence, we can obtain that

$$\rho_{T_1}(T_2^{m-1}x)\cap\rho_{T_4}(T_3^{k-1}y)=\rho_{T_1\oplus T_4}(T_2^{m-1}x\oplus T_3^{k-1}y).$$

That is, $\sigma_{T_1}(T_2^{m-1}x)\cup\sigma_{T_4}(T_3^{k-1}y)=\sigma_{T_1\oplus T_4}(T_2^{m-1}x\oplus T_3^{k-1}y)$, and so Proposition 6.2 implies $\sigma_{T_1\oplus T_4}(T_2^{m-1}x\oplus T_3^{k-1}y)\subset F$. Hence $T_2^{m-1}x\oplus T_3^{k-1}y\in H_{T_1\oplus T_4}(F)$. Thus $(T_2^{m-1}\oplus T_3^{k-1})H_A(F)\subset H_{T_1\oplus T_4}(F)$. □

THEOREM 6.4. Let $A = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ be an operator matrix defined on $\mathscr{H} \oplus \mathscr{H}$, where T_j are mutually commuting operators on $\mathscr{H}$ for $j = 1,2,3,4$. Suppose that A has the property (β).

(a) If T_3 is nilpotent, then T_1 has the property (β).

(b) If T_2 is nilpotent, then T_4 has the property (β).

(c) If both T_2 and T_3 are nilpotent, then T_1 and T_4 have the property (β).

Conversely, suppose that T_1 and T_4 have the property (β). If T_2 or T_3 is nilpotent, then A has the property (β).

Proof. (a) Suppose that A has the property (β). Let $T_3^k = 0$ and let $\{f_n\}$ be any sequence of $\mathscr{H}$-valued analytic functions on an open set G in $\mathbb{C}$ such that $\{(T_1 - z)f_n(z)\}$ converges uniformly to 0 on every compact subset of G. Let K be any compact subset of G. Then

$$\lim_{n\to\infty} \|(T_1 - z)f_n(z)\| = 0 \tag{55}$$

uniformly on K. Since

$$(A - z)\begin{pmatrix} T_3^{k-1} f_n(z) \\ 0 \end{pmatrix} = \begin{pmatrix} (T_1 - z)T_3^{k-1} f_n(z) \\ T_3^k f_n(z) \end{pmatrix} = \begin{pmatrix} T_3^{k-1}(T_1 - z)f_n(z) \\ 0 \end{pmatrix},$$

from (55) we get that $\lim_{n\to\infty} \|(A-z)(T_3^{k-1} f_n(z) \oplus 0)\| = 0$ uniformly on K. Since A has the property (β), we obtain

$$\lim_{n\to\infty} \|T_3^{k-1} f_n(z)\| = 0 \tag{56}$$

uniformly on K. Similarly, since

$$(A - z)\begin{pmatrix} T_3^{k-2} f_n(z) \\ 0 \end{pmatrix} = \begin{pmatrix} T_3^{k-2}(T_1 - z)f_n(z) \\ T_3^{k-1} f_n(z) \end{pmatrix},$$

(55) and (56) imply that $\lim_{n\to\infty} \|(A-z)(T_3^{k-2} f_n(z) \oplus 0)\| = 0$ uniformly on K. Since A has the property (β), it holds that

$$\lim_{n\to\infty} \|T_3^{k-2} f_n(z)\| = 0$$

uniformly on K. By continuing this procedure, we can conclude $\{f_n(z)\}$ eventually converges uniformly to 0 on any compact subset K of G. Therefore, T_1 has the property (β).

(b) The proof is analogous to the above.

(c) It follows immediately from (a) and (b).

In order to prove the last statement, assume that T_1 and T_4 have the property (β) and T_2 is nilpotent of order k for some positive integer k. Let $\{f_n\}$ and $\{g_n\}$ be sequences of $\mathscr{H}$-valued analytic functions on an open subset G of $\mathbb{C}$ such that $\{(A-z)(f_n(z) \oplus g_n(z))\}$ converges uniformly to 0 on every compact subset of G. Let K be any compact subset of G. Note that

$$(A - z)\begin{pmatrix} f_n(z) \\ g_n(z) \end{pmatrix} = \begin{pmatrix} (T_1 - z)f_n(z) + T_2 g_n(z) \\ T_3 f_n(z) + (T_4 - z)g_n(z) \end{pmatrix},$$

which implies that

$$\begin{cases} \lim_{n\to\infty}\|(T_1-z)f_n(z)+T_2g_n(z)\|=0 \\ \lim_{n\to\infty}\|T_3f_n(z)+(T_4-z)g_n(z)\|=0 \end{cases} \tag{57}$$

uniformly on K. Since $T_2^k=0$, (57) induces that $\lim_{n\to\infty}\|(T_1-z)T_2^{k-1}f_n(z)\|=0$ uniformly on K. Since T_1 has the property (β),

$$\lim_{n\to\infty}\|T_2^{k-1}f_n(z)\|=0 \tag{58}$$

uniformly on K. From (58) we obtain that

$$\lim_{n\to\infty}\|(T_4-z)T_2^{k-1}g_n(z)\|=0$$

uniformly on K, as multiplying the second equation of (57) by T_2^{k-1}. Since T_4 has the property (β), we have

$$\lim_{n\to\infty}\|T_2^{k-1}g_n(z)\|=0 \tag{59}$$

uniformly on K. Therefore, multiplying the first equation of (57) by T_2^{k-2}, it holds from (59) that

$$\lim_{n\to\infty}\|(T_1-z)T_2^{k-2}f_n(z)\|=0$$

uniformly on K. Since T_1 has the property (β),

$$\lim_{n\to\infty}\|T_2^{k-2}f_n(z)\|=0$$

uniformly on K, which ensures

$$\lim_{n\to\infty}\|(T_4-z)T_2^{k-2}g_n(z)\|=0$$

uniformly on K. Since T_4 has the property (β), it follows that

$$\lim_{n\to\infty}\|T_2^{k-2}g_n(z)\|=0$$

uniformly on K. By repeating this procedure, we finally achieve

$$\lim_{n\to\infty}\|f_n(z)\|=\lim_{n\to\infty}\|g_n(z)\|=0$$

uniformly on K. Hence $\{f_n\oplus g_n\}$ converges uniformly to 0 on any compact subset K of G, and so A has the property (β). The above proof is applicable for the case when T_3 is nilpotent. $\square$

REMARK. Theorem 6.4 still holds even if we replace the property (β) by the single-valued extension property.

Recall that for an operator $T\in\mathscr{L}(\mathscr{H})$, we define *a spectral maximal space* of T to be a closed T-invariant subspace $\mathscr{M}$ of $\mathscr{H}$ with the property that $\mathscr{M}$ contains any closed T-invariant subspace $\mathscr{N}$ of $\mathscr{H}$ such that $\sigma(T|_{\mathscr{N}})\subset\sigma(T|_{\mathscr{M}})$.

COROLLARY 6.5. Let $A = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ be an operator matrix defined on $\mathscr{H} \oplus \mathscr{H}$, where T_j are mutually commuting operators on $\mathscr{H}$ for $j = 1,2,3,4$. Suppose that T_1 and T_4 have the property (β). If T_2 or T_3 is nilpotent, then $H_A(F)$ is a spectral maximal space of A and $\sigma(A|_{H_A(F)}) \subset \sigma(A) \cap F$ for any closed subset F in $\mathbb{C}$.

Proof. Since A has the property (β) from Theorem 6.4, $H_A(F)$ is closed. Hence the proof follows from [3] or [13]. □

COROLLARY 6.6. Under the same hypothesis as Corollary 6.5, if $XB = AX$ where X is a quasiaffinity, then B has the single-valued extension property and $XH_B(F) \subset H_A(F)$ for any subset F in $\mathbb{C}$.

Proof. Let $f : D \to \mathscr{H}$ be an analytic function on an open set D such that $(B - z)f(z) \equiv 0$. Then $(A - z)Xf(z) = X(B - z)f(z) \equiv 0$ on D. Since A has the single-valued extension property be Theorem 6.4, $Xf(z) \equiv 0$ on D. Since X is a quasiaffinity, $f(z) \equiv 0$ on D. Hence B has the single-valued extension property. To prove the last conclusion, it suffices to show that $\sigma_A(Xx) \subset \sigma_B(x)$ for any $x \in \mathscr{H}$; in fact, if it holds, then $x \in H_B(F)$ implies $\sigma_A(Xx) \subset F$, which means that $Xx \in H_A(F)$. If $z_0 \in \rho_B(x)$, then we can choose an $\mathscr{H}$-valued analytic function f on some neighborhood of z_0 for which $(B - z)f(z) \equiv x$. Since $XB = AX$, we have $X(B - z)f(z) = (A - z)Xf(z) \equiv Xx$, and so $z_0 \in \rho_A(Xx)$. □

COROLLARY 6.7. Under the same hypothesis as Corollary 6.5, let F be any closed set in $\mathbb{C}$ and $x \in H_A(F)$. If $f : \rho_A(x) \to \mathscr{H} \oplus \mathscr{H}$ is an analytic function such that $(A - z)f(z) \equiv x$, then $O_A(x) \subset H_A(F)$, where $O_A(x)$ is the linear closed subspace generated by all the values $f(z)$ with $z \in \rho_A(x)$.

Proof. The proof follows from Corollary 6.5 and [3]. □

Recall that an operator $T \in \mathscr{L}(\mathscr{H})$ is *totally $*$-paranormal* if $\|(T - z)^* x\|^2 \leqslant \|(T - z)^2 x\| \|x\|$ for all $x \in \mathscr{H}$ and all $z \in \mathbb{C}$ (see [12] for more details). The following proposition whose proof is based on the method of [22] gives an example of an operator matrix which has the property (β).

PROPOSITION 6.8. Let $A = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ be an operator matrix defined on $\mathscr{H} \oplus \mathscr{H}$, where T_j are mutually commuting operators on $\mathscr{H}$ for $j = 1,2,3,4$. Suppose that T_1 and T_4 are totally $*$-paranormal. If T_2 or T_3 is nilpotent, then A has the property (β).

Proof. From Theorem 6.4, it suffices to show that every totally $*$-paranormal operator has the property (β). Suppose that $T \in \mathscr{L}(\mathscr{H})$ is totally $*$-paranormal. Let G be any open subset of $\mathbb{C}$, and let $f_n : G \to \mathscr{H}$ be a sequence of analytic functions such that

$$\lim_{n\to\infty} \|(T - z)f_n(z)\| = 0 \tag{60}$$

uniformly on every compact subset K of G. From now, let K be any compact disk in G with $K = \overline{B(z_0;R)}$ for some $z_0 \in G$ and $R > 0$, and let $M = \sup_n \|f_n\|_{\overline{B(z_0;R)}} < \infty$.

Then for all n and $z \in \overline{B(z_0;r)}$ with $0 < r < R$, by Cauchy's integral formula we get the following inequality

$$\begin{aligned} \| f_n(z) - f_n(z_0) \| &= \left\| \frac{1}{2\pi i} \int_{|\xi - z_0| = R} \frac{f_n(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \int_{|\xi - z_0| = R} \frac{f_n(\xi)}{\xi - z_0} d\xi \right\| \\ &\leqslant \frac{1}{2\pi} \int_{|\xi - z_0| = R} \frac{| z - z_0 | \, \| f_n(\xi) \|}{| \xi - z | \, | \xi - z_0 |} \, | d\xi | \\ &\leqslant \frac{Mr}{R - r}. \end{aligned} \tag{61}$$

For all n and all $z \in \overline{B(z_0;r)}$ with $0 < r < R$, (61) implies that

$$\begin{aligned} \| f_n(z_0) \|^2 &= \langle f_n(z_0) - f_n(z), f_n(z_0) \rangle + \langle f_n(z), f_n(z_0) \rangle \\ &\leqslant \| f_n(z_0) - f_n(z) \| \| f_n(z_0) \| + | \langle f_n(z), f_n(z_0) \rangle | \\ &\leqslant \frac{M^2 r}{R - r} + | \langle f_n(z), f_n(z_0) \rangle | . \end{aligned} \tag{62}$$

Also the inequality

$$\| f_n(z) \| \leqslant \| f_n(z) - f_n(z_0) \| + \| f_n(z_0) \| \tag{63}$$

holds. Choose a sufficiently small $r > 0$ such that $\frac{Mr}{R-r} < \frac{\varepsilon}{2}$ and $\frac{M^2 r}{R-r} < \frac{\varepsilon^2}{8}$. Then by the above inequalities from (61) to (63) we get that

$$\begin{cases} \| f_n(z_0) \|^2 < \frac{\varepsilon^2}{8} + | \langle f_n(z), f_n(z_0) \rangle | \\ \| f_n(z) \| < \frac{\varepsilon}{2} + \| f_n(z_0) \|. \end{cases} \tag{64}$$

On the other hand, let $z_1 \in \overline{B(z_0;r)} \setminus \{z_0\}$. Then

$$\begin{cases} \lim_{n\to\infty} \| (T - z_0) f_n(z_0) \| = 0 \\ \lim_{n\to\infty} \| (T - z_1) f_n(z_1) \| = 0. \end{cases} \tag{65}$$

Since T is totally $*$-paranormal,

$$\lim_{n\to\infty} \| (T - z_1)^* f_n(z_1) \| = 0. \tag{66}$$

Note that

$$\begin{aligned} & (z_0 - z_1) \langle f_n(z_0), f_n(z_1) \rangle \\ = & \langle (z_0 - T) f_n(z_0), f_n(z_1) \rangle + \langle (T - z_1) f_n(z_0), f_n(z_1) \rangle \\ = & \langle (z_0 - T) f_n(z_0), f_n(z_1) \rangle + \langle f_n(z_0), (T - z_1)^* f_n(z_1) \rangle. \end{aligned} \tag{67}$$

Hence from (65), (66) and (67) we have

$$\lim_{n\to\infty} \langle f_n(z_0), f_n(z_1) \rangle = 0. \tag{68}$$

Thus there exists a positive integer N such that for all $n \geqslant N$

$$|\langle f_n(z_0), f_n(z_1)\rangle| < \frac{\varepsilon^2}{8}. \tag{69}$$

Combining (64) and (69), we can conclude that $\|f_n(z)\| < \varepsilon$ for all $z \in \overline{B(z_0;r)}$ with $0 < r < R$. Hence T has the property (β). $\square$

REMARK. From the proof of Proposition 6.8 we observe that every totally $*$-paranormal operator has the property (β).

Finally, we shall consider the special case of 2×2 operator matrices whose entries do not commute. For this, recall that for a bounded sequence $\{\alpha_n\}_{n=1}^{\infty}$ in $\mathbb{C}$ an operator $W \in \mathscr{L}(\mathscr{H})$ is called a (*unilateral*) *weighted shift* with weight $\{\alpha_n\}$ if $We_n = \alpha_n e_{n+1}$ for $n \in \mathbb{N}$.

PROPOSITION 6.9. Let $T = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix}$ be an operator matrix in $\mathscr{L}(\mathscr{H}\oplus\mathscr{H})$ where W_i are weighted shifts with weights $\{\alpha_k^{(i)}\}$ for $i = 1,2,3,4$. Then T has the property (β) and the single-valued extension property.

Proof. If T has the property (β), then it has the single-valued extension property. Hence we only have to show that T has the property (β). Let G be any open subset of $\mathbb{C}$, and let $\{f_n \oplus g_n\}_{n=1}^{\infty}$ be a sequence of $\mathscr{H}\oplus\mathscr{H}$-valued analytic functions on G such that

$$\lim_{n\to\infty} \|(T-z)(f_n(z)\oplus g_n(z))\| = 0 \tag{70}$$

uniformly on every compact subset K of G. Since

$$\begin{aligned}(T-z)(f_n(z)\oplus g_n(z)) &= \begin{pmatrix} W_1 - z & W_2 \\ W_3 & W_4 - z \end{pmatrix}\begin{pmatrix} f_n(z) \\ g_n(z) \end{pmatrix} \\ &= \begin{pmatrix} (W_1 - z)f_n(z) + W_2 g_n(z) \\ W_3 f_n(z) + (W_4 - z)g_n(z) \end{pmatrix},\end{aligned}$$

from (70) we get that

$$\begin{cases} \lim_{n\to\infty} \|(W_1 - z)f_n(z) + W_2 g_n(z)\| = 0 \\ \lim_{n\to\infty} \|W_3 f_n(z) + (W_4 - z)g_n(z)\| = 0 \end{cases} \tag{71}$$

uniformly on every compact subset K of G. For the orthonormal basis $\{e_k\}_{k=1}^{\infty}$ of $\mathscr{H}$, we set $f_n(z) = \sum_{k=1}^{\infty} f_{n,k}(z)e_k$ and $g_n(z) = \sum_{k=1}^{\infty} g_{n,k}(z)e_k$ where $f_{n,k} : G \to \mathbb{C}$ and $g_{n,k} : G \to \mathbb{C}$ are analytic functions. For any $k \in \mathbb{N}$, from (71) we obtain that

$$\begin{cases} \lim_{n\to\infty} z f_{n,1}(z) = 0 \\ \lim_{n\to\infty} (\alpha_k^{(1)} f_{n,k}(z) - z f_{n,k+1}(z) + \alpha_k^{(2)} g_{n,k}(z)) = 0, \text{ and} \end{cases} \tag{72}$$

$$\begin{cases} \lim_{n\to\infty} zg_{n,1}(z) = 0 \\ \lim_{n\to\infty}(\alpha_k^{(3)} f_{n,k}(z) - zg_{n,k+1}(z) + \alpha_k^{(4)} g_{n,k}(z)) = 0 \end{cases} \tag{73}$$

uniformly on every compact subset K of G. Since a zero operator is hyponormal and hyponormal operators satisfy the property (β), the equations (72) and (73) imply that $f_{n,1}(z)$ and $g_{n,1}(z)$ converge uniformly to 0 on every compact subset K of G. Then from (72) and (73) we get that for all $k \in \mathbb{N}$

$$\lim_{n\to\infty} zf_{n,k+1}(z) = \lim_{n\to\infty} zg_{n,k+1}(z) = 0 \tag{74}$$

uniformly on every compact subset K of G. By the hyponormality of a zero operator, we can apply the property (β) of hyponormal operators to (74). Then $f_{n,k+1}(z)$ and $g_{n,k+1}(z)$ converge uniformly to 0 on every compact subset K of G. Thus $f_n(z)$ and $g_n(z)$ converge uniformly to 0 on every compact subset K of G. Hence T has the property (β). $\square$

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