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ESTIMATING EIGENVALUES OF MATRICES BY INDUCED NORMS

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Abstract. A classical result of König in terms of matrices states that for $1 \leqslant p < q \leqslant \infty$ the eigenvalues $\lambda_1(A),\ldots,\lambda_n(A)$ of an $n\times n$ square matrix A satisfy $\max_k k^{\frac{1}{p}-\frac{1}{q}}|\lambda_k(A)| \leqslant C_{q,p}\|A\|_{q,p}$ for some absolute constant $C_{q,p}>0$ not depending on the matrix A, where $\|A\|_{q,p}$ denotes the norm of A viewed as an operator from ℓ_q^n into ℓ_p^n. We refine this result for $1 \leqslant p < q \leqslant 2$ by means of interpolation of Banach spaces.

1. Introduction

For $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$ let $A=(a_{ij})\in\mathbb{K}^{n\times n}$ be a square matrix. Denote by $\lambda_1(A),\ldots,\lambda_n(A)$ the eigenvalues of A (viewed as a matrix over the field $\mathbb{C}$ if necessary) counted according to their algebraic multiplicity and satisfying $|\lambda_1(A)| \geqslant |\lambda_2(A)| \geqslant \ldots \geqslant |\lambda_n(A)|$, and by $s_1(A)\geqslant s_2(A)\geqslant\ldots\geqslant s_n(A)\geqslant 0$ the square roots of the eigenvalues of the selfadjoint and positive matrix $\overline{A}^t A$, called the singular values of A. Furthermore, for $1\leqslant r<\infty$ we set

$$\Lambda_r(A) := \Big(\sum_{k=1}^n |\lambda_k(A)|^r\Big)^{\frac{1}{r}} \qquad \text{and} \qquad \Lambda_{r,\infty}(A) := \max_{k=1,\ldots,n} k^{\frac{1}{r}}|\lambda_k(A)|$$

as well as

$$\sigma_r(A) := \Big(\sum_{k=1}^n s_k(A)^r\Big)^{\frac{1}{r}} \qquad \text{and} \qquad \sigma_{r,\infty}(A) := \max_{k=1,\ldots,n} k^{\frac{1}{r}} s_k(A).$$

For $1\leqslant p,q\leqslant\infty$ we set

$$\|A\|_{q,p} := \sup_{x\neq 0}\frac{\|Ax\|_p}{\|x\|_q} \qquad \text{and} \qquad |A|_{q,p} := \Big\|\Big(\|Ae_j\|_q\Big)_{j=1,\ldots,n}\Big\|_p,$$

where, as usual, for $x_1,\ldots,x_n\in\mathbb{K}$ we denote

$$\|(x_1,\ldots,x_n)\|_p := \Big(\sum_{i=1}^n |x_i|^p\Big)^{\frac{1}{p}}$$

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if $1 \leqslant p < \infty$, and

$$\|(x_1,\dots,x_n)\|_\infty := \max_{k=1,\dots,n} |x_k|,$$

e_j the j-th standard unit vector in $\mathbb{K}^n$. For $1 \leqslant p \leqslant \infty$, we then set $\ell_p^n := (\mathbb{K}^n, \|\cdot\|_p)$. For $1 \leqslant p,q < \infty$, a more explicit expression for $|A|_{q,p}$ is given by

$$|A|_{q,p} = \Big(\sum_{j=1}^n \Big(\sum_{i=1}^n |a_{ij}|^q \Big)^{\frac{p}{q}} \Big)^{\frac{1}{p}},$$

and if $1 \leqslant q < \infty$, then $|A|_{q,\infty}$ (which will frequently appear in our formulas) reads as

$$|A|_{q,\infty} = \max_{j=1,\dots,n} \Big(\sum_{i=1}^n |a_{ij}|^q \Big)^{\frac{1}{q}}.$$

Classical work due to König (see also the surveys [16, Corollary 12] and [17, Proposition 20], as well as [9] for a recent generalization) gives the following connections between eigenvalues and induced norms :

THEOREM 1.1. ([15], 2.b.11) For $1 \leqslant p < q \leqslant \infty$ define by $\alpha := \frac{1}{p} - \frac{1}{q}$, and let $A \in \mathbb{C}^{n\times n}$.

(a) If $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$, then $\Lambda_{\frac{1}{\alpha}}(A) \leqslant 4\,\|A\|_{q,p}$.

(b) If $1 \leqslant p < q = 2$, then $\sigma_{\frac{1}{\alpha}}(A) \leqslant \sqrt{2}\,\|A\|_{2,p}$.

(c) If $1 \leqslant p < q < 2$, then $\Lambda_{\frac{1}{\alpha},\infty}(A) \leqslant 2e\sqrt{2}\,\|A\|_{q,p}$.

Clearly, these estimates are optimal (up to a multiplicative constant) for a wide class of matrices, in particular for diagonal matrices. However, for other matrices this is not the case. Take, e.g., the matrix J_n where all entries are equal to 1. It has the only non-zero eigenvalue n with multiplicity 1, so that the left-hand side in the above is always of the order n, whereas $\|J_n\|_{q,p} = n^{1+\alpha}$. For such a matrix, the following celebrated result (see also [5] for an elementary approach and [8] for a generalization within the framework of Orlicz norms, respectively) is more suitable (as usual, we denote by r' the conjugate number of r, i.e., $\frac{1}{r} + \frac{1}{r'} = 1$):

THEOREM 1.2. ([14]) Let $2 \leqslant r < \infty$. Then for any matrix $A \in \mathbb{C}^{n\times n}$

$$\Lambda_r(A) \leqslant |A|_{r',r}. \tag{1.1}$$

While giving once again the exact result for diagonal matrices, the obvious disadvantage of (1.1) is the fact that this estimate only depends on the modulus of the entries of the matrix. For instance, for an arbitrary matrix with moduli of its entries all equal to 1 one would get the same estimate as for J_n, although the behaviour of the eigenvalues may be completely different (take, e.g., a matrix orthogonal up to the factor $\sqrt{n}$, for which all its eigenvalues have modulus equal to $\sqrt{n}$).

We would now like to pursue simultaneously the following two different tasks here:

(i) Find upper estimates for $\Lambda_r(A)$ or $\Lambda_{r,\infty}(A)$ in terms of the induced norms $\|A\|_{q,p}$ which are better suited for matrices far from diagonal matrices (e.g., matrices for which the moduli of the entries (almost) all coincide).

(ii) Find upper estimates for $\Lambda_r(A)$ or $\Lambda_{r,\infty}(A)$ in terms of the induced norms $\|A\|_{q,p}$ for *all* $r > \frac{1}{\alpha}$.

Let us briefly describe how our resulting estimates will look like: Using König's result and the trivial estimate $\Lambda_\infty(A) := |\lambda_1(A)| \leqslant \|A\|_{1,1} = |A|_{1,\infty}$, standard interpolation techniques then yield for any $r > \frac{1}{\alpha}$

$$\Lambda_{r,\infty}(A) \leqslant \Lambda_{\frac{1}{\alpha},\infty}(A)^{\frac{1}{\alpha r}} \Lambda_\infty(A)^{1-\frac{1}{\alpha r}} \leqslant C_\alpha^{\frac{1}{\alpha r}} \|A\|_{q,p}^{\frac{1}{\alpha r}} |A|_{1,\infty}^{1-\frac{1}{\alpha r}},$$

where $C_\alpha > 0$ is the constant occuring in Theorem 1.1, respectively. We will show that in the case $1 \leqslant p < q \leqslant 2$ the above can be improved upon in the sense that $|A|_{1,\infty}$ can be replaced by $|A|_{s(\alpha,r),\infty}$, where $1 < s(\alpha,r) \leqslant 2$ is determined by α and r. Moreover, changing the corresponding exponents on the right-hand side, we will also state a related estimate where $|A|_{1,\infty}$ will be replaced by $|A|_{s,\infty}$, $2 < s < \infty$. In the particular case $1 = p < q = 2$, we will derive related estimates for $\sigma_r(A)$, $2 \leqslant r \leqslant 4$.

2. More on induced norms

If $A \in \mathbb{R}^{n\times n}$, we sometimes write $\|A\|_{q,p}^{\mathbb{R}}$ and $\|A\|_{q,p}^{\mathbb{C}}$ in order to distinguish between A viewed as a real matrix and A viewed as a complex matrix. In general, one has $\|A\|_{q,p}^{\mathbb{R}} \leqslant \|A\|_{q,p}^{\mathbb{C}}$, whereas the reverse inequality (in its whole generality) only holds if either $q \leqslant p$ or if A is non-negative (see, e.g., [7, p. 347]). For $p < q$, in general it holds $\|A\|_{q,p}^{\mathbb{C}} \leqslant \sqrt{2}\,\|A\|_{q,p}^{\mathbb{R}}$ ([20]). Furthermore, for $1 \leqslant p \leqslant q \leqslant 2$, the following improvement is known, where the occuring multiplicative constant is best possible:

LEMMA 2.1. ([6]) Let $1 \leqslant p \leqslant q \leqslant 2$. Then for any matrix $A \in \mathbb{R}^{n\times n}$ it holds

$$\|A\|_{q,p}^{\mathbb{C}} \leqslant (\sqrt{\pi})^{\frac{1}{p}-\frac{1}{q}} \left(\frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})}\right)^{\frac{1}{q}} \left(\frac{\Gamma(\frac{p+2}{2})}{\Gamma(\frac{p+1}{2})}\right)^{\frac{1}{p}} \|A\|_{p,q}^{\mathbb{R}}. \tag{2.1}$$

In order to obtain sharper results as well as smaller multiplicative constants, a standard tensor product trick will be used. For this, we need to know the behaviour of $\|\cdot\|_{p,q}$ under taking Kronecker products of matrices. For two square matrices $A = (a_{ij}) \in \mathbb{K}^{n\times n}$ and $B = (b_{ij}) \in \mathbb{K}^{m\times m}$ we denote their Kronecker/tensor product $A \otimes B \in \mathbb{K}^{nm\times nm}$ (see [13, 4.2.1]) by

$$A \otimes B := \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \vdots \\ a_{n1}B & \dots & a_{nn}B \end{pmatrix}.$$

LEMMA 2.2. Let $1 \leqslant p,q \leqslant \infty$. For two matrices $A,B \in \mathbb{K}^{n\times n}$ let $A \otimes B$ be their Kronecker product. Then the following hold:

(i) $\Lambda_r(A)\,\Lambda_r(B) = \Lambda_r(A\otimes B)$ for all $1 \leqslant r < \infty$.

(ii) $\|A\otimes B\|_{q,p} \leqslant \|A\|_{q,p}\|B\|_{q,p}$, if either $q \leqslant p$, or A and B are non-negative real matrices.

(iii) $\|A\otimes B\|_{2,p} \leqslant \Gamma\Big(\frac{p+2}{2}\Big)^{-\frac{1}{p}}\|A\|_{2,p}\|B\|_{2,p}$, if $1 \leqslant p < 2$ and $\mathbb{K} = \mathbb{C}$.

Proof. (ii) can be found in, e.g., [7, p. 80 and p. 87], and (iii) in [2, Proposition 10.2]. Part (i) is standard (see, e.g., [5, (1.5)] or [8, Lemma 11]), but we give a sketch for the convenience of the reader. It is easy to see that the set of eigenvalues of $A\otimes B$ equals $\{\lambda_i(A)\lambda_j(B);\ 1 \leqslant i,j \leqslant n\}$ and that the multiplicity of each these products is the product of the multiplicities of $\lambda_i(T)$ and $\lambda_j(T)$, respectively (see e. g. [13, 4.2.12]). The conclusion then follows by the definition of the norms in ℓ_r^n and $\ell_r^{n^2}$, respectively. $\square$

In what follows, we set $|A| = (|a_{ij}|)$.

LEMMA 2.3. (Tensor product trick) Assume that for $1 \leqslant p,q,s,t \leqslant \infty$ and any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that

$$\Lambda_{r,\infty}(A) \leqslant C(\varepsilon)\,n^{\varepsilon}\,\|A\|_{q,p}^{\alpha}\|A\|_{t,s}^{\beta}$$

holds for some fixed $\alpha,\beta > 0$, $1 \leqslant r < \infty$ and all $A \in \bigcup_n \mathbb{K}^{n\times n}$. Then

$$\Lambda_r(A) \leqslant \||A|\|_{q,p}^{\alpha}\||A|\|_{t,s}^{\beta}$$

for all $A \in \bigcup_n \mathbb{K}^{n\times n}$. Moreover, if $k_{q,p}, k_{t,s} > 0$ are constants such that

$$\|A\otimes A\|_{q,p} \leqslant k_{q,p}\,\|A\|_{q,p}^2 \text{ and } \|A\otimes A\|_{t,s} \leqslant k_{t,s}\,\|A\|_{t,s}^2$$

for all $A \in \bigcup_n \mathbb{K}^{n\times n}$, then

$$\Lambda_r(A) \leqslant k_{q,p}^{\alpha}k_{t,s}^{\beta}\,\|A\|_{q,p}^{\alpha}\|A\|_{t,s}^{\beta}$$

for all $A \in \bigcup_n \mathbb{K}^{n\times n}$.

Proof. The proof is standard (see, e.g., [7, p. 464]), but we sketch it for the convenience of the reader. However, we only consider the first part of the statement, the second one is proved similarly.

Fix $A \in \mathbb{K}^{n\times n}$ and $\varepsilon > 0$. Elementary calculus shows that

$$\Lambda_r(A) \leqslant D(\varepsilon)\,n^{\varepsilon}\,\Lambda_{r,\infty}(A)$$

for some constant $D(\varepsilon) > 0$ independently of $A \in \mathbb{K}^{n\times n}$. Thus, since $\|A\|_{q,p} \leqslant \||A|\|_{q,p}$ and $\|A\|_{t,s} \leqslant \||A|\|_{t,s}$ (an easy exercise), it follows

$$\Lambda_r(A) \leqslant E(\varepsilon)\,n^{2\varepsilon}\,\||A|\|_{q,p}^{\alpha}\,\||A|\|_{t,s}^{\beta},$$

where $E(\varepsilon)=C(\varepsilon)D(\varepsilon)$. Hence, by Lemma 2.2 (i) and (ii) and the obvious equality $|A\otimes A|=|A|\otimes|A|$ it follows that

$$\begin{aligned}\Lambda_r(A)^2=\Lambda_r(A\otimes A)&\leqslant E(\varepsilon)\,n^{4\varepsilon}\,\||A|\otimes|A|\|_{q,p}^{\alpha}\,\||A|\otimes|A|\|_{t,s}^{\beta}\\&\leqslant E(\varepsilon)\,n^{4\varepsilon}\,\||A|\|_{q,p}^{2\alpha}\,\||A|\|_{t,s}^{2\beta},\end{aligned}$$

whence

$$\Lambda_r(A)\leqslant\sqrt{E(\varepsilon)}n^{2\varepsilon}\,\||A|\|_{q,p}^{\alpha}\,\||A|\|_{t,s}^{\beta}.$$

Iterating the argument leads to

$$\Lambda_r(A)\leqslant E(\varepsilon)^{2^{-k}}n^{2\varepsilon}\,\||A|\|_{q,p}^{\alpha}\,\||A|\|_{t,s}^{\beta}.$$

Taking first $k\to\infty$ and then $\varepsilon\to 0$ shows the result. $\square$

Furthermore, we will use the following simple observation which shows that some quantities in the upcoming theorems can be easily calculated:

LEMMA 2.4. For any matrix $A\in\mathbb{K}^{n\times n}$ and any $1\leqslant p\leqslant\infty$ it is

$$\|A\|_{1,p}=|A|_{p,\infty}\text{ and }\|A\|_{p,1}=\max_{|\theta_i|=1}|A^t(\theta_1,\dots,\theta_n)^t|_{1,p'}\leqslant|A|_{1,p'}.$$

In particular, if $A\in\mathbb{R}^{n\times n}$ is non-negative, then $\|A\|_{p,1}=|A|_{1,p'}$.

Proof. This follows from the Krein-Milman Theorem (see [19]). Indeed, the extremal points in $B_{\ell_1^n}$ are all of the form αe_i for some $|\alpha|=1$ and $i\in\{1,\dots,n\}$, hence

$$\|A\|_{1,p}=\sup_{\|x\|_1\leqslant 1}\|Ax\|_p=\sup_{|\alpha|=1,\,j\in\{1,\dots,n\}}\|A(\alpha e_j)\|_p=\max_{j=1,\dots,n}\|Ae_j\|_p=|A|_{p,\infty}.$$

Furthermore, the extremal points in $B_{\ell_\infty^n}$ are all of the form $(\theta_1,\dots,\theta_n)$ with $|\theta_i|=1$ for all $i=1,\dots,n$. Thus, the remaining formula follows from $\|A\|_{p,1}=\|A^t\|_{\infty,p'}$ similarly as in the above. $\square$

3. Interpolation of spaces of operators

In this section, very special concepts from functional analysis will be presented and exploited. We try to keep this as short as possible, albeit hoping that non-experts in this field will get an impression about the ideas behind our eigenvalue estimates to be presented later.

One crucial ingredient will be the theory of complex interpolation of Banach spaces. For a compatible Banach couple (X_0,X_1) of complex Banach spaces and $0<\theta<1$ we denote by $[X_0,X_1]_\theta$ the resulting complex interpolation space in the sense of [3]. We do not want to trouble the reader with details on what compatible means; in this article, we will always consider couples of finite-dimensional spaces of the same dimension, i.e., $X_0=(\mathbb{C}^m,\|\cdot\|_0)$ and $X_1=(\mathbb{C}^m,\|\cdot\|_1)$ simply being $\mathbb{C}^m$

equipped with two different norms. We will also use the notation $\ell_p^n(X)$ for the linear space X^n of all X-valued n-tuples, X a Banach space, equipped with the norm $\|(x_1,\ldots,x_n)\|_{\ell_p^n(X)} := \|(\|x_1\|,\ldots,\|x_n\|)\|_p$ for $1 \leqslant p \leqslant \infty$ and $x_1,\ldots,x_n \in X$. We will frequently use the following basic facts (see, e.g., [3]):

LEMMA 3.1. Let (X_0,X_1) be a compatible Banach couple and $0<\theta<1$.

(i) The complex interpolation functor is of power type θ, i.e., for any $x \in X_0 \cap X_1$ it holds
$$\|x\|_{[X_0,X_1]_\theta} \leqslant \|x\|_{X_0}^{1-\theta}\|x\|_{X_1}^{\theta}. \tag{3.1}$$

(ii) For $1 \leqslant p_0, p_1 \leqslant \infty$ it is
$$[\ell_{p_0}^m, \ell_{p_1}^m]_\theta = \ell_{p_\theta}^m \tag{3.2}$$
and
$$[\ell_{p_0}^m(X_0), \ell_{p_1}^m(X_1)]_\theta = \ell_{p_\theta}^m([X_0,X_1]_\theta) \tag{3.3}$$
with equal norms, respectively, where $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

For two Banach spaces X and Y, we denote by $\mathscr{L}(X,Y)$ the space of all bounded and linear operators between X and Y, equipped with the usual operator norm $\|T\| := \sup_{x\neq 0}\frac{\|Tx\|_Y}{\|x\|_X}$ (so that $\|A\|_{q,p}$ for an $n\times n$ matrix A simply is the norm of A viewed as an operator $T_A : \ell_q^n \to \ell_p^n$, where T_A denotes the standard interpretation of A as a linear operator from $\mathbb{K}^n$ into $\mathbb{K}^n$).

The following highly non-trivial result about complex interpolation of spaces of bounded operators is due to [18] (see also [10] and [23] for related work); however, an analysis of the proof easily shows that the constant given there (which would have been $\sqrt{\frac{8}{\pi}}$) can be improved upon when we consider operators starting from a Hilbert space. Note that a more general version of the result below has been exploited in [26] in order to present an interpolation approach to Hardy–Littlewood inequalities.

LEMMA 3.2. ([18]) Let $1 \leqslant p_0, p_1 \leqslant 2$. Then for all $0<\theta<1$
$$\|\mathrm{id} : \mathscr{L}(\ell_2^m, \ell_{p_\theta}^n) \to [\mathscr{L}(\ell_2^m, \ell_{p_0}^n), \mathscr{L}(\ell_2^m, \ell_{p_1}^n)]_\theta\| \leqslant \frac{2}{\sqrt{\pi}},$$
where $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{1-\theta}{p_1}$.

A crucial tool will be the notion of absolutely summing norms. If X and Y are Banach spaces and $1 \leqslant s \leqslant r \leqslant \infty$, then for every operator $T : X \to Y$ and each $m \in \mathbb{N}$ we define
$$\pi_{r,s}^{(m)}(T) := \sup\Big\{\Big(\sum_{k=1}^m \|Tx_k\|_Y^r\Big)^{\frac{1}{r}};\ \sup_{\|x'\|_{X'}\leqslant 1}\Big(\sum_{k=1}^m |x'(x_k)|^s\Big)^{1/s} \leqslant 1\Big\}.$$

If $\pi_{r,s}(T) := \sup_{m\in\mathbb{N}} \pi_{r,s}^{(m)}(T) < \infty$, then T is called absolutely (r,s)-summing. In this case, we write $T \in \Pi_{r,s}$, and $T \in \Pi_r$ if $r = s$. For our purposes here, we will concentrate on the case $2 = s \leqslant r \leqslant \infty$. For $r = \infty$, this definition gives the usual operator norm. See, e.g., [11], for more information on absolutely summing operators. For our purposes, we denote by $\Pi^{(m)}_{r,2}(X,Y)$ the space $\mathscr{L}(X,Y)$ equipped with the norm $\pi^{(m)}_{r,2}(T)$ for an operator $T : X \to Y$.

The following in the case $s = 2$ is known as the *Little Grothendieck Theorem* (see, e.g., [7, p. 140]) and in the general case due to Kwapień [21] (see also [7, p. 462]):

LEMMA 3.3. Let $A \in \mathbb{C}^{n\times n}$ and $1 < s < \infty$. Then

$$\pi_{\max(s,s'),2}(T_A : \ell_1^n \to \ell_s^n) \leqslant \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{2}{\max(s,s')}} \|A\|_{1,s}.$$

Proof. This can be seen exactly as in [21], interpolating between the cases $s = 2$, where the Little Grothendieck Theorem states that

$$\pi_{2,2}(T_A : \ell_1^n \to \ell_2^n) \leqslant \frac{2}{\sqrt{\pi}} \|T_A : \ell_1^n \to \ell_2^n\|,$$

and the trivial cases $s = 1$ and $s = \infty$, respectively, where $\max(s,s') = \infty$. Here, recall that by definition $\pi_{\infty,2}(T) = \|T\|$ for any bounded linear operator T between two Banach spaces X and Y. $\square$

Complex interpolation of spaces of summing operators is fairly easy when the domain space is fixed – the following interpolation formula is more involved (we state a finite-dimensional version only):

LEMMA 3.4. Let $1 \leqslant p_0, p_1 \leqslant 2 \leqslant r_0, r_1 \leqslant \infty$ and $1 \leqslant q_0, q_1 \leqslant \infty$. Then for $0 < \theta < 1$

$$\|\mathrm{id} : [\Pi^{(m)}_{r_0,2}(\ell^n_{p_0}, \ell^n_{q_0}), \Pi^{(m)}_{r_1,2}(\ell^n_{p_1}, \ell^n_{q_1})]_\theta \to \Pi^{(m)}_{r_\theta,2}(\ell^n_{p_\theta}, \ell^n_{q_\theta})\| \leqslant \frac{2}{\sqrt{\pi}},$$

where $\frac{1}{t_\theta} = \frac{1-\theta}{t_0} + \frac{\theta}{t_1}$ for $t \in \{r,p,q\}$.

Proof. This is pretty standard, but we state briefly the idea for the convenience of the reader. It is well-known that

$$\pi^{(m)}_{r,2}(T : X \to Y) = \|\Phi^m_T : \mathscr{L}(\ell^m_2, X) \to \ell^m_r(Y)\|,$$

where $\Phi^m_T(S) := ((TS)(e_i))_i$ and e_i denotes the i-th standard unit vector. Thus, it suffices to show that

$$\|\Phi^m : \mathscr{L}(\ell^m_2, \ell^n_{p_\theta}) \times [\Pi^{(m)}_{r_0,2}(\ell^n_{p_0}, \ell^n_{q_0}), \Pi^{(m)}_{r_1,2}(\ell^n_{p_1}, \ell^n_{q_1})]_\theta \to \ell^m_{r_\theta}(\ell^n_{q_\theta})\| \leqslant \frac{2}{\sqrt{\pi}},$$

where $\Phi^m(S,T) := ((TS)(e_i))_i$. This now follows from the border cases

$$\|\Phi^m : \mathscr{L}(\ell^m_2, \ell^n_{p_i}) \times \Pi^{(m)}_{r_i,2}(\ell^n_{p_i}, \ell^n_{q_i}) \to \ell^m_{r_i}(\ell^n_{q_i})\| = 1,$$

$i=0,1$ together with (3.3), Lemma 3.2 and the fact that the complex interpolation functor behaves nicely with respect to bilinear interpolation (see, e.g., [3, 4.4.1]). $\square$

The following now is the crucial result needed for our eigenvalue estimates – see, e.g., [24], for related work.

PROPOSITION 3.5. Let $1 \leqslant p,q \leqslant 2$ and $1<s<\infty$. Then for $0<\theta<1$

$$\|\mathrm{id}:[\mathscr{L}(\ell_q^n,\ell_p^n),\mathscr{L}(\ell_1^n,\ell_s^n)]_\theta \to \Pi^{(n)}_{\frac{\max(s,s')}{\theta},2}(\ell_u^n,\ell_v^n)\| \leqslant \left(\frac{2}{\sqrt{\pi}}\right)^{1+\frac{2\theta}{\max(s,s')}},$$

where $\frac{1}{u}=\frac{1-\theta}{q}+\frac{\theta}{1}$ and $\frac{1}{v}=\frac{1-\theta}{p}+\frac{\theta}{s}$.

Proof. The claim immediately follows from Lemma 3.4 and Lemma 3.3 (taking $p_0=q$, $p_1=1$, $q_0=p$, $q_1=s$, $r_0=\infty$, and $r_1=s'$ (if $s\leqslant 2$) or $r_1=s$ (if $s>2$). $\square$

4. Estimates for eigenvalues

The following classical result will now be the link to eigenvalues. The constants in the statement below – for $r>2$ smaller than the constant $2e$ stated in the cited article – follow from a short analysis of the proof given there:

LEMMA 4.1. ([17], Corollary 16) Let $2\leqslant r<\infty$ and $A\in\mathbb{C}^{n\times n}$. Then for any given norm $\|\cdot\|$ on $\mathbb{C}^n$ and $X_n:=(\mathbb{C}^n,\|\cdot\|)$ it holds

$$\Lambda_{r,\infty}(A)\leqslant (2e)^{\frac{1}{r}+\frac{1}{2}}\pi^{(n)}_{r,2}(T_A:X_n\to X_n). \tag{4.1}$$

Now we are prepared to take the next crucial step towards our most general result – essentially, this is now an immediate consequence of Proposition 3.5. Although we do not want to downplay the role of the parameter s, we would like to point out that the case $s=2$ seems to be the most important one to us.

PROPOSITION 4.2. Let $1\leqslant p<q\leqslant 2$, $1<s\leqslant 2$ and $r=\frac{1}{\alpha}+s'$. Then for all $A\in\mathbb{K}^{n\times n}$

$$\Lambda_{r,\infty}(A)\leqslant C_{q,p,s}\,\|A\|_{q,p}^{1-\frac{s'}{r}}\,|A|_{s,\infty}^{\frac{s'}{r}}, \tag{4.2}$$

where

$$C_{q,p,s}=\begin{cases}\left(\frac{8e}{\pi}\right)^{\frac{1}{r}+\frac{1}{2}}\leqslant 4.27 & \text{if } \mathbb{K}=\mathbb{C},\\ \left[\left(\frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})}\right)^{\frac{1}{q}}\left(\frac{\Gamma(\frac{p+2}{2})}{\Gamma(\frac{p+1}{2})}\right)^{\frac{1}{p}}\right]^{1-\frac{s'}{r}}\left(\frac{8e}{\pi}\right)^{\frac{1}{r}+\frac{1}{2}}\sqrt{\pi}^{\frac{1}{r}}\leqslant 4.75 & \text{if } \mathbb{K}=\mathbb{R}.\end{cases}$$

Proof. Choose $0<\theta<1$ such that $\frac{1}{u}:=\frac{1-\theta}{q}+\frac{\theta}{1}=\frac{1-\theta}{p}+\frac{\theta}{s}$; this is always possible by the restrictions on the parameters involved. More precisely, $\theta:=\frac{s'}{r}$ does the job (and is the only possible choice). Then $u = v$ in Proposition 3.5. Thus, the claim then follows from this by (3.1) (the complex interpolation functor is of power type θ), (4.1) (the link from summing norms to eigenvalues), (2.1) (the complexification procedure), and Lemma 2.4. $\square$

THEOREM 4.3. Let $1 \leqslant p < q \leqslant 2 < r < \infty$ such that $\frac{1}{r} < \alpha := \frac{1}{p} - \frac{1}{q}$, and let $A \in \mathbb{K}^{n\times n}$. Then

$$\Lambda_{r,\infty}(A) \leqslant C_{q,p,r} \|A\|_{q,p}^{\frac{1}{\alpha r}} |A|_{s(\alpha,r),\infty}^{1-\frac{1}{\alpha r}}, \tag{4.3}$$

where

$$s(\alpha,r) = \begin{cases} 2 & \text{if } r \leqslant \frac{1}{\alpha}+2, \\ \frac{\alpha r-1}{\alpha r-1-\alpha} & \text{if } r > \frac{1}{\alpha}+2, \end{cases}$$

and, if $\frac{1}{\alpha} < r < \frac{1}{\alpha}+2$,

$$C_{q,p,r} = \begin{cases} (2e)^{\frac{1}{r}+\frac{1}{2}} \left(\frac{2}{\sqrt{\pi}}\right)^{2-\frac{1}{\alpha r}} \leqslant 7.61 & \text{if } \mathbb{K}=\mathbb{C}, \\ \left[\left(\frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})}\right)^{\frac{1}{q}} \left(\frac{\Gamma(\frac{p+2}{2})}{\Gamma(\frac{p+1}{2})}\right)^{\frac{1}{p}}\right]^{\frac{1}{\alpha r}} (2e)^{\frac{1}{r}+\frac{1}{2}} \left(\frac{2}{\sqrt{\pi}}\right)^{2-\frac{1}{\alpha r}} \sqrt{\pi}^{\frac{1}{r}} & \\ \qquad \leqslant 8.45 & \text{if } \mathbb{K}=\mathbb{R}, \end{cases}$$

and, if $\frac{1}{\alpha}+2 \leqslant r < \infty$,

$$C_{q,p,r} = \begin{cases} \left(\frac{8e}{\pi}\right)^{\frac{1}{r}+\frac{1}{2}} \leqslant 4.27 & \text{if } \mathbb{K}=\mathbb{C}, \\ \left[\left(\frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})}\right)^{\frac{1}{q}} \left(\frac{\Gamma(\frac{p+2}{2})}{\Gamma(\frac{p+1}{2})}\right)^{\frac{1}{p}}\right]^{\frac{1}{\alpha r}} \left(\frac{8e}{\pi}\right)^{\frac{1}{r}+\frac{1}{2}} \sqrt{\pi}^{\frac{1}{r}} \leqslant 4.75 & \text{if } \mathbb{K}=\mathbb{R}. \end{cases}$$

The parameter $s(\alpha,r)$ is best possible as a function depending on α and r.

Proof. The case $\mathbb{K}=\mathbb{R}$ always follows from the case $\mathbb{K}=\mathbb{C}$ by the complexification procedure, so let $\mathbb{K}=\mathbb{C}$.

We start with the case $\frac{1}{\alpha}+2 \leqslant r < \infty$. An easy calculation shows that $s = \frac{\alpha r-1}{\alpha r-1-\alpha}$ satisfies $r = \frac{1}{\alpha}+s'$. Thus, (4.3) follows immediately from the preceding proposition. Coming to the case $\frac{1}{\alpha} < r < \frac{1}{\alpha}+2$, it is well-known that when taking $\theta = \left(1-\frac{1}{\alpha r}\right)\left(\frac{1+2\alpha}{2\alpha}\right)$, it holds

$$\|x\|_{\ell^n_{r,\infty}} \leqslant \|x\|_{\ell^n_{\frac{1}{\alpha},\infty}}^{1-\theta} \|x\|_{\ell^n_{\frac{1}{\alpha}+2,\infty}}^{\theta}$$

for any $x \in \mathbb{C}^n$ (see, e.g., (1.85) on page 167 in [28]). Thus, (4.3) follows from case (iii) of Theorem 1.1 and the case $r = \frac{1}{\alpha}+2$ in (4.3). Note that the constant occuring in (iii) of Theorem 1.1 can be slightly improved to be $(2e)^{\alpha+\frac{1}{2}} \frac{2}{\sqrt{\pi}}$ which we have used in the above formulation; since this slight improvement of the constants is not really essential here (but nice anyway), we omit the details. The fact that the parameter $s(\alpha,r)$ is best possible as a function depending on α and r follows from the upcoming examples (ii) and (iii). $\square$

REMARK 4.4. Note that due to the tensor product trick, for a non-negative matrix A one can substitute in the above $\Lambda_{r,\infty}(A)$ by $\Lambda_r(A)$ and $C_{q,p,r}$ by 1, respectively. Moreover, if $\frac{1}{\alpha} < r < \frac{1}{\alpha} + 2$, then there exists a constant $K_{q,p,r} > 0$ (with $K_{q,p,r}$ diverging to infinity as $r \to \frac{1}{\alpha}$ or as $r \to \frac{1}{\alpha} + 2$) such that for any square matrix A it holds

$$\Lambda_r(A) \leqslant K_{q,p,r} C_{q,p,r} \|A\|_{q,p}^{\frac{1}{\alpha r}} |A|_{2,\infty}^{1-\frac{1}{\alpha r}}. \tag{4.4}$$

This follows in a way similar to the above from the fact that ℓ_r^n can be represented as the real interpolation space $(\ell^n_{\frac{1}{\alpha},\infty}, \ell^n_{\frac{1}{\alpha}+2,\infty})_{\theta,r}$ (same θ as in the above), with an equivalence of norms constant $K_{q,p,r}$ not depending on the dimension n (but diverging to infinity as $r \to \frac{1}{\alpha}$ or as $r \to \frac{1}{\alpha} + 2$), so that it follows that

$$\|x\|_{\ell_r^n} \leqslant K_{q,p,r} \, \|x\|_{\ell^n_{\frac{1}{\alpha},\infty}}^{1-\theta} \, \|x\|_{\ell^n_{\frac{1}{\alpha}+2,\infty}}^{\theta}$$

for any $x \in \mathbb{C}^n$ (see, e.g., [3, 5.3.1] for this fact and more on real interpolation); for those who do not want to dive even further into interpolation theory we recommend to take a look at [28], in particular (1.87) on page 167.

Clearly, the upper estimates for $\Lambda_{r,\infty}(A)$ in (4.3) can differ from its actual value with an arbitrarily large error; take, e.g., any upper triangular matrix with zeros in the diagonal and arbitrarily large entries in the upper triangle. However, the classical upper bounds stated at the beginning bear the same problem. Down below we will see that for matrices which are closer to symmetric matrices the occurring error will be significantly smaller.

EXAMPLE 4.5. (i) Let $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n \geqslant 0$ and D_μ be the associated diagonal matrix. Then the right-hand side in (4.3) (with constant 1) equals

$$\|\mu\|_{\frac{1}{\alpha}}^{\frac{1}{\alpha r}} \|\mu\|_\infty^{1-\frac{1}{\alpha r}} \leqslant n^{\frac{1}{r}-\frac{1}{\alpha r^2}} \|\mu\|_r \leqslant (1+\log n)^{\frac{1}{r}} n^{\frac{1}{r}-\frac{1}{\alpha r^2}} \Lambda_{r,\infty}(D_\mu).$$

Taking $\mu_k = k^{-\frac{1}{r}}$, $k = 1,\ldots,n$, one can see that the maximal (multiplicative) error for diagonal matrices indeed is at least $c n^{\frac{1}{r}-\frac{1}{\alpha r^2}}$ for some universal constant $c > 0$. However, it is quite reassuring that the estimate is optimal at least for the identity matrix E_n.

(ii) The matrix J_n where all entries are 1 has the only non-zero eigenvalue n with multiplicity 1. It satisfies $\|J_n\|_{q,p} = n^{\frac{1}{p}+\frac{1}{q'}}$ and $|J_n|_{s,\infty} = n^{\frac{1}{s}}$ (see also [12, Lemma 2]). Thus, the above inequality (4.3) is optimal for this matrix in case (ii) and all choices of $1 \leqslant p < q \leqslant 2$.

(iii) The Vandermonde matrix of the n roots of unity

$$V_n := \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & \ldots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \ldots & \omega^{2n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2n-2} & \ldots & \omega^{(n-1)^2} \end{pmatrix},$$

where $\omega = e^{2\pi i/n}$, has the property that $\overline{V_n}^t V_n = nE_n$, thus $|\lambda_k(V_n)| = s_k(V_n) = \sqrt{n}$ for all $k = 1,\dots,n$. Moreover, this implies $\|V_n\|_{2,2} = \sqrt{n}$ and $\|V_n\|_{q,p} \leqslant n^{\frac{1}{p}}$ for all $1 \leqslant p \leqslant q \leqslant 2$. Furthermore (see above), $|V_n|_{s,\infty} = |J_n|_{s,\infty} = n^{\frac{1}{s}}$. Hence, the above inequality (4.3) is optimal (up to a constant independent of the dimension) for this matrix in case (i) for $1 \leqslant p < q = 2$.

(iv) The matrix R_n where all the entries of its first row are 1 and all other entries are 0 has the only non-zero eigenvalue 1 with multiplicity 1 and the only non-zero singular value $\sqrt{n}$ with multiplicity 1; clearly the same holds for its transpose which we denote by C_n. It is $\|R_n\|_{q,p} = n^{\frac{1}{q'}}$, $\|C_n\|_{q,p} = n^{\frac{1}{p}}$ and $|R_n|_{s,\infty} = 1$, $|C_n|_{s,\infty} = n^{\frac{1}{s}}$ (see also [12, Lemma 2]). Hence, the right-hand side in the above inequality (4.3) for both matrices yields $n^{\frac{1}{q'\alpha r}}$ which is minimal for $p = 1$ with value $n^{\frac{1}{r}}$. Although this is still far from the exact value $\Lambda_r(J_n) = \Lambda_r(C_n) = 1$, it is for large r also far away (in the positive sense) from the exact value $\sigma_r(R_n) = \sigma_r(C_n) = \sqrt{n}$ which shows that most likely the right-hand side in the above might be considerably smaller than $\sigma_r(A)$ for a large class of (non-normal) matrices A. It will become clear in the upcoming section on singular values, where we derive similar estimates, why this is somewhat remarkable.

REMARK 4.6. As mentioned within the proof of the theorem, examples (ii) and (iii) in the above show that an inequality of the type

$$\Lambda_r(A) \leqslant C_{q,p,r} \|A\|_{q,p}^{\frac{1}{\alpha r}} |A|_{s(\alpha,r),\infty}^{1-\frac{1}{\alpha r}}$$

(all parameters chosen as in the theorem and s a suitable function of the parameters α and r) necessarily implies

$$s(\alpha,r) = \begin{cases} 2 & \text{if } r \leqslant \frac{1}{\alpha}+2, \\ \frac{\alpha r-1}{\alpha r-1-\alpha} & \text{if } r > \frac{1}{\alpha}+2. \end{cases}$$

Hence, in this sense our estimates are optimal, and, in particular, the breaking point $r = \frac{1}{\alpha}+2$ in the formulation is not artificial. Furthermore, the exponent $\frac{1}{\alpha r}$ is also natural when demanding exactness for the identity matrix E_n. However, we do not know what happens (apart from the case $q = 2$) if one replaces $s(\alpha,r)$ (depending only on the difference α) by $s(q,p,r)$ (a function depending on q and p themselves).

Using the tensor product trick stated in Lemma 2.3, the above gives the following inequality with $\Lambda_r(A)$ instead of $\Lambda_{r,\infty}(A)$ on the left-hand side and constant 1 on the right-hand side which seems to be particularly interesting for non-negative real matrices.

COROLLARY 4.7. Let $1 \leqslant p < q \leqslant 2 < r < \infty$ such that $\frac{1}{r} < \alpha := \frac{1}{p} - \frac{1}{q}$, and let $A \in \mathbb{K}^{n\times n}$. Then

$$\Lambda_r(A) \leqslant \||A|\|_{q,p}^{\frac{1}{\alpha r}} |A|_{s(\alpha,r),\infty}^{1-\frac{1}{\alpha r}}, \tag{4.5}$$

where

$$s(\alpha,r)=\begin{cases}2 & \text{if } r\leqslant \frac{1}{\alpha}+2,\\ \frac{\alpha r-1}{\alpha r-1-\alpha} & \text{if } r>\frac{1}{\alpha}+2.\end{cases}$$

In particular, taking $p=1$ it follows for $r\geqslant q'+2$

$$\Big(\sum_i|\lambda_i(A)|^r\Big)^{\frac{1}{r}}\leqslant\Big[\sum_j\Big(\sum_i|a_{ij}|\Big)^{q'}\max_\ell\Big(\sum_k|a_{k\ell}|^{\frac{r-q'}{r-q'-1}}\Big)^{r-q'-1}\Big]^{\frac{1}{r}}. \tag{4.6}$$

Furthermore, using once more the tensor product trick on the above theorem when $q=2$, things also improve a lot – note that here we have $\|A\|_{2,p}$ instead of $\||A|\|_{2,p}$ on the right-hand side (we leave the calculations to the reader):

COROLLARY 4.8. Let $1\leqslant p<2$ and $\frac{2p}{2-p}<r<\infty$ and $A\in\mathbb{K}^{n\times n}$. Then

$$\Lambda_r(A)\leqslant\kappa_{p,r}\,\|A\|_{2,p}^{\frac{2p}{(2-p)r}}\,|A|_{s(p,r),\infty}^{1-\frac{2p}{(2-p)r}}, \tag{4.7}$$

where

$$s(p,r)=\begin{cases}2 & \text{if } r\leqslant\frac{2p}{2-p}+2,\\ \frac{r(2-p)-2p}{r(2-p)-p-2} & \text{if } r>\frac{2p}{2-p}+2.\end{cases}$$

and

$$\kappa_{p,r}\begin{cases}\Gamma\Big(\frac{p+2}{2}\Big)^{-\frac{2}{(2-p)r}}\leqslant\frac{2}{\sqrt{\pi}}\simeq 1.13 & \text{if } \mathbb{K}=\mathbb{C},\\ \Big(\sqrt{\pi}\,\Gamma\big(\frac{p+1}{2}\big)^{-1}\Big)^{\frac{2}{(2-p)r}}\leqslant 1.26 & \text{if } \mathbb{K}=\mathbb{R}.\end{cases}$$

As seen in the examples above, the accuracy of our upper estimates seems to increase when r tends to infinity. Indeed, when doing so, we recover a well-known (and easy) estimate for the largest eigenvalue:

COROLLARY 4.9. Let $A\in\mathbb{K}^{n\times n}$. Then

$$|\lambda_1(A)|\leqslant\inf_{\alpha>0}\max_{j=1,\dots,n}\lim_{s\to 1}\Big(\sum_{i=1}^n|a_{ij}|^s\Big)^{\frac{\alpha}{(\alpha+1)s-1}}\leqslant|A|_{1,\infty}=\|A\|_{1,1}.$$

Proof. Fix $0<\alpha\leqslant\frac{1}{2}$ and choose any $1\leqslant p<q\leqslant 2$ such that $\alpha=\frac{1}{p}-\frac{1}{q}$. Now for $1<s\leqslant 2$ and $r=s'+\frac{1}{\alpha}$ it is $\frac{s'}{r}=\frac{\alpha s}{(\alpha+1)s-1}$. Then by (4.2) and Lemma 2.3

$$\Lambda_r(A)\leqslant\||A|\|_{q,p}^{\frac{s-1}{s(\alpha+1)-1}}\,|A|_{s,\infty}^{\frac{\alpha s}{(\alpha+1)s-1}}.$$

Now letting s tend to 1 (and r therefore tend to infinity), we obtain

$$|\lambda_1(A)|\leqslant\max_{j=1,\dots,n}\lim_{s\to 1}\Big(\sum_{i=1}^n|a_{ij}|^s\Big)^{\frac{\alpha}{(\alpha+1)s-1}}.$$

Now taking the infimum over all α, we arrive at the first inequality in the above (note that $\inf_{\alpha>0} = \lim_{\alpha\to 0}$ since the function $\alpha \mapsto \frac{\alpha}{(\alpha+1)s-1}$ is increasing). The second follows immediately from $|A|_{s,\infty} \leqslant |A|_{1,\infty}$ and $\frac{\alpha s}{(\alpha+1)s-1} \to 1$ as $s \to 1$. $\square$

For completeness sake, we state the following inequality resulting from Kwapień's theorem in the case where $s > 2$. Note that the result is optimal for the matrices J_n; however, for the matrices V_n and even for E_n it is not.

PROPOSITION 4.10. Let $1 \leqslant p < q \leqslant 2$, $\alpha := \frac{1}{p} - \frac{1}{q}$, $\frac{1}{\alpha} + 2 \leqslant r < \infty$, and let $A \in \mathbb{K}^{n\times n}$. Then

$$\Lambda_{r,\infty}(A) \leqslant D_{q,p,r} \|A\|_{q,p}^{\frac{r-1}{\alpha r+r}} |A|_{\frac{\alpha r+1}{\alpha+1},\infty}^{\frac{\alpha r+1}{\alpha r+r}}, \tag{4.8}$$

where

$$D_{q,p,r} = \begin{cases} \left(\frac{8e}{\pi}\right)^{\frac{1}{r}+\frac{1}{2}} \leqslant 4.27 & \text{if } \mathbb{K} = \mathbb{C}, \\ \left[\left(\frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})}\right)^{\frac{1}{q}} \left(\frac{\Gamma(\frac{p+2}{2})}{\Gamma(\frac{p+1}{2})}\right)^{\frac{1}{p}}\right]^{\frac{r-1}{\alpha r+r}} \left(\frac{8e}{\pi}\right)^{\frac{1}{r}+\frac{1}{2}} \sqrt{\pi}^{\frac{\alpha(r-1)}{\alpha r+r}} \leqslant 4.75 & \text{if } \mathbb{K} = \mathbb{R}. \end{cases}$$

Proof. First, analogously to the proof of (4.2), one establishes

$$\Lambda_{r,\infty}(A) \leqslant C_{q,p,s} \|A\|_{q,p}^{1-\frac{s}{r}} |A|_{s,\infty}^{\frac{s}{r}}, \tag{4.9}$$

where

$$C_{q,p,s} = \begin{cases} \left(\frac{8e}{\pi}\right)^{\frac{1}{r}+\frac{1}{2}} \leqslant 4.27 & \text{if } \mathbb{K} = \mathbb{C}, \\ \left[\left(\frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})}\right)^{\frac{1}{q}} \left(\frac{\Gamma(\frac{p+2}{2})}{\Gamma(\frac{p+1}{2})}\right)^{\frac{1}{p}}\right]^{1-\frac{s}{r}} \left(\frac{8e}{\pi}\right)^{\frac{1}{r}+\frac{1}{2}} \sqrt{\pi}^{\frac{1}{r}} \leqslant 4.75 & \text{if } \mathbb{K} = \mathbb{R}, \end{cases}$$

whenever $1 \leqslant p < q \leqslant 2 < s < \infty$ and $r > s$ such that $\alpha = \frac{1}{p} - \frac{1}{q} = \frac{s-1}{r-s}$. Then for given parameters p, q and r determine such a number s, which turns out to be $\frac{\alpha r+1}{\alpha+1}$. $\square$

Although the above does not seem to be very accurate for fixed α and r, it is asymptotically consistent with the trivial estimate $|\lambda_1(A)| \leqslant \|A\|_{p,p}$ for any square matrix A and $1 \leqslant p \leqslant \infty$:

COROLLARY 4.11. Let $A \in \mathbb{C}^{n\times n}$, and for $1 \leqslant p < q \leqslant 2$ denote $\alpha_{q,p} := \frac{1}{p} - \frac{1}{q}$. Then for fixed $1 < q \leqslant 2$

$$|\lambda_1(A)| \leqslant C \lim_{p\uparrow q} \|A\|_{q,p}^{\frac{1}{1+\alpha_{q,p}}} \leqslant C \|A\|_{q,q},$$

and for fixed $1 \leqslant p < 2$

$$|\lambda_1(A)| \leqslant C \lim_{q\downarrow p} \|A\|_{q,p}^{\frac{1}{1+\alpha_{q,p}}} \leqslant C \|A\|_{p,p}.$$

In particular,

$$|\lambda_1(A)| \leqslant C \min\Big(\lim_{p\uparrow 2} \|A\|_{2,p}^{\frac{2p}{2+p}}, \lim_{q\downarrow 1} \|A\|_{q,1}^{\frac{q}{2q-1}} \Big).$$

In all of the above, one may choose $C = \sqrt{\frac{8e}{\pi}} \leqslant 2.64$ (and $C = 1$ if the matrix is non-negative).

Proof. Fix $1 \leqslant p < q \leqslant 2$, and then let $r \to \infty$ in (4.8). This gives

$$|\lambda_1(A)| \leqslant \sqrt{\frac{8e}{\pi}}\, \|A\|_{q,p}^{\frac{1}{1+\alpha_{q,p}}} \lim_{r\to\infty} |A|_{\frac{1+r\alpha_{q,p}}{1+\alpha_{q,p}},\infty}^{\frac{1+r\alpha_{q,p}}{r+r\alpha_{q,p}}}.$$

An elementary calculation shows that if $A \neq 0$ and $M := |A|_{\infty,\infty}$, then

$$|A|_{\frac{1+r\alpha_{q,p}}{1+\alpha_{q,p}},\infty}^{\frac{1+r\alpha_{q,p}}{r+r\alpha_{q,p}}} \leqslant M^{\frac{1+r\alpha_{q,p}}{r+r\alpha_{q,p}}} \left|\frac{A}{M}\right|_{\frac{1+r\alpha_{q,p}}{1+\alpha_{q,p}},\infty}^{\frac{1+r\alpha_{q,p}}{r+r\alpha_{q,p}}} \leqslant M^{\frac{1+r\alpha_{q,p}}{r+r\alpha_{q,p}}}\, n^{\frac{1}{r}} \to M^{\frac{\alpha_{q,p}}{1+\alpha_{q,p}}}$$

as $r \to \infty$. Hence,

$$|\lambda_1(A)| \leqslant \sqrt{\frac{8e}{\pi}}\, \|A\|_{q,p}^{\frac{1}{1+\alpha_{q,p}}}\, |A|_{\infty,\infty}^{\frac{\alpha_{q,p}}{1+\alpha_{q,p}}}.$$

The first inequalities now follow by letting $p \to q$ or $q \to p$. Since

$$\|A\|_{q,p}^{\frac{1}{1+\alpha_{q,p}}} \leqslant n^{\frac{\alpha}{1+\alpha_{q,p}}} \min(\|A\|_{q,q}, \|A\|_{p,p})^{\frac{1}{1+\alpha_{q,p}}},$$

the second inequalities are also clear by letting $p \to q$ or $q \to p$. $\square$

EXAMPLE 4.12. (i) Let $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n \geqslant 0$ and D_μ be the associated diagonal matrix. Then

$$\lim_{p\uparrow q} \|D_\mu\|_{q,p}^{\frac{1}{1+\alpha_{q,p}}} \leqslant \lim_{p\uparrow q} \mu_1^{\frac{1}{1+\alpha_{q,p}}}\, n^{\frac{\alpha_{q,p}}{1+\alpha_{q,p}}} = \mu_1 = |\lambda_1(D_\mu)|.$$

(ii) For the matrix J_n with all entries equal to 1 we obtain straightaway

$$\lim_{p\uparrow q} \|J_n\|_{q,p}^{\frac{1}{1+\alpha_{q,p}}} = n = |\lambda_1(J_n)|.$$

(iii) For the Vandermonde matrix V_n we get

$$\lim_{p\uparrow 2} \|V_n\|_{2,p}^{\frac{1}{1+\alpha_{2,p}}} \leqslant \lim_{p\uparrow 2} n^{\frac{\frac{1}{2}+\alpha_{2,p}}{1+\alpha_{2,p}}} = n^{\frac{1}{2}} = |\lambda_1(V_n)|.$$

(iv) For the matrix R_n the above yields

$$\lim_{q\downarrow 1} \|R_n\|_{q,1}^{\frac{1}{1+\alpha_{q,1}}} = \lim_{q\downarrow 1} n^{\frac{\alpha_{q,1}}{1+\alpha_{q,1}}} = 1 = |\lambda_1(R_n)|.$$

5. Estimates for singular values

It seems that (4.7) might be quite appropriate for normal matrices. Indeed, let $A \in \mathbb{C}^{n\times n}$. Then for $p < 2$ it is $\|A\|_{2,p} \leqslant n^{\frac{1}{p}-\frac{1}{2}}\|A\|_{2,2}$ and altogether the right-hand side in the above less than or equal to $\kappa_r n^{\frac{1}{r}}\|A\|_{2,2} \leqslant \kappa_r n^{\frac{1}{r}}\sigma_r(A)$. Thus, for A normal (4.7) is correct up to a multiplicative constant in $O(n^{\frac{1}{r}})$.

However, what about matrices which are not normal? A result of Le Merdy yields a direct estimate in the case $2 \leqslant r \leqslant 4$ which is somewhat similar to the above – notice that for $p = 1$ and $2 \leqslant r \leqslant 4$, (4.7) reads

$$\Lambda_r(A) \leqslant \kappa_{2,r}\|A\|_{2,1}^{\frac{2}{r}}\,|A|_{2,\infty}^{1-\frac{2}{r}}.$$

PROPOSITION 5.1. Let $2 \leqslant r \leqslant 4$ and $A \in \mathbb{K}^{n\times n}$. Then

$$\sigma_r(A) \leqslant \kappa_r\,\|A\|_{2,1}^{\frac{2}{r}}\,|A^t|_{2,\infty}^{1-\frac{2}{r}}, \tag{5.1}$$

where

$$\kappa_r = \begin{cases} \left(\frac{2}{\sqrt{\pi}}\right)^{1-\frac{2}{r}} \leqslant 1.07 & \text{if } \mathbb{K} = \mathbb{C}, \\ \frac{\pi^{\frac{3}{r}-\frac{1}{2}}}{2^{\frac{5}{r}-1}} \leqslant 1.19 & \text{if } \mathbb{K} = \mathbb{R}. \end{cases}$$

Proof. We first establish the case $r = 4$. Let $A \in \mathbb{K}^{n\times n}$. Then

$$\sigma_4(A) \leqslant \kappa_4\sqrt{\|A\|_{2,1}\,|A^t|_{2,\infty}}, \tag{5.2}$$

where

$$\kappa_4 = \begin{cases} \sqrt{\frac{2}{\sqrt{\pi}}} \leqslant 1.07 & \text{if } \mathbb{K} = \mathbb{C}, \\ \sqrt[4]{\frac{\pi}{2}} \leqslant 1.12 & \text{if } \mathbb{K} = \mathbb{R}. \end{cases}$$

This can be seen as follows: Denote by $\mathscr{S}_r^n$ the linear space of all complex $n\times n$ matrices equipped with the Schatten-r-norm σ_r. The proof of $(1)'$ in [22] then shows that

$$\|\mathrm{id} : [\mathscr{L}(\ell_2^n,\ell_1^n),\mathscr{L}(\ell_2^n,\ell_\infty^n)]_{\frac{1}{2}} \to \mathscr{S}_4^n\| \leqslant \sqrt{\frac{2}{\sqrt{\pi}}}.$$

Thus, (5.2) follows from the fact that the complex interpolation functor is of power type θ, and from Lemma 2.4: $\|A\|_{2,\infty} = \|A^t\|_{1,2} = |A^t|_{2,\infty}$.

Next observe that by Schur's (in)equality for matrices and [30, Lemma 2.3] we have $\sigma_2(A) = |A|_{2,2} \leqslant \|A\|_{2,1}$ which settles the case $r = 2$ and $\mathbb{K} = \mathbb{C}$. The remaining parts now follow from the interpolation formula $[\mathscr{S}_2^n,\mathscr{S}_4^n]_{2-\frac{4}{r}} = \mathscr{S}_r^n$ (cf. [29]) and once again from the fact that the complex interpolation functor is of power type θ, and the complexification procedure. □

In contrast to our eigenvalue estimates, the above estimate for singular values cannot have an arbitrarily large error. Indeed, since $\|A\|_{2,1} \leqslant \sqrt{n}\,\|A\|_{2,2}$ it follows that the right-hand side in the above is less than or equal to $\kappa_r n^{\frac{1}{r}} \sigma_r(A)$; we do not know if this really is the asymptotically maximal (multiplicative) error of the right-hand side approximating $\sigma_r(A)$ when using the identity $\sigma_r(A) = \sigma_r(A^t)$ (see below for R_n and C_n) – note that from the observations below one has to expect a maximal error of magnitude at least $\max(\frac{n^{\frac{2}{r^2}-\frac{1}{r}}}{\sqrt{1+\log n}}, n^{\frac{2}{r}-\frac{1}{2}})$.

EXAMPLE 5.2. (i) Similarly to our first set of examples, it can be seen that for $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n \geqslant 0$ the right-hand side in the above applied to the diagonal matrix D_μ is less than or equal to $n^{\frac{2}{r^2}-\frac{1}{r}} \sigma_4(D_\mu)$ and that the maximal (multiplicative) error for diagonal matrices indeed is at least $c\,\frac{n^{\frac{2}{r^2}-\frac{1}{r}}}{\sqrt{1+\log n}}$ for some universal constant $c > 0$.

(ii) For the matrix J_n the above estimate is only optimal for $r = 4$:

$$\|J_n\|_{2,1}^{\frac{2}{r}} \, |J_n^t|_{2,\infty}^{1-\frac{2}{r}} = n^{\frac{2}{r}+\frac{1}{2}} = n^{\frac{2}{r}-\frac{1}{2}} \sigma_r(J_n).$$

(iii) For the Vandermonde matrix V_n the above estimate is optimal up to the constant κ_r:

$$\|V_n\|_{2,1}^{\frac{2}{r}} \, |V_n^t|_{2,\infty}^{1-\frac{2}{r}} = n^{\frac{1}{r}+\frac{1}{2}} = \sigma_r(V_n).$$

(iv) When considering R_n and C_n, this once again shows that for matrices which are not normal it might be essential to recall the identity $\sigma_r(A) = \sigma_r(A^t)$:

$$\|R_n\|_{2,1}^{\frac{2}{r}} \, |R_n^t|_{2,\infty}^{1-\frac{2}{r}} = n^{\frac{1}{2}} = \sigma_r(R_n),$$

whereas

$$\|C_n\|_{2,1}^{\frac{2}{r}} \, |C_n^t|_{2,\infty}^{1-\frac{2}{r}} = n^{\frac{2}{r}} = n^{\frac{2}{r}-\frac{1}{2}} \sigma_r(C_n).$$

REMARK 5.3. Example (iv) also shows that the range of r – in contrast to the case of eigenvalues – in the proposition above cannot be extended to any value beyond 4. Indeed, let $r > 4$. Then regardless of $1 \leqslant s \leqslant \infty$ one has $\|C_n\|_{2,1}^{\frac{2}{r}} \, |C_n^t|_{s,\infty}^{1-\frac{2}{r}} = n^{\frac{2}{r}} < n^{\frac{1}{2}} = \sigma_r(C_n)$.

The considerations above with regard to (4.7) suggest that the answer to the following question might be affirmative:

PROBLEM 5.4. Let $1 < p < 2$ and $\frac{2p}{2-p} \leqslant r \leqslant \frac{4}{2-p}$. Does the inequality

$$\sigma_r(A) \leqslant \kappa_{p,r} \, \|A\|_{2,p}^{\frac{2p}{(2-p)r}} \, |A^t|_{2,\infty}^{1-\frac{2p}{(2-p)r}},$$

$\kappa_{p,r}$ as before, hold true for $A \in \mathbb{K}^{n \times n}$?

Note that as in the above this would immediately follow by interpolation from Theorem 1.1 (ii) and an affirmative answer to the following question:

PROBLEM 5.5. Let $1 < p < 2$. Does

$$\sup_n \|\mathrm{id} : [\mathscr{L}(\ell_2^n, \ell_p^n), \mathscr{L}(\ell_2^n, \ell_\infty^n)]_{1-\frac{p}{2}} \to \mathscr{S}^n_{\frac{4}{2-p}}\| < \infty$$

hold true? Note that the calculations made in [24] (where for other choices of the indices the answer to similar questions very often is negative) do not deny this possibility.

6. Concluding remarks

The estimates in the above might be very interesting from the following computational point of view:

PROBLEM 6.1. Fix $1 \leqslant p \leqslant q \leqslant \infty$ and a matrix $A \in \mathbb{K}^{n\times n}$. Does there exist an efficient algorithm for computing $\|A\|_{q,p}$, or at least approximating this quantity up to a constant factor?

Using a semidefinite programming relaxation, Nesterov [25] (see also [27]) gave an affirmative answer when $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$ (approximation up to a factor smaller than 2.3), and very recently Bhaskara and Vijayaraghavan [4] have provided an efficent algorithm for the general case and non-negative matrices (approximation up to a factor arbitrarily close to 1). However, the latter have also shown in the case of arbitrary matrices and $1 \leqslant p \leqslant q < 2$ not simultaneously equal to 1 (or, by duality, $2 < p \leqslant q \leqslant \infty$ not simultaneously equal to ∞) that approximating $\|A\|_{q,p}$ up to any constant factor is NP-hard; moreover, they have even proved inapproximability of almost polynomial factor (for a precise formulation we refer to their article). For non-negative matrices and $p = q$, Bhaskara and Vijayaraghavan applied their results to the oblivious routing problem. Other applications involve, e.g., robust optimization (see, e.g., [27]).

Thus, our upper estimates (and clearly also the ones due to König) for $\Lambda_{r,\infty}(A)$ (which clearly is efficiently computable) might be used to obtain lower estimates for $\|A\|_{q,p}$ in these critical cases and compare them with upper estimates coming from other sources (see, e.g., [27]). As we have already seen, the accuracy of these lower estimates may vary for particular classes of matrices with respect to the choice of the parameters q, p and r. With regard to the results for J_n and V_n, respectively, for given q and p the best choice for r seems to be $\frac{1}{\alpha}+2$, which then would give (in the complex case)

$$\|A\|_{q,p} \geqslant \Big(\frac{\pi}{8e}\Big)^{\frac{1}{2}+2(\frac{1}{p}-\frac{1}{q})} \Big(\frac{\sum_k |\lambda_k(A)|^{\frac{pq}{q-p}+2}}{\max_j \sum_i |a_{ij}|^2}\Big)^{\frac{1}{p}-\frac{1}{q}}.$$

Moreover, the proof of Corollary 4.11 shows that $\|A\|_{q,p}$ can be estimated from below in terms of the largest eigenvalue of A:

$$\|A\|_{q,p} \geqslant \Big(\frac{\pi}{8e}\Big)^{\frac{1}{2}(\frac{1}{p}+\frac{1}{q'})} \frac{|\lambda_1(A)|^{\frac{1}{p}+\frac{1}{q'}}}{\max_{i,j} |a_{ij}|^{\frac{1}{p}-\frac{1}{q}}}.$$

A very special and interesting case seems to be $q = \infty$ and $p > 2$ which can be reformulated in the following sense as a Longest Vector Problem: Find the length of the longest column of a matrix A. This is somewhat a counterpart to the Shortest Vector Problem (with integer values) which has received a lot of attention in the cryptography community; see [4] and the references therein.

Finally, notice that our proofs are based on the so-called *Little Grothendieck Theorem* in the case $\mathbb{K} = \mathbb{C}$, which, in terms of summing operators, reads $\pi_2(T) \leqslant \frac{2}{\sqrt{\pi}} \|T\|$ for any operator $T : \ell_1 \to \ell_2$ (see, e.g., [7]). *Grothendieck's Theorem* in terms of summing operators reads $\pi_1(T) \leqslant K_G \|T\|$ for any operator $T : \ell_1 \to \ell_2$ (see, e.g., [7]), where $\frac{\pi}{2} \leqslant K_G \leqslant \frac{\pi}{2\ln(1+\sqrt{2})}$ in the real case. This stronger result – also known as *Grothendieck's inequality* – is used in [1] to obtain an efficient randomized $\frac{2\ln(1+\sqrt{2})}{\pi}$-approximation algorithm for approximating $\|A\|_{\infty,1}$ and thus providing an efficient algorithm which approximates the so-called *cut-norm* of a matrix up to the same constant multiplicative factor. This concept plays a major role in the design of efficient approximation algorithms for dense graph and matrix problems.

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