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SIGN PRESERVATION PROPERTIES OF SOME NONLINEAR TRANSFORMATIONS

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Abstract. It is shown that the number of sign changes in certain transformations does not increase with each iteration in time. These transformations are composed of linear components defined in terms of totally positive matrices and semi linear components similar to $u \to ku^3$. In particular the analysis shows that for certain semi linear parabolic equations discretized using finite difference methods, the number of sign changes does not increase.

1. Introduction

This paper is partially motivated a well known fact on parabolic equations. Consider the heat equation in one spatial variable

$$u_t = \alpha^2 u_{xx}, \quad 0 < x < 1, \quad \text{and } \ t > 0,$$
$$u(0,t) = u(1,t) = 0, \quad u(x,0) = u^0(x),$$

where α^2 is the diffusivity constant. The number of sign changes in $u(.,t)$ does not increase with time, that is if $t_1 < t_2$ then the number of sign changes in $u(.,t_2)$ in the interval $[0,1]$ is less than or equal to the number of sign changes in $u(.,t_1)$.

If a finite difference scheme is used to discretize equation (1) then it is desirable that the resulting matrix equation, say $U^{k+1} = AU^k$ where k is the discrete time, also has this property, namely, that the number of sign changes in U^{k+1} is less than or equal to that in U^k. Such results can be found in literature on numerical PDE, see [8].

Results on non-increase in sign changes are also well documented in literature on total positivity, whose main motivation comes from applications in probability and statistics.The idea of total positivity is applicable to matrices, integral kernels and differential operators, and is in fact, one of the most important properties of many operators, see [5].

In our paper we consider the one dimensional semilinear parabolic equation

$$u_t = \alpha^2 u_{xx} + f(u) \quad 0 < x < L, \quad \text{and } \ t > 0,$$
$$u(0,t) = u(L,t) = 0, \quad u(x,0) = u^0(x),$$

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where $f(u)$ is a given function. Discretizing the above equation with forward difference scheme with uniform space mesh h and time mesh k, we get the following equation

$$U^{k+1} = AU^k + F(U).$$

Here U^k denotes the discrete $u(x,t)$ at the k_{th} time step and for $0 < x < L$,

$$A = \begin{bmatrix} 1-2r & r & 0 & \dots & 0 \\ r & 1-2r & r & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & r \\ 0 & \dots & 0 & r & 1-2r \end{bmatrix}, \qquad r = \alpha^2(k/h^2) \tag{1}$$

We show that under certain conditions on r and F the number of sign changes in U^{k+1} is less or equal to the sign changes in U^k, and extend the result to equations with arbitrary totally positive matrices. Our proofs are linear-algebraic in nature, and are more simple and more transparent than proofs in PDE or total positivity literature.

In this paper we say that there is a sign change whenever two consecutive non zero entries in a vector have opposite signs, and if an entry is zero then it has no sign. For example both the vectors

$$\begin{bmatrix} + \\ + \\ - \\ - \end{bmatrix}, \quad \begin{bmatrix} + \\ + \\ 0 \\ 0 \\ - \end{bmatrix}$$

change sign only once and

$$\begin{bmatrix} + \\ + \\ + \end{bmatrix}, \quad \begin{bmatrix} + \\ 0 \\ 0 \\ + \end{bmatrix}$$

do not change sign.

2. Number of Oscillations in the Finite Difference Methods

2.1. Forward Difference Method

CLAIM 1. *Let*

$$E = \begin{bmatrix} a_1 & & & & & 0 \\ 0 & \ddots & & & & \\ & \ddots & a_{i-1} & & & \\ & & b & a_i & & \\ & & & \ddots & \ddots & \\ 0 & & & & 0 & a_n \end{bmatrix}$$

be a $n\times n$ matrix with $b>0$, and $a_i>0$ for $i=1,\dots,n$. Also let $U=[u_1,\dots,u_n]$ be any real vector and

$$F(U)=\begin{bmatrix} f_1(u_1)\\ \vdots\\ f_n(u_n)\end{bmatrix}$$

with the property that $f_i(x_i)>0$ if $x_i>0$ and $f_i(x_i)<0$ if $x_i>0$ and $f(x_i)=0$ if $x_i=0$ then the number of sign changes in $EU+F(U)$ is less than or equal to the number of sign changes in U.

Proof. Partition the matrix E as follows:

$$E=\left[\begin{array}{c|cc|c} A & & & \\ \hline & a_{i-1} & 0 & \\ & b & a_i & \\ \hline & & & \tilde{A}\end{array}\right],\quad \text{where}\quad A=\begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_{i-2}\end{bmatrix},\qquad \tilde{A}=\begin{bmatrix} a_{i+1} & & \\ & \ddots & \\ & & a_n\end{bmatrix}.$$

Similarly,

$$U=\left[\begin{array}{c} \hat{U}\\ \hline u_{i-1}\\ u_i\\ \hline \tilde{U}\end{array}\right],$$

where

$$\hat{U}=\begin{bmatrix} u_1\\ \vdots\\ u_{i-2}\end{bmatrix}\quad\text{and}\quad \tilde{U}=\begin{bmatrix} u_{i+1}\\ \vdots\\ u_n\end{bmatrix},\quad\text{and}\quad F(U)=\left[\begin{array}{c} F(\hat{U})\\ \hline f(u_{i-1})\\ f(u_i)\\ \hline F(\tilde{U})\end{array}\right],$$

where

$$F(\hat{U})=\begin{bmatrix} f(u_1)\\ \vdots\\ f(u_{i-2})\end{bmatrix},\qquad F(\tilde{U})=\begin{bmatrix} f(u_{i+1})\\ \vdots\\ f(u_n)\end{bmatrix}.$$

Then

$$\begin{aligned} EU+F(U)&=\left[\begin{array}{c|cc|c} \mathrm{A} & & & \\ \hline & a_{i-1} & 0 & \\ & b & a_i & \\ \hline & & & \tilde{A}\end{array}\right]\left[\begin{array}{c} \hat{U}\\ \hline u_{i-1}\\ u_i\\ \hline \tilde{U}\end{array}\right]+\left[\begin{array}{c} F(\hat{U})\\ \hline u_{i-1}\\ u_i\\ \hline F(\tilde{U})\end{array}\right]\\ &=\left[\begin{array}{c} A\hat{U}+F(\hat{U})\\ \hline a_{i-1}u_{i-1}+f(u_{i-1})\\ bu_{i-1}+a_iu_i+f(u_i)\\ \hline \tilde{A}\tilde{U}+F(\tilde{U})\end{array}\right].\end{aligned}$$

Observe that multiplication with A and $\tilde{A}$ will not change sign because $a_i > 0$ for $i = 1...n$ and $F(\hat{U})$ and $F(\tilde{U})$ will have the same sign as $\hat{U}$ and $\tilde{U}$. Also $a_{i-1}u_{i-1} + f(u_{i-1})$ will have the same sign as u_{i-1}. Now consider the following possibilities:

Case 1. Let $u_{i-1}, u_i > 0$. Then $bu_{i-1} + a_i u_i + f(u_i) > 0$. It follows that then there is no sign change in the vector, hence the claim is proved. Similarly when both $u_{i-1}, u_i < 0$ the number of sign changes does not change.

Case 2. If $u_{i-1}, u_i \neq 0$, and u_{i-1} and u_i have different signs. Suppose that $u_{i-1} > 0$ and $u_i < 0$ then $a_{i-1}u_{i-1} + fu_{i-1} > 0$ if $bu_{i-1} + a_i u_i + f(u_i)$ is negative, then the number of sign change stays the same. In case $bu_{i-1} + a_i u_i + f(u_i)$ is zero or positive, the number of sign change can either stays the same or decreases by one. The possibility when $u_{i-1} < 0$ and $u_i > 0$ can be proved in the same way.

Case 3. If $u_{i-1} = 0, u_i \neq 0$, then $a_{i-1}u_{i-1} + f(u_{i-1}) = 0$ and $bu_{i-1} + a_i u_i + f(u_i)$ will have the same sign as u_i and the number of sign changes in $EU + F(U)$ will be less or equal to the number of sign changes in U.

Case 4. If $u_{i-1} \neq 0, u_i = 0$, then $a_{i-1}u_{i-1} + f(u_{i-1})$ and $bu_{i-1} + a_i u_i + f(u_i)$ will have the same sign as u_{i-1}. Hence the number of sign changes will be less or equal to the number of sign changes in U.

Case 5. If $u_{i-1} = u_i = 0$, then $a_{i-1}u_{i-1} + f(u_{i-1}), bu_{i-1} + a_i u_i + f(u_i) = 0$ and the number of sign changes in $EU + F(U)$ will equal to the number of sign changes in U. $\square$

CLAIM 2. *Let*

$$G = \begin{bmatrix} a_1 & 0 & & & & 0 \\ & \ddots & \ddots & & & \\ & & a_{i-1} & & & \\ & & & a_i & b & \\ & & & & \ddots & 0 \\ 0 & & & & & a_n \end{bmatrix}$$

be a $n \times n$ matrix with $b > 0$, and $a_i > 0$ for $i = 1...n$. With the same assumptions on U and $F(U)$ as in claim (1) the number of sign changes in $GU + F(U)$ less than or equal to the number of sign changes in U.

The proof is similar to the proof of Claim 1. $\square$

CLAIM 3. *Let a_i, b_i, c_i be positve and define*

$$\Pi_n = \begin{bmatrix} a_1 & c_1 & & 0 \\ b_2 & \ddots & \ddots & \\ & \ddots & \ddots & c_{i-1} \\ 0 & & b_i & a_i \end{bmatrix} \qquad \textit{for } i = 1, \dots n.$$

Assume $\det \Pi_i > 0$. *Then the Crout factorization (see* [6]*) of* Π_n *is given by*

$$\Pi_n = \begin{bmatrix} d_1 & & & 0 \\ b_2 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & b_n & d_n \end{bmatrix} \begin{bmatrix} 1 & f_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & f_{n-1} \\ 0 & & & 1 \end{bmatrix}.$$

Then $d_i, f_i > 0$ *for* $i = 1, \dots n$.

Proof. It is clear that $d_i = \det \Pi_i > 0$. Assume $d_1, \dots, d_{i-1} > 0$. Since $0 < \det \Pi_i = d_1 \dots d_i$, it follows that $d_i > 0$ and consequently $f_i = c_i / d_i > 0$. $\square$

THEOREM 1. *Suppose*

$$U^{(k+1)} = \Pi U^{(k)} + F(U^{(k)}), \tag{2}$$

where Π *is the matrix* Π_n *as defined in Claim 3, and* $F(U)$ *is defined in Claim 1. Then the number of sign changes in* $U^{(k+1)}$ *is less than or equal to the number of sign changes in* $U^{(k)}$.

Proof. Factorizing Let $e_i = b_i / d_i > 0$, then

$$\Pi = \begin{bmatrix} 1 & & & 0 \\ e_2 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & e_n & 1 \end{bmatrix} \begin{bmatrix} d_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & f_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & f_{n-1} \\ 0 & & & 1 \end{bmatrix} = \mathscr{L} \mathscr{D} \mathscr{U}$$

It is known that $\mathscr{L} = E_{21} E_{31} \dots E_{n1} E_{32} \dots E_{n2} \dots E_{nn-1}$, where

$$E_{i,i-1} = \begin{bmatrix} 1 & & & & & 0 \\ 0 & \ddots & & & & \\ & \ddots & \ddots & & & \\ & & a_{i,i-1} & 1 & & \\ & & & \ddots & \ddots & \\ 0 & & & & 0 & 1 \end{bmatrix}$$

with $a_{i,i-1} > 0$. Similarly $\mathscr{U} = G_{12} G_{13} \dots G_{1n} G_{23} \dots G_{2n} \dots G_{n-1n}$, where

$$G_{i,i+1} = \begin{bmatrix} 1 & 0 & & & 0 \\ & \ddots & \ddots & & \\ & & 1 & b_{i,i+1} & \\ & & & \ddots & \ddots \\ & & & & \ddots & 0 \\ 0 & & & & & 1 \end{bmatrix}$$

with $b_{i,i+1} > 0$. Therefore equation (3) can be written as

$$U^{(k+1)} = AU^{(k)} + F(U^{(k)}) = E_{21}E_{32}\dots E_{nn-1}DG_{12}G_{23}\dots G_{n-1n}U^{(k)} + F(U^{(k)}).$$

Using Claims 1 and 2, multiplication with any of the matrix $E_{i+1,i}$ or $G_{i,i+1}$ will not increase the number of sign changes in $U^{(k)}$ and the same is true for multiplication by a diagonal matrix D with positive entries. Hence it follows that the number of sign changes in $U^{(k+1)}$ will be less or equal to the number of sign changes in $U^{(k)}$. $\square$

REMARK. For matrix A in (1) to be Π in Theorem 1, it is sufficient to require $r < 1/4$. Then $A = \Pi$ is a symmetric positive definite matrix and will satisfy $\det\Pi_i > 0$ for $i = 1,\dots n$.

2.2. Backward Difference Method

THEOREM 2. *Suppose*

$$AU^{(k+1)} = U^{(k)} + F(U^{(k)}),$$

where $A = \begin{bmatrix} a_1 & -c_1 & & 0 \\ -b_2 & \ddots & \ddots & \\ & \ddots & \ddots & -c_{n-1} \\ 0 & & -b_n & a_n \end{bmatrix}$ *and* $a_i, b_i, c_i > 0$ *for* $i = 1,\dots n$. *Let* A_i *be the* $i \times i$ *principal submatrix. Assume* $\det A_i > 0$ *for* $i = 1,\dots n$. *Then the number of sign changes in* $U^{(k+1)}$ *is less than or equal to the number of sign changes in* $U^{(k)}$.

Proof. Similarly to the proof of Theorem 1,we have

$$A = E_{21}E_{32}\dots E_{nn-1}DG_{12}G_{23}\dots\dots G_{n-1n}U^{(k)},$$

here

$$E_{i+1,i} = \begin{bmatrix} 1 & & & & 0 \\ 0 & \ddots & & & \\ & \ddots & \ddots & & \\ & & -e_{i+1,i} & 1 & \\ & & & \ddots & \ddots \\ 0 & & & 0 & 1 \end{bmatrix}, \qquad G_{i,i+1} = \begin{bmatrix} 1 & 0 & & & 0 \\ & \ddots & \ddots & & \\ & & 1 & -g_{i,i+1} & \\ & & & \ddots & \ddots \\ & & & & \ddots\ 0 \\ 0 & & & & 1 \end{bmatrix},$$

with $e_{i+1,i}$ and $g_{i,i+1}$ being positive. Now

$$E_{i+1,i}^{-1} = \begin{bmatrix} 1 & & & & 0 \\ 0 & \ddots & & & \\ & \ddots & \ddots & & \\ & & e_{i+1,i} & 1 & \\ & & & \ddots & \ddots \\ 0 & & & 0 & 1 \end{bmatrix}$$

and

$$G_{i,i+1}^{-1} = \begin{bmatrix} 1 & 0 & & & & 0 \\ & \ddots & \ddots & & & \\ & & 1 & g_{i,i+1} & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & 0 \\ 0 & & & & & 1 \end{bmatrix}.$$

Thus

$$\begin{aligned} U^{(k+1)} &= A^{-1}(U^{(k)} + F(U^{(k)})) \\ &= G_{n-1n}^{-1} \cdots G_{1n}^{-1} \cdots G_{12}^{-1} D^{-1} E_{nn-1}^{-1} \cdots E_{n1}^{-1} \cdots E_{21}^{-1}(U^{(k)} + F(U^{(k)})). \end{aligned}$$

The result follows by using Claims 1 and 2. □

REMARK. There is no restriction on $r = \alpha^2 k/h^2$ for the backward difference method in order for number of sign changes in U^k be non increasing.

2.3. Crank-Nicolson Method

THEOREM 3. *Suppose*

$$AU^{(k+1)} = \Pi U^{(k)} + F(U^{(k)}),$$

where $\Pi = \begin{bmatrix} a_1 & c_1 & & 0 \\ b_2 & \ddots & \ddots & \\ & \ddots & \ddots & c_{n-1} \\ 0 & & b_n & a_n \end{bmatrix}$ *and* $A = \begin{bmatrix} \tilde{a}_1 & -\tilde{c}_1 & & 0 \\ -\tilde{b}_2 & \ddots & \ddots & \\ & \ddots & \ddots & -\tilde{c}_{n-1} \\ 0 & & -\tilde{b}_n & \tilde{a}_n \end{bmatrix}$. *For* $i = 1, \ldots, n$, *let* $a_i, b_i, c_i, \tilde{a}_i, \tilde{b}_i, \tilde{c}_i > 0$, *and assume that both* $\det \Pi_i, \det A_i$ *are positive, where* Π_i *and* A_i *are* $i \times i$ *principal submatrices of* Π *and* A *respectively. Then the number of sign changes in* $U^{(k+1)}$ *is less than or equal to the number of sign changes in* $U^{(k)}$.

Proof. Combine the proofs of Theorems 1 and 2. □

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