Operators
and
Matrices
Volume 5, Number 3 (2011), 417–423

QUASI–SIMILAR k–PARANORMAL OPERATORS

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(*Communicated by R. Curto*)

Abstract. It is proved in this paper that k-paranormal operators satisfy (Bishop's) property (β); and also that if S and T are k-paranormal contractions such that the completely non-unitary part S_c of S has finite multiplicity, then S is quasi-similar to T if and only if their unitary parts are unitarily equivalent and their completely non-unitary parts are quasi-similar. This generalizes a result of W.W. Hastings [4] on subnormal operators and P.Y. Wu [11] on hyponormal operators.

1. Introduction

Let $B(\mathscr{H})$ denote the algebra of operators on an infinite dimensional complex Hilbert space. An operator $T \in B(\mathscr{H})$ is k-paranormal for some integer $k \geqslant 1$ if

$$\|Tx\|^{k+1} \leqslant \|T^{k+1}x\|$$

for every unit vector $x \in \mathscr{H}$. Let $P(k)$ denote the class of all k-paranormal operators. $T \in B(\mathscr{H})$ is a quasi-affinity if it is injective and has a dense range; $S,T \in B(\mathscr{H})$ are quasi-similar, $S \sim T$, if there exist quasi-similarities $X,Y \in B(\mathscr{H})$ such that

$$SX = XT \quad \text{and} \quad TY = YS.$$

Let T_u denote the unitary part and T_c denote the cnu (completely non-unitary) part of a contraction $T \in B(\mathscr{H})$. Nagy–Foiaş classes of contractions, C_{00}, C_{01}, C_{10} and C_{11} [8, p. 72] are defined as usual. Let $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$, and $\sigma_e(T)$ stand for spectrum, point spectrum, approximate point spectrum, and essential spectrum (or Fredholm spectrum) of $T \in B(\mathscr{H})$, respectively. Let $\mathbb{N}$ denote the set of non–negative integers. The ascent $\operatorname{asc}(T)$ and descent $\operatorname{dsc}(T)$ of $T \in B(\mathscr{H})$ are given by

$$\operatorname{asc}(T) = \inf\{n \in \mathbb{N} : T^{-n}(0) = T^{-(n+1)}(0)\}$$

and

$$\operatorname{dsc}(T) = \inf\{n \in \mathbb{N} : T^{n}(\mathscr{H}) = T^{n+1}(\mathscr{H})\}$$

(if no such integer n exists, then $\operatorname{asc}(T) = \infty$, respectively $\operatorname{dsc}(T) = \infty$). We say that T has the single valued extension property, or SVEP, at $\lambda \in \mathbb{C}$ if for every open neighborhood U of λ, the only analytic solution f to the equation

Mathematics subject classification (2010): Primary 47A45, Secondary 47B20.
Keywords and phrases: Hilbert space operators, k-paranormal operators, quasi-similarity.

Paper OaM-05-29

$$(T-\mu)f(\mu)=0$$

for all $\mu \in U$ is the constant function $f \equiv 0$; we say that T has SVEP if T has a SVEP at every $\lambda \in \mathbb{C}$. It is well known that finite ascent implies SVEP; also, an operator has SVEP at every isolated point of its spectrum (as well as at every isolated point of its approximate point spectrum). An operator $T \in B(\mathscr{H})$ satisfies (Bishop's) property (β) if, for every open subset U of the complex plane $\mathbb{C}$ and every sequence of analytic functions $f_n : U \to \mathscr{H}$ with the property that

$$(T-\lambda)f_n(\lambda) \to 0 \quad \text{as} \quad n \to \infty$$

uniformly on all compact subsets of U, $f_n(\lambda) \to 0$ as $n \to \infty$ locally uniformly on U.

2. Bishop's property (β) for $P(k)$ operators

Recall that operators $S \in P(k)$ are normaloid, i.e., $\|S\| = r(S)$.

LEMMA 2.1. ([12, Lemma 2.3 and Corollary 2.6]). *If $T \in P(k)$ and $(0 \neq)\lambda \in \sigma_p(T)$, then*

$$T = \begin{pmatrix} \lambda & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} (T-\lambda)^{-1}(0) \\ \{(T-\lambda)^{-1}(0)\}^{\perp} \end{pmatrix},$$

where $T_{22} \in P(k) \cap B(\{(T-\lambda)^{-1}(0)\}^{\perp})$ is such that $\lambda \notin \sigma_p(T_{22})$.

LEMMA 2.2. ([3, Corollary 1]). *If $T \in P(k)$ is a contraction, then it has a decomposition $T = T_u \oplus T_c$, where $T_c \in C_{.0}$.*

LEMMA 2.3. *Operators $T \in P(k)$ have finite ascent $\leqslant 1$.*

Proof. Since $\operatorname{asc}(T-\lambda) = 0$ for every $\lambda \in \sigma(T) \setminus \sigma_p(T)$, we consider points $\lambda \in \sigma_p(T)$. If $\lambda = 0$, then the definition of k-paranormality implies that $T^{-(k+1)}(0) \subseteq T^{-1}(0)$; since $T^{-1}(0) \subseteq T^{-2}(0) \subseteq \ldots$, $T^{-(k+1)}(0) = T^{-1}(0)$. Now let $\lambda \neq 0$. Then

$$T-\lambda = \begin{pmatrix} 0 & T_{12} \\ 0 & T_{22}-\lambda \end{pmatrix} \begin{pmatrix} (T-\lambda)^{-1}(0) \\ \{(T-\lambda)^{-1}(0)\}^{\perp} \end{pmatrix}.$$

Recall, [10, Exercise 7, p. 293], that $\operatorname{asc}(T-\lambda) \leqslant \operatorname{asc}(0) + \operatorname{asc}(T_{22}-\lambda)$. Since $\operatorname{asc}(T_{22}-\lambda) = 0$, we have that $\operatorname{asc}(T-\lambda) = 1$. $\square$

An immediate consequence of Lemma 2.3 is the following:

COROLLARY 2.4. *Operators $T \in P(k)$ have SVEP.*

Given an open subset U of $\mathbb{C}$, let $H(U,\mathscr{H})$ denote the Fréchet space of analytic functions from U to $\mathscr{H}$. Then $T \in B(\mathscr{H})$ satisfies property (β) precisely when the operator $T_U : H(U,\mathscr{H}) \to H(U,\mathscr{H})$, $(T_U f)(\lambda) := (T-\lambda)f(\lambda)$, (is injective and) has closed range [7, Proposition 3.3.5].

Let $\ell^\infty(\mathscr{H})$ denote the space of all bounded sequences of elements of $\mathscr{H}$, and let $c_0(\mathscr{H})$ denote the space of all null sequences of $\mathscr{H}$. Endowed with the canonical norm, the quotient space $\mathscr{K} = \ell^\infty(\mathscr{H})/c_0(\mathscr{H})$ can be made into a Hilbert space [1], into which $\mathscr{H}$ may be isometrically embedded. The Berberian–Quigley extension theorem, [7, p. 255], says that given an operator $T \in B(\mathscr{H})$ there exists an isometric *-isomorphism $T \to T^o \in B(\mathscr{K})$ preserving order such that $\sigma(T) = \sigma(T^o)$ and $\sigma_a(T) = \sigma_a(T^o) = \sigma_p(T^o)$. Let $[x_n] \in \mathscr{K}$ denote the equivalence class of the sequence $\{x_n\} \subset \mathscr{H}$. If $T \in P(k)$, then

$$\|T^o[x]\|^{k+1} = \|Tx\|^{k+1} \leqslant \|T^{k+1}x\|\|x\|^k = \|T^{ok+1}[x]\|\,\|[x]\|^k$$

for each $x \in \mathscr{H}$. Hence the Berberian–Quigley extension T^o of an operator $T \in P(k)$ is again k-paranormal.

THEOREM 2.5. *Operators $T \in P(k)$ satisfy property (β).*

Proof. Let U be an open subset of $\mathbb{C}$, and assume that

$$(T-\lambda)f_n(\lambda) \to 0 \ \text{ on } \ H(U,\mathscr{H})$$

for every $\lambda \in U$. Then

$$(T^o - \lambda I^o)[f_n(\lambda)] = 0 \ \text{ on } \ H(U,\mathscr{K})$$

for every $\lambda \in U$. Since the k-paranormal operator T^o has SVEP, $[f_n(\lambda)] = 0$ (i.e., $\{f_n\} \in c_0(\mathscr{H})$). We claim that $f_n(\lambda) \to 0$ on $H(U,\mathscr{H})$. Start by observing that if $D(\lambda;r) = \{\mu \in \mathbb{C} : |\lambda-\mu| < r\}$ is such that $\overline{D(\lambda;r)} \subset U$, then the analytic sequence $\{f_n(\lambda)\}$ is uniformly bounded on $\overline{D(\lambda;r)}$; furthermore, for every $\varepsilon > 0$, there exists a natural number N and $0 < \rho < r$ such that

$$\|f_n(\mu)\| < \frac{\varepsilon}{2} \quad \text{and} \quad \|f_n(\lambda) - f_n(\mu)\| < \frac{\varepsilon}{2}$$

for all $n > N$ and $\mu \in D(\lambda;\rho)$. Indeed, considering $\frac{f_n}{1+\|f_n\|}$ instead of f_n if need be, we may assume that $\sup|f_n| = M < \infty$ on $\overline{D(\lambda;r)}$. The function f_n being analytic, $f_n(\mu) - f_n(\lambda) = \sum_{m=1}^{\infty} a_{nm}(\mu-\lambda)^m$, and then $\|f_n(\lambda) - f_n(\mu)\| \leqslant \frac{M\rho}{r-\rho}$ for all $\mu \in \overline{D(\lambda;\rho)}$ such that $0 < \rho < r$. Now choose N and ρ such that $|f_n(\lambda)| < \frac{\varepsilon}{4}$ (recall that $f_n(\lambda) \in c_0$) and $\frac{M\rho}{r-\rho} < \frac{\varepsilon}{4}$. Then

$$\|f_n(\mu)\| \leqslant \|f_n(\lambda)\| + \|f_n(\lambda) - f_n(\mu)\| < \frac{\varepsilon}{2}$$

for all $n > N$ and $\mu \in D(\lambda;\rho)$. Consequently, $f_n(\lambda) \to 0$ in $H(U,\mathscr{H})$, i.e., T satisfies property (β). $\square$

The conclusion that $P(k)$ operators satisfy property (β) generalizes an observation by Uchiyama and Takahashi [9] that paranormal operators (i.e., $P(1)$ operators) satisfy property (β). Property (β) has a number of consequences: we list below but a couple of these. Let $\mathbf{D}$ denote the closed unit disc in $\mathbb{C}$.

COROLLARY 2.6. *If $S \in P(k)$ is quasi-similar to an operator $T \in B(\mathscr{H})$ satisfying property (β), then $\sigma_x(S) = \sigma_x(T)$, where $\sigma_x = \sigma$ or σ_e. In particular, if $S \in P(k)$ is quasi-similar to an isometry $V \in B(\mathscr{H})$, then S is a contraction such that $\sigma_x(S) = \sigma_x(V) = \mathbf{D}$.*

Proof. That $\sigma_x(S) = \sigma_x(T)$ follows from an application of [7, Theorem 3.7.15]. In the particular case in which $T = V$, it follows that $\sigma_x(S) = \sigma_x(V) = \mathbf{D}$. Hence, since S is normaloid, $r(T) = \|T\| = 1$, i.e., S is a contraction. $\square$

A number of the commonly considered classes of operators in $B(\mathscr{H})$ (for example, hyponormal, M-hyponormal, p-hyponormal for $0 < p \leqslant 1$, w-hyponormal, (p,k)-quasihyponormal operators for $0 < p \leqslant 1$ and integers $k \geqslant 1$) are known to satisfy property (β); Corollary 2.6 applies to operators T belonging to one of these classes. An operator T on a separable Hilbert space $\mathscr{H}$ is said to be supercyclic if the homogeneous orbit $\{\lambda T^n x : \lambda \in \mathbb{C},\ n \in \mathbf{N} \cup \mathbf{0}\}$ is dense in $\mathscr{H}$ for some $x \in \mathscr{H}$. It is known that paranormal operators (i.e., operators in $P(1)$) are not supercyclic [2]. Does this extend to operators in $P(k)$ for $k \geqslant 2$?

We have a partial result for invertible k-paranormal operators. Recall that the inverse of an invertible paranormal is again paranormal. It is, however, an open question whether the inverse of an invertible k-paranormal operator for $k \geqslant 2$ is k-paranormal [6].

COROLLARY 2.7. *Operators $T \in P(k)$ such that T^{-1}, whenever it exists, is also a $P(k)$ operator are not supercyclic.*

Proof. Suppose that $T \in P(k)$ is supercyclic. The class $P(k)$ being closed under multiplication by non-zero scalars, we may assume that $\|T\| = 1$. Since the supercyclic contraction T satisfies property (β), $\sigma(T)$ is contained in the boundary $\partial\mathbf{D}$ of $\mathbf{D}$ [7, Proposition 3.3.18]. Thus T is invertible, and hence (by hypothesis) $T^{-1} \in P(k)$. But then $\|T^{-1}\| = 1$ $(= \|T\|)$. Consequently, T is a unitary. Since no unitary on an infinite dimensional Hilbert space can be supercyclic, we have a contradiction. $\square$

Next, we state a couple of corollaries to Corollary 2.7

COROLLARY 2.8. *Invertible operators in $P(k)$ such that their inverse lies in $P(k-1)$ are not supercyclic.*

Proof. [6, Theorem 1] implies that $T^{-1} \in P(k)$; apply Corollary 2.7. $\square$

COROLLARY 2.9. *If $T \in P(k)$ is invertible, and if*

$$\|T^k x\|^{k+1} \leqslant \|Tx\|^{k+1} \|T^{k+1}x\|^{k-1}$$

for every unit vector $x \in \mathscr{H}$, then T is not supercyclic.

Proof. [6, Theorem 2] implies that $T^{-1} \in P(k)$; apply Corollary 2.7. $\square$

3. Quasi-similar $P(k)$ operators

The multiplicity μ_T of an operator $T \in B(\mathscr{H})$ is the minimum cardinality of a set $K \subseteq \mathscr{H}$ such that $\mathscr{H} = \bigvee_{n=0}^{\infty} T^n K$. Evidently, if $S, T \in B(\mathscr{H})$ and $SX = XT$ for some operator $X \in B(\mathscr{H})$ with dense range, then $\mu_S \leqslant \mu_T$; hence, if there exist operators $X, Y \in B(\mathscr{H})$ with dense range such that $SX = XT$ and $TY = YS$, then $\mu_S = \mu_T$. The following technical lemma will be required.

LEMMA 3.1. ([11, Theorem 3.7]). *If $X \in B(\mathscr{H})$ has dense range and is in the commutant of a $C_{1.}$-contraction $T \in B(\mathscr{H})$, then X is injective.*

In the following we shall denote the normal part and the pure part (i.e., completely non-normal part) of an operator $S \in B(\mathscr{H})$ by S_n and S_p, respectively; if S is a contraction, then we shall denote its unitary and cnu parts by S_u and S_c, respectively.

THEOREM 3.2. *Let $S, T \in B(\mathscr{H})$ be $P(k)$ contractions such that $\mu_{S_c} < \infty$. Then $S \sim T$ if and only if S_u, T_u are unitarily equivalent and $S_c \sim T_c$.*

Proof. The "if" part being obvious, we prove the "only if" part. Since S and T have $C_{.0}$ cnu parts by Lemma 2.2,

$$S = S_u \oplus S_c = \begin{pmatrix} S_{11} & 0 & 0 \\ 0 & S_{22} & * \\ 0 & 0 & S_{33} \end{pmatrix} \quad \text{and} \quad T = T_u \oplus T_c = \begin{pmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & * \\ 0 & 0 & T_{33} \end{pmatrix},$$

where $S_{11} = S_u$, $T_{11} = T_u$, S_{22} and $T_{22} \in C_{00}$, and S_{33} and $T_{33} \in C_{10}$ [8, Chapter II, Theorem 4.1]. Let $SX = XT$ and $TY = YS$, where $X, Y \in B(\mathscr{H})$ are quasi-affinities. Then X and Y have representations $X = [X_{ij}]_{i,j=1}^3$ and $Y = [Y_{ij}]_{i,j=1}^3$. Observe that $S_{11}X_{12} = X_{12}T_{22}$; since S_{11} is unitary and $T_{22} \in C_{00}$,

$$\|X_{12}x\| = \|S_{11}^n X_{12}x\| \leqslant \|X_{12}\| \, \|T_{22}^n x\| \to 0$$

as $n \to \infty$ for all x. Hence $X_{12} = 0$. A similar argument shows that indeed $X_{21} = X_{31} = X_{32} = 0 = Y_{12} = Y_{21} = Y_{31} = Y_{32}$. Thus X_{11} and Y_{11} are injective, and

$$X_0 = \begin{pmatrix} X_{22} & X_{23} \\ 0 & X_{33} \end{pmatrix} \quad \text{and} \quad Y_0 = \begin{pmatrix} Y_{22} & Y_{23} \\ 0 & Y_{33} \end{pmatrix}$$

have dense range. The equalities $S_{11}X_{11} = X_{11}T_{11}$ and $T_{11}Y_{11} = Y_{11}S_{11}$ imply that $\overline{\mathrm{ran}X_{11}}$ reduces S, $\overline{\mathrm{ran}Y_{11}}$ reduces T, T_{11} is unitarily equivalent to $S_{11}|_{\overline{\mathrm{ran}X_{11}}}$ and S_{11} is unitarily equivalent to $T_{11}|_{\overline{\mathrm{ran}Y_{11}}}$. Thus, S_{11} and T_{11} are unitarily equivalent to direct summands of each other. Hence, [5], S_{11} and T_{11} are unitarily equivalent.

By hypothesis, $\mu_{S_c} < \infty$. Since $S_cX_0 = X_0T_c$ and $T_cY_0 = Y_0S_c$, and X_0 and Y_0 have dense range, $\mu_{S_c} = \mu_{T_c} < \infty$; this, since $\mu_{S_{33}} \leqslant \mu_{S_c}$ and $\mu_{T_{33}} \leqslant \mu_{T_c}$, implies that both $\mu_{S_{33}}$ and $\mu_{T_{33}}$ are finite. Evidently, $S_{33}X_{33}Y_{33} = X_{33}Y_{33}S_{33}$ and $T_{33}Y_{33}X_{33} = Y_{33}X_{33}T_{33}$, where $X_{33}Y_{33}$ and $Y_{33}X_{33}$ have dense range. Applying Lemma 3.1 it follows that $X_{33}Y_{33}$ and $Y_{33}X_{33}$ are quasi-affinities; hence X_{33} and Y_{33} are quasi-affinities. But then X_0 and Y_0 are quasi-affinities; hence $S_c \sim T_c$. □

Theorem 3.2 extends a result on hyponormal contractions of Wu [11, Corollary 3.10], see also [4], to k-paranormal contractions. The following corollary extends [11, Corollary 3.11]. Recall that every isometry $V \in B(\mathscr{H})$ has a decomposition $V = V_u \oplus V_c$, where $V_c \in C_{10}$ is a unilateral shift.

COROLLARY 3.3. *Let $S \in P(k)$ be such that (its pure part) S_p has finite multiplicity. Then $S \sim V$ for some isometry $V \in B(\mathscr{H})$ if and only if S_n is unitarily equivalent to V_u and $S_p \sim V_c$.*

Proof. Since every isometry satisfies property (β), $S \sim V$ implies that $\sigma(S) = \sigma(V) = \mathbf{D}$. Consequently, S is a contraction. Decompose S into its normal and pure parts by $S = S_n \oplus S_p$; then $S_p \in C_{.0}$. Let $V = V_u \oplus V_c$. If $SX = XV$ and $VY = YS$, $X = [X_{ij}]_{i,j=1}^2$ and $Y = [Y_{ij}]_{i,j=1}^2$, then $S_cX_{21} = X_{21}V_u$ and $V_cY_{21} = Y_{21}S_n$. Clearly, $X_{21} = 0$. Applying the Putnam–Fuglede theorem to $V_cY_{21} = Y_{21}S_n$ it is seen that $\overline{\mathrm{ran}Y_{21}}$ reduces V_c and $V_c|_{\overline{\mathrm{ran}Y_{21}}}$ is unitary. Consequently, $Y_{21} = 0$, Y_{11} is injective and $V_uY_{11} = Y_{11}S_n$. Another application of the Putnam–Fuglede theorem to $V_uY_{11} = Y_{11}S_u$ now shows that $\overline{\mathrm{ran}Y_{11}}$ reduces V_u and S_n is unitarily equivalent to $V_u|_{\overline{\mathrm{ran}Y_{11}}}$. Hence S_n is unitary (and unitarily equivalent to V_n). Applying Theorem 3.2, $S_p \sim V_c$, and the proof is complete. □

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(Received September 21, 2009)

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