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SPECTRAL MEASURES OF JACOBI OPERATORS WITH RANDOM POTENTIALS

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Abstract. Let H_ω be a self-adjoint Jacobi operator with a potential sequence $\{\omega(n)\}_n$ of independently distributed random variables with continuous probability distributions and let μ_ϕ^ω be the corresponding spectral measure generated by H_ω and the vector ϕ. We consider sets $\mathscr{A}(\omega)$ which depend on ω, but are independent of two consecutive given entries of the secuence ω, and prove that $\mu_\phi^\omega(\mathscr{A}(\omega)) = 0$ for almost every ω. This result is applied to show equivalence relations between spectral measures for random Jacobi matrices and to study the interplay of the eigenvalues of these matrices and their submatrices.

1. Introduction

Let H_0 be a Jacobi operator with zero main diagonal in a Hilbert space with an orthonormal basis $\{\delta_k\}_{k\in I}$, where I is a finite or countable index set. We consider the random self-adjoint operator given by

$$H_\omega = H_0 + \sum_{n\in I} \omega(n)\langle\delta_n,\cdot\rangle\,\delta_n\,,$$

where $\omega(n)$ are independent random variables with continuous probability distributions, that is, distributions such that the probability of any single point is zero. Note that a continuous probability distribution may be singular, i. e., there may be sets of zero Lebesgue measure with positive probability.

It is a well known fact regarding Schrödinger and Jacobi operators with ergodic potentials, that the probability of a given $\lambda\in\mathbb{R}$ being an eigenvalue is zero [3, 4, 12]. Here we present an extended result (Theorem 3.1) for H_ω, which is not necessarily ergodic, when the point λ depends on the sequence ω except for two entries $\omega(n_0)$ and $\omega(n_0+1)$, $n_0\in I$. This is complemented by Theorem 3.2 when λ is a measurable function of ω. Since λ is allowed to depend on ω, it is possible to apply these results to obtain information about the spectral behavior of the above mentioned operators.

As a first application, we study equivalence relations of spectral measures $\mu_n^\omega(\cdot) := \langle\delta_n, E_{H_\omega}(\cdot)\delta_n\rangle$, where E_{H_ω} is the family of spectral projections for H_ω given by the spectral theorem. By applying Theorems 3.1 and 3.2, we obtain equivalence of spectral

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measures for one-sided infinite random Jacobi matrices with continuous (could be singular) probability distributions, that is, $\mu_n^\omega \sim \mu_m^\omega$ for a. e. ω and any n,m in I. When these distributions are not only continuous but absolutely continuous, the equivalence of spectral measures was proven in [9] by different methods. For spectral measures of double-sided infinite Jacobi operators, the equivalence relations $\mu_k^\omega + \mu_l^\omega \sim \mu_m^\omega + \mu_n^\omega$ for a. e. ω and any $k,l,m,n \in I$ are established.

A second application concerns the interplay of the eigenvalues of Jacobi matrices and their submatrices. This has been studied in the context of orthogonal polynomials, in particular, there are results describing the behavior of eigenvalues of submatrices near a neighborhood of an eigenvalue of the whole matrix [5] [14, Sec. 1.2.11]. Here we show, as a consequence of Theorems 3.1 and 3.2, that eigenvalues of a Jacobi matrix do not coincide with eigenvalues, moments or entries of its submatrices almost surely. Thus, it is not only true that one point is an eigenvalue of H_ω for at most a set of zero measure as mentioned above, but an arbitrary eigenvalue of any submatrix (which depends on ω) is not an eigenvalue of H_ω almost surely.

This work is organized as follows. In Section 2 the notation is introduced along with some preliminary concepts. Section 3 is devoted to the proof of the main results (Theorems 3.1 and 3.2), where measurability conditions play a key role. In Section 4, we apply the results of the previous section to study equivalence relations between spectral measures and the possible coincidence of eigenvalues with sets of real numbers associated with submatrices.

2. Preliminaries

In this section we fix the notation and introduce the setting of the model. Mainly we use a notation similar to that in [15]. Fix n_1, n_2 in $\mathbb{Z} \cup \{+\infty\} \cup \{-\infty\}$ define an interval I of $\mathbb{Z}$ as follows

$$I := \{n \in \mathbb{Z} : n_1 < n < n_2\} .$$

The linear space of M-valued sequences $\{\xi(n)\}_{n \in I}$ will be denoted by $l(I,M)$, that is,

$$l(I,M) := \{\xi : I \to M\} .$$

If M is itself a Hilbert space, then one has a Hilbert space

$$l^2(I,M) := \{\xi \in l(I,M) : \sum_{n \in I} \|\xi(n)\|_M^2 < \infty\} ,$$

with inner product given by

$$\langle \xi, \eta \rangle := \sum_{n \in I} \langle \xi(n), \eta(n) \rangle_M .$$

Now, let us introduce a measure in $l(I,\mathbb{R})$ as follows. Let $\{p_n\}_{n \in I}$ be a sequence of arbitrary probability measures on $\mathbb{R}$ and consider the product measure $\mathbb{P} = \times_{n \in I} p_n$ defined on the product σ-algebra $\mathscr{F}$ of $l(I,\mathbb{R})$ generated by the cylinder sets, i. e, by sets of the form $\{\omega : \omega(i_1) \in A_1, \dots, \omega(i_n) \in A_n\}$ for $i_1, \dots, i_n \in I$, where $A_1, \dots, A_n$ are Borel sets in $\mathbb{R}$. We have thus constructed a measure space $\Omega = (l(I,\mathbb{R}), \mathscr{F}, \mathbb{P})$.

Consider $a \in l(I,\mathbb{R})$ with $a(n) > 0$ for all $n \in I$, and let $\omega \in \Omega$. Define, for $\xi \in l^2(I,\mathbb{C})$,

$$(H\xi)(n) := \begin{cases} \omega(n)\xi(n) + a(n)\xi(n+1) & n = n_1 + 1, \quad n_1 > -\infty, \\ (\tau\xi)(n) & n_1 + 1 < n < n_2 - 1, \\ a(n-1)\xi(n-1) + \omega(n)\xi(n) & n = n_2 - 1, \quad n_2 < +\infty, \end{cases} \tag{2.1}$$

where

$$(\tau\xi)(n) := a(n-1)\xi(n-1) + \omega(n)\xi(n) + a(n)\xi(n+1). \tag{2.2}$$

In the Hilbert space $l^2(I,\mathbb{C})$, one can uniquely associate a closed symmetric operator with H (see [1, Sec. 47]) which we shall denote by H_ω to emphasize the dependence on the sequence $\omega \in \Omega$. The operator H_ω is a Jacobi operator having a Jacobi matrix as its matrix representation with respect to the canonical basis $\{\delta_k\}_{k\in I}$ in $l^2(I,\mathbb{C})$, where

$$\delta_k(n) = \begin{cases} 0 & n \neq k \\ 1 & n = k. \end{cases} \tag{2.3}$$

H_ω is defined so that $\{\delta_k\}_{k\in I} \subset \operatorname{dom}(H_\omega)$.

As in the case of differential equations, one defines the Wronskian associated with the difference equation (2.1) by

$$W_n(\xi,\eta) := a(n)((\xi(n)\eta(n+1) - \eta(n)\xi(n+1)), \qquad n_1 < n < n_2 - 1.$$

It turns out that, for all n, m such that $n_1 < m < n < n_2 - 1$, the Green formula (see [15, Eq. 1.20]) holds

$$\sum_{k=m+1}^{n} (\xi(\tau\eta) - (\tau\xi)\eta)(k) = W_n(\xi,\eta) - W_m(\xi,\eta). \tag{2.4}$$

Besides this formula, the Wronskian shares some properties with the Wronskian of the theory of differential equations, in particular, if $W_n(\xi,\eta) = 0$ for all n in a subinterval of I, then ξ and η are linearly dependent in that subinterval. This is verified directly from the definition of the Wronskian.

Now, assume that $I = \mathbb{Z}$ and consider the second-order difference equation

$$(\tau u)(n) = zu(n), \qquad n \in \mathbb{Z}, z \in \mathbb{C}, \tag{2.5}$$

where τ is defined in (2.2). Fix the numbers $m \in \mathbb{Z}$ and $z \in \mathbb{C}$. In the space $l(\mathbb{Z},\mathbb{C})$, take the sequences

$$c_m(z) = \{c_m(z,n)\}_{n\in\mathbb{Z}}, \qquad s_m(z) = \{s_m(z,n)\}_{n\in\mathbb{Z}}$$

which are solutions of (2.5) and satisfy the following initial conditions:

$$c_m(z, m-1) = 1, \qquad c_m(z,m) = 0, \tag{2.6}$$

$$s_m(z, m-1) = 0, \qquad s_m(z,m) = 1. \tag{2.7}$$

Because of the linear independence of $c_m(z), s_m(z)$, they constitute a fundamental system of solutions of (2.5). Note that for any $n \in \mathbb{Z}$, $c_m(z,n), s_m(z,n)$ are polynomials of z. The roots of these polynomials are measurable functions of ω.

By means of the polynomials defined above we state the following result [15], [7, Prop. A.1].

LEMMA 2.1. *Consider the operator H_ω with fixed $\omega \in \Omega$. For any fixed $n \in I$, we have*

$$\delta_n = \begin{cases} s_{n_1+1}(H_\omega, n)\delta_{n_1+1} & -\infty < n_1 \\ c_{n_2}(H_\omega, n)\delta_{n_2-1} & n_2 < +\infty \\ s_{m+1}(H_\omega, n)\delta_{m+1} + c_{m+1}(H_\omega, n)\delta_m & -\infty = n_1, n_2 = +\infty \quad \forall m \in I, \end{cases} \tag{2.8}$$

where the polynomials $c_m(z,n), s_m(z,n)$ have been evaluated at the operator H_ω.

The symmetric operator H_ω is not always self-adjoint. However, in this work, we always consider H_ω to be a self-adjoint operator for each $\omega \in \Omega$. If one of the numbers n_1, n_2 is not finite, conditions for self-adjointness should be assumed. For instance, when both n_1 and n_2 are infinite, the so called Carleman criterion (cf. [2, Chap. 7 Sec. 3.2])

$$\sum_{n \in \mathbb{N}} \frac{1}{\max\{a(-n-1), a(n-1)\}} = \infty \tag{2.9}$$

entails self-adjointness of H_ω.

Notice that the operator H_ω can be written as

$$H_\omega = H_0 + \sum_{n \in I} \omega(n) \langle \delta_n, \cdot \rangle \delta_n,$$

where H_0 is a self-adjoint Jacobi operator with zero main diagonal.

For the self-adjoint operator H_ω, we have the following remarks.

REMARK 1. For every pair ξ, η in the domain of the self-adjoint operator H_ω,

$$\lim_{n \to \infty} W_n(\xi, \eta) = 0$$

(see [15, Sec. 2.6]).

REMARK 2. From (2.8), it follows that a self-adjoint Jacobi operator, whose corresponding matrix is finite or one-sided infinite, has simple spectrum (see [1, Sec. 69]). Moreover, the last equation in (2.8) shows that, when both n_1, n_2 are infinite, two consecutive elements of the canonical basis constitute a generating basis for H_ω (see [1, Sec. 72]).

Let μ_ϕ^ω be the spectral measure for H_ω and the vector ϕ, viz., the unique Borel measure on $\mathbb{R}$ such that

$$\langle \phi, f(H_\omega)\phi \rangle = \int_{\mathbb{R}} f(\lambda) d\mu_\phi^\omega(\lambda)$$

for any bounded function f. Equivalently,

$$\mu_\phi^\omega(\cdot) = \langle \phi, E_{H_\omega}(\cdot)\phi \rangle \,, \tag{2.10}$$

where E_{H_ω} is the family of spectral projections for H_ω given by the spectral theorem.

NOTATION. Below, we shall repeatedly deal with $\mu_{\delta_n}^\omega$ (see (2.3)) and we denote it by μ_n^ω for short.

DEFINITION 1. Given two measures ν and μ with the same collection of measurable sets, we say that μ is absolutely continuous with respect to ν, denoted $\mu \prec \nu$, if for every measurable Δ such that $\nu(\Delta) = 0$, it follows that $\mu(\Delta) = 0$. Also, ν and μ are said to be equivalent, denoted $\nu \sim \mu$, if they are mutually absolutely continuous, that is, if they have the same zero sets.

Suppose that at least one of the numbers n_1, n_2 is finite. By inserting (2.8) into (2.10), one obtains, for an arbitrary Borel set $\Delta \subset \mathbb{R}$ [7, Cor. A.2],

$$\mu_n^\omega(\Delta) = \begin{cases} \int_\Delta s_{n_1+1}^2(\lambda, n) d\mu_{n_1+1}^\omega(\lambda) & n_1 > -\infty \\ \int_\Delta c_{n_2}^2(\lambda, n) d\mu_{n_2-1}^\omega(\lambda) & n_2 < +\infty . \end{cases} \tag{2.11}$$

When both numbers n_1, n_2 are infinite, let us define, for any Borel $\Delta \subset \mathbb{R}$ and $n \in \mathbb{Z}$, the matrix

$$\boldsymbol{\mu}_n(\Delta) := \begin{pmatrix} \mu_n^\omega(\Delta) & \langle E_{H_\omega}(\Delta)\delta_n, \delta_{n+1} \rangle \\ \langle E_{H_\omega}(\Delta)\delta_{n+1}, \delta_n \rangle & \mu_{n+1}^\omega(\Delta) \end{pmatrix} .$$

The third equation in (2.8) implies

$$\mu_n^\omega(\Delta) = \int_\Delta \left\langle d\boldsymbol{\mu}_m(\lambda) \begin{pmatrix} c_{m+1}(\lambda, n) \\ s_{m+1}(\lambda, n) \end{pmatrix}, \begin{pmatrix} c_{m+1}(\lambda, n) \\ s_{m+1}(\lambda, n) \end{pmatrix} \right\rangle_{\mathbb{C}^2} . \tag{2.12}$$

There exists a matrix (see comment after [15, Lem. B.13])

$$\boldsymbol{R}_m(\lambda) = \begin{pmatrix} a_m(\lambda) & b_m(\lambda) \\ b_m(\lambda) & 1 - a_m(\lambda) \end{pmatrix}$$

such that

$$\boldsymbol{\mu_m}(\Delta) = \int_\Delta \boldsymbol{R}_m(\lambda) d(\mu_m^\omega + \mu_{m+1}^\omega)(\lambda) . \tag{2.13}$$

REMARK 3. Notice that from Remark 2, (2.13), and [1, Sec. 72] it follows that $\mu_k^\omega + \mu_{k+1}^\omega \sim \mu_l^\omega + \mu_{l+1}^\omega$ for any $k, l \in \mathbb{Z}$.

3. Main results

Under the assumption that H_ω is ergodic, it is well known that a fixed $r \in \mathbb{R}$ is an eigenvalue of H_ω with probability zero [12, Thm.2.12], [3, Prop.V.2.8] [4, Thm. 9.5]. In the case of H_ω considered here, the following result holds.

THEOREM 3.1. *Assume that I contains at least three integers and suppose n_0, n_0+1 are in I. Let the measures p_{n_0}, p_{n_0+1} be continuous (a continuous measure evaluated at a single point of $\mathbb{R}$ equals zero). Consider a finite or infinite sequence of real functions $\{r\}_k$ ($r_k : \Omega \to \mathbb{R}$), not necessarily measurable, such that, for $\omega, \widetilde{\omega} \in \Omega$,*

$$r_k(\omega) = r_k(\widetilde{\omega}) \tag{3.1}$$

whenever $\omega(n) = \widetilde{\omega}(n)$ for all $n \in I \setminus \{n_0, n_0+1\}$. For any non-zero element ϕ in the Hilbert space $l^2(I, \mathbb{C})$, either

$$\mu_\phi^\omega(\cup_k r_k(\omega)) = 0 \tag{3.2}$$

for $\mathbb{P}$ a. e. ω, or the set of ω where (3.2) holds is not measurable.

Proof. We consider two cases:

A) One of the numbers n_1, n_2 is finite.

Without loss of generality let us assume that n_1 is finite. By Remark 2, δ_{n_1+1} is a cyclic vector of H_ω for any $\omega \in \Omega$.

Fix an element r_{k_0} of the sequence $\{r_k\}_k$. Define the set

$$\mathscr{Q}^{r_{k_0}} := \{\omega \in \Omega : \mu_{n_1+1}^\omega(\{r_{k_0}(\omega)\}) > 0\}.$$

Let us construct a partition of $\mathscr{Q}^{r_{k_0}}$. If $\omega_0 \in \mathscr{Q}^{r_{k_0}}$, then $r_{k_0}(\omega_0)$ is an eigenvalue of H_{ω_0} with corresponding eigenvector $\psi = E_{H_{\omega_0}}(\{r_{k_0}(\omega_0)\})\delta_{n_1+1}$. Due to the cyclicity of δ_{n_1+1}, the converse is true, that is, if we have an eigenvalue r of H_{ω_0}, then $\mu_{n_1+1}^{\omega_0}(\{r\}) > 0$.

Analogously, if $\omega_0 + t\delta_{n_0} \in \mathscr{Q}^{r_{k_0}}$ for some $t \in \mathbb{R} \setminus \{0\}$, there is a non-zero element ξ of the domain of $H_{\omega_0 + t\delta_{n_0}}$ (which coincides with the domain of H_{ω_0}) such that

$$H_{\omega_0 + t\delta_{n_0}}\xi = r_{k_0}(\omega_0)\xi, \tag{3.3}$$

where (3.1) is used. From (2.1), it is clear that both ξ and ψ satisfy the difference equation

$$(\tau u)(n) = r_{k_0}(\omega_0)u(n)$$

for all n such that $n_1 + 1 < n < n_2 - 1$ and $n \neq n_0$. So, by (2.4), $W_n(\xi, \psi)$ is constant for all n such that $n_0 \leqslant n < n_2$. Now, when n_2 is finite, both ξ and ψ satisfy the difference equation (see (2.1))

$$a(n-1)u(n-1) + \omega(n)u(n) = r_{k_0}(\omega_0)u(n), \quad \text{for } n = n_2 - 1.$$

This implies that $W_{n_2-2}((\xi, \psi)) = 0$, so the constant $W_n(\xi, \psi)$, for all n such that $n_0 \leqslant n < n_2 - 1$, is in fact zero. If n_2 is infinite, then, from what was said in Section 2 (see Remark 1) one concludes that $W_n(\xi,\psi)=0$ for all $n \geqslant n_0$. Therefore, in both cases, n_2 finite or infinite, there exists $c \in \mathbb{C}$ such that $\xi(n) = c\psi(n)$ for all n such that $n_0 \leqslant n < n_2 - 1$. This implies that ξ cannot satisfy (3.3) for $t \neq 0$ when ψ is an eigenvector with $\psi(n_0) \neq 0$. If $\psi(n_0) = 0$, then one may repeat the reasoning above for n_0+1, since, in this case, it follows from (2.1) that $\psi(n_0+1) \neq 0$. Thus we assert that either

$$\mu_{n_1+1}^{\omega_0+t\delta_{n_0}}(\{r_{k_0}(\omega_0)\}) = 0, \qquad \forall t \in \mathbb{R}\setminus\{0\}, \tag{3.4}$$

or

$$\mu_{n_1+1}^{\omega_0+s\delta_{n_0+1}}(\{r_{k_0}(\omega_0)\}) = 0, \qquad \forall s \in \mathbb{R}\setminus\{0\}, \tag{3.5}$$

for any $\omega_0 \in \mathscr{Q}^{r_{k_0}}$. Let $\mathscr{Q}_1$ be the set of $\omega_0 \in \mathscr{Q}^{r_{k_0}}$ such that (3.4) holds, and $\mathscr{Q}_2 = \mathscr{Q}^{r_{k_0}} \setminus \mathscr{Q}_1$. Thus we have the partition $\mathscr{Q}^{r_{k_0}} = \mathscr{Q}_1 \cup \mathscr{Q}_2$. Notice that, if $\psi(n_0) = 0$, then ψ is an eigenvector of $H_{\omega_0+t\delta_{n_0}}$ for all $t \in \mathbb{R}$. Thus, for any $\omega_0 \in \mathscr{Q}_2$,

$$\mu_{n_1+1}^{\omega_0+t\delta_{n_0}}(\{r_{k_0}(\omega_0)\}) > 0 \qquad \forall t \in \mathbb{R}. \tag{3.6}$$

Let us denote by $\chi_{\mathscr{A}}$ the characteristic function of $\mathscr{A}$, that is,

$$\chi_{\mathscr{A}}(\omega) = \begin{cases} 1 & \text{if } \omega \in \mathscr{A} \\ 0 & \text{if } \omega \notin \mathscr{A}\,. \end{cases} \tag{3.7}$$

Since $\mu_{n_1+1}^{\omega}(\{r\})$ is a measurable function of $\omega \in \Omega$ for any fixed $r \in \mathbb{R}$ (see [3, Sec. 5.3]), we know that $\mu_{n_1+1}^{\omega+t\delta_{n_0}+s\delta_{n_0+1}}(\{r\})$ is a measurable function of $(t,s) \in \mathbb{R}^2$ (see [13, Thm. 7.5]) for any fixed $\omega \in \Omega$. Therefore, using (3.1), one establishes that

$$\chi_{\mathscr{Q}^{r_{k_0}}}^{-1}(\{1\}) = \{(t,s) \in \mathbb{R}^2 : \mu_{n_1+1}^{\omega+t\delta_{n_0}+s\delta_{n_0+1}}(\{r_{k_0}(\omega)\}) > 0\}$$

is measurable. Hence

$$(t,s) \to \chi_{\mathscr{Q}^{r_{k_0}}}(\omega + t\delta_{n_0} + s\delta_{n_0+1})$$

is a measurable function for any fixed $\omega \in \Omega$. Thus, by Fubini

$$\begin{aligned} &\int_{\mathbb{R}^2} \chi_{\mathscr{Q}^{r_{k_0}}}(\omega + t\delta_{n_0} + s\delta_{n_0+1}) d(p_{n_0} \times p_{n_0+1})(t,s) \\ &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \chi_{\mathscr{Q}^{r_{k_0}}}(\omega + t\delta_{n_0} + s\delta_{n_0+1}) dp_{n_0}(t) \right] dp_{n_0+1}(s)\,. \end{aligned}$$

The following equality holds

$$\int_{\mathbb{R}} \chi_{\mathscr{Q}^{r_{k_0}}}(\omega + t\delta_{n_0} + s\delta_{n_0+1}) dp_{n_0}(t) = \chi_{\mathscr{Q}_2}(\omega + s\delta_{n_0+1})\,. \tag{3.8}$$

When $\omega + s\delta_{n_0+1} \in \mathscr{Q}^{r_{k_0}}$, (3.8) is verified using (3.4), (3.6), $p_{n_0}(\mathbb{R}) = 1$ and the continuity of p_{n_0}. If $\omega + s\delta_{n_0+1} \notin \mathscr{Q}^{r_{k_0}}$, then either $\omega + t\delta_{n_0} + s\delta_{n_0+1} \notin \mathscr{Q}^{r_{k_0}}$ for every $t \in \mathbb{R}$ and (3.8) follows, or there exists $t_0 \in \mathbb{R}$ such that $\omega + t_0\delta_{n_0} + s\delta_{n_0+1} \in \mathscr{Q}^{r_{k_0}}$.

If $\omega + t_0\delta_{n_0} + s\delta_{n_0+1} \in \mathscr{Q}_1$, (3.8) follows from (3.4) and continuity of p_{n_0}. The case $\omega + t_0\delta_{n_0} + s\delta_{n_0+1} \in \mathscr{Q}_2$ is not possible since (3.6) would imply $\omega + s\delta_{n_0+1} \in \mathscr{Q}^{r_{k_0}}$.

Notice that $\mathscr{Q}_2$ does not need to be measurable and nevertheless the equality (3.8) shows that $\chi_{\mathscr{Q}_2}(\omega + s\delta_{n_0+1})$ is a measurable function of s. Hence

$$\int_{\mathbb{R}^2} \chi_{\mathscr{Q}^{r_{k_0}}}(\omega + t\delta_{n_0} + s\delta_{n_0+1}) d(p_{n_0} \times p_{n_0+1})(t,s) = \int_{\mathbb{R}} \chi_{\mathscr{Q}_2}(\omega + s\delta_{n_0+1}) dp_{n_0+1}(s) = 0$$

since the support of $\chi_{\mathscr{Q}_2}(\omega + s\delta_{n_0+1})$ is only one point as a consequence of (3.5). So we arrive at the conclusion that, for any fixed $\omega \in \Omega$,

$$\mu_{n_1+1}^{\omega + t\delta_{n_0} + s\delta_{n_0+1}}(\{r_{k_0}(\omega)\}) = 0$$

for $p_{n_0} \times p_{n_0+1}$-a. e. (t,s). Note that, since

$$\sum_k \mu_{n_1+1}^{\omega}(\{r_k(\omega)\}) \geqslant \mu_{n_1+1}^{\omega}(\cup_k r_k(\omega)),$$

we actually have that

$$\mu_{n_1+1}^{\omega + t\delta_{n_0} + s\delta_{n_0+1}}(\cup_k r_k(\omega)) = 0 \tag{3.9}$$

for any fixed $\omega \in \Omega$, for $p_{n_0} \times p_{n_0+1}$-a. e. (t,s).

Now, let $Q := \{\omega \in \Omega : \mu_{n_1+1}^{\omega}(\cup_k r_k(\omega)) > 0\}$ and assume that it is measurable. Then

$$\begin{aligned}\mathbb{P}(Q) &= \int_{\Omega} \chi_Q(\omega) d\mathbb{P}(\omega)\\ &= \int_{\mathbb{R}^{I\setminus\{n_0,n_0+1\}}} \left[\int_{\mathbb{R}^2} \chi_Q(\widetilde{\omega} + t\delta_{n_0} + s\delta_{n_0+1}) d(p_{n_0} \times p_{n_0+1})(t,s)\right] \underset{n\in I\setminus\{n_0,n_0+1\}}{\times} dp_n(\widetilde{\omega}),\end{aligned}$$

where $\omega = \widetilde{\omega} + t\delta_{n_0} + s\delta_{n_0+1}$ and we have used Fubini's theorem. From (3.9) and the definition of Q, we have

$$\chi_Q(\widetilde{\omega} + t\delta_{n_0} + s\delta_{n_0+1}) = 0$$

for $p_{n_0} \times p_{n_0+1}$ a. e. (t,s). Therefore $\mathbb{P}(Q) = 0$.

Thus we have proven (3.2) with $\phi = \delta_{n_1+1}$. To prove it for an arbitrary $\phi \in l^2(I,\mathbb{C})$ observe that $\mu_\phi^\omega \prec \mu_{n_1+1}^\omega$ [1, Sec. 70 Thm. 1].

B) The numbers n_1, n_2 are infinite.

It follows from [1, Sec. 72] and (2.13) (cf. [15, Eq. 2.141]) that r is an eigenvalue of H_ω if and only if $(\mu_m^\omega + \mu_{m+1}^\omega)(\{r\}) > 0$ for any fixed $m \in \mathbb{Z}$. Thus, one can repeat the proof for A) with $\mu_m^\omega + \mu_{m+1}^\omega$ instead of $\mu_{n_1+1}^\omega$. Hence one proves that either

$$(\mu_m^\omega + \mu_{m+1}^\omega)(\cup_k r_k(\omega)) = 0$$

for $\mathbb{P}$ a. e. ω, or the set of ω where the equality above holds is not measurable. The proof is then completed by recalling that, for all $\phi \in l^2(\mathbb{Z},\mathbb{C})$, $\mu_\phi^\omega \prec \mu_m^\omega + \mu_{m+1}^\omega$ (this follows as in the first part of the proof of [1, Sec. 70 Thm. 1] using [1, Sec. 72]). $\square$

THEOREM 3.2. *Let* $\{r_k\}_k$ *be a finite or infinite sequence of measurable functions* ($r_k : \Omega \to \mathbb{R}$). *The function* $h : \Omega \to \mathbb{R}$ *given by*

$$h(\omega) := \mu_\phi^\omega(\cup_k r_k(\omega))$$

is measurable.

Proof. Consider a simple function $s(\omega) = \sum_{j=1}^N \alpha_j \chi_{A_j}(\omega)$, where $\chi_{A_j}(\omega)$ is the characteristic function of A_j (see (3.7)). Note that $A_j = s^{-1}(\{\alpha_j\})$ and the sets $\{A_j\}_{j=1}^N$ form a partition of Ω.

Let $V \subset \mathbb{R}$ be an open set. The set

$$A := \{\omega \in \Omega : \langle \phi, E_{H_\omega}(\{s(\omega)\})\phi \rangle \in V\}$$

is measurable. Indeed,

$$A = \cup_{j=1}^N \left[A_j \cap \{\omega \in \Omega : \left\langle \phi, E_{H_\omega}(\{\alpha_j\})\phi \right\rangle \in V\}\right]$$

and each $\{\omega \in \Omega : \left\langle \phi, E_{H_\omega}(\{\alpha_j\})\phi \right\rangle \in V\}$ is measurable (cf. the commentary after [3, Prop. V.3.1]). Thus, the function $\mu_\phi^\omega(s(\omega))$ is measurable. We approximate the measurable function $r_1(\omega)$ by simple functions to obtain the assertion of the theorem for $r_1(\omega)$.

Now, suppose that

$$h_m(\omega) := \mu_\phi^\omega(\cup_{k=1}^m r_k(\omega))$$

is a measurable function. Clearly,

$$h_{m+1}(\omega) = \begin{cases} h_m(\omega) & r_{m+1}(\omega) \in \cup_{k=1}^m r_k(\omega) \\ h_m(\omega) + \mu_\phi^\omega(r_{m+1}(\omega)) & \text{otherwise.} \end{cases}$$

So from the measurability of $h_m(\omega)$ and $\mu_\phi^\omega(r_{m+1}(\omega))$, the measurability of $h_{m+1}(\omega)$ follows. By induction we prove the assertion of the theorem for any finite sequence of measurable functions $\{r_k\}_k$. The case of an infinite sequence is proven by taking a pointwise limit w.r.t. $\omega \in \Omega$ of $h_m(\omega)$ when m tends to ∞. $\square$

Let $\sigma_p(H_\omega)$ denote the set of eigenvalues of the operator H_ω.

COROLLARY 3.1. *If* H_ω *is measurable* [3, Def. V.3.1], *then* $h(\omega) := \mu_\phi^\omega(\sigma_p(H_\omega))$ *is a measurable function.*

Proof. Since the operator H_ω is measurable, we can apply a result of [8] and give a measurable enumeration of the points in $\sigma_p(H_\omega)$. Then the assertion follows from Theorem 3.2. $\square$

4. Applications to spectral theory

We begin this section by stating an elementary result.

LEMMA 4.1. *Let μ be a measure on X and let*

$$\gamma(\Delta) := \int_{\Delta} f(\lambda) d\mu(\lambda),$$

where f is a non-negative measurable function. Then

$$\gamma \sim \mu \qquad \Longleftrightarrow \qquad \mu(\{\lambda \in X : f(\lambda) = 0\}) = 0.$$

Proof. ($\Leftarrow$) γ is absolutely continuous w.r.t μ by definition. Now, assume $\gamma(\Delta) = 0$, then $f(\lambda) = 0$ for μ-a. e. λ on Δ and

$$\mu(\Delta) = \mu(\Delta \setminus \{\lambda \in X : f(\lambda) = 0\}) + \mu(\{\lambda \in X : f(\lambda) = 0\}) = 0.$$

($\Rightarrow$) If $\mu(\{\lambda \in X : f(\lambda) = 0\}) > 0$, then $\gamma(\{\lambda \in X : f(\lambda) = 0\}) = 0$, so the measures are not equivalent. □

THEOREM 4.1. *Assume that at least one of the numbers n_1, n_2 is finite, I contains at least three integers, and the measures of the sequence $\{p_n\}_{n\in I}$ are continuous. For any fixed $n, m \in I$ and $\mathbb{P}$-a. e. ω,*

$$\mu_n^\omega \sim \mu_m^\omega .$$

Proof. Let $n_1 > -\infty$. Under this assumption we proceed stepwise. Firstly, we show that $\mu_n^\omega \sim \mu_{n_1+1}^\omega$ for $n_1 < n < n_2 - 1$. Secondly, it is proven that $\mu_{n_2-2}^\omega \sim \mu_{n_2-1}^\omega$ when n_2 is finite.

In view of the first equation in (2.11), $\mu_n^\omega \sim \mu_{n_1+1}^\omega$ if and only if (see Lemma 4.1)

$$\mu_{n_1+1}^\omega(\{\lambda : s_{n_1+1}(\lambda, n) = 0\}) = 0,$$

for $\mathbb{P}$-a. e. ω. Due to the initial conditions (2.6) and (2.7), it is straightforward to verify that the polynomial $s_{n_1+1}(\lambda, n)$ is completely determined by the sequences $\{a(k)\}_{k=n_1+1}^{n-1}$ and $\{\omega(k)\}_{k=n_1+1}^{n-1}$. Now, the finite sequence $\{\lambda_k(\omega)\}_k$ of zeros of $s_{n_1+1}(\lambda, n)$ satisfies the conditions imposed on the sequence $\{r_k(\omega)\}_k$ in the statement of Theorem 3.1 when $n_0 \geqslant n$. By applying Theorem 3.1 and 3.2, one completes the first step. Now, suppose that n_2 is finite, and use the second equation in (2.11) to express $\mu_{n_2-2}^\omega$. The polynomial involved here, $c_{n_2}(\lambda, n_2 - 2)$, is completely determined by $a(n_2 - 2)$ and $\omega(n_2 - 1)$. The only root of this polynomial, satisfies the conditions imposed on the sequence $\{r_k(\omega)\}_k$ in Theorem 3.1 taking $n_0 < n_2 - 2$.

The statement of the theorem is completely proven after noticing that, when n_1 is not finite, one repeats the reasoning above, with n_1, n_2, $s_{n_1+1}(\lambda, n)$, $c_{n_2}(\lambda, n_2 - 2)$ replaced by n_2, n_1, $c_{n_2}(\lambda, n)$, $s_{n_1+1}(\lambda, n_1 + 2)$, respectively. □

REMARK 4. Theorem 4.1 is proven in [9] for the case of absolutely continuous probability distributions in a more general setting. Our approach is different. In particular we do not need Poltoratskii's theorem used in [9].

REMARK 5. When I is unbounded and $n_1 > -\infty$ ($n_2 < +\infty$), the assertion of Theorem 4.1 remains true if we require only that there is $n^* \in I$ such that p_n is continuous for $n > n^*$ ($n < n^*$). If I is bounded, it is sufficient to require that three consecutive measures in the finite sequence $\{p_n\}_{n\in I}$ are continuous. This refinement of Theorem 4.1 follows directly from its proof.

REMARK 6. One may construct self-adjoint Jacobi operators for which $\mu^{\omega}_{n_1+1} \not\sim \mu^{\omega}_{n_1+2n}$ for all $n \in \mathbb{N}$ and fixed ω. Indeed, as mentioned in [5, Example 1] for n_1 finite and n_2 infinite, there are self-adjoint Jacobi matrices such that $\mu^{\omega}_{n_1+1}(\{0\}) \neq 0$ and $s_{n_1+1}(0, n_1 + 2n) = 0$. On the other hand, there exist Jacobi operators for which $\mu^{\omega}_n \sim \mu^{\omega}_m$ when n and m are sufficiently big. This is the case of the self-adjoint Jacobi operator studied in [11] (see the proof of Corollary 5.2 in [11]).

We now turn to the case, when neither of the numbers n_1, n_2 is finite. Observe that by inserting (2.13) into (2.12) one has

$$\mu^{\omega}_n(\Delta) = \int_{\Delta} g_{(m,n)}(\lambda)\, d(\mu^{\omega}_m + \mu^{\omega}_{m+1})(\lambda)\,, \tag{4.1}$$

where

$$g_{(m,n)}(\lambda) := \left\langle \boldsymbol{R}_m(\lambda) \begin{pmatrix} c_{m+1}(\lambda, n) \\ s_{m+1}(\lambda, n) \end{pmatrix}, \begin{pmatrix} c_{m+1}(\lambda, n) \\ s_{m+1}(\lambda, n) \end{pmatrix} \right\rangle_{\mathbb{C}^2} \tag{4.2}$$

THEOREM 4.2. *Assume that neither of the numbers n_1, n_2 is finite and there is $n^* \in I$ such that p_n is continuous for every n either greater or less than n^*. For any fixed $k, l, m, n \in \mathbb{Z}$ and $\mathbb{P}$-a. e. ω,*

$$\mu^{\omega}_k + \mu^{\omega}_l \sim \mu^{\omega}_m + \mu^{\omega}_n\,.$$

Proof. It follows from (4.1) that

$$(\mu^{\omega}_m + \mu^{\omega}_n)(\Delta) = \int_{\Delta} \left(g_{(m,m)}(\lambda) + g_{(m,n)}(\lambda)\right) d(\mu^{\omega}_m + \mu^{\omega}_{m+1})(\lambda)\,. \tag{4.3}$$

Let us show that $\mu^{\omega}_m + \mu^{\omega}_n \sim \mu^{\omega}_m + \mu^{\omega}_{m+1}$ for $\mathbb{P}$-a. e. ω. Due to (4.3) and Lemma 4.1, this will be done if one proves that

$$(\mu^{\omega}_m + \mu^{\omega}_{m+1})(\mathscr{B}) = 0 \quad \text{for } \mathbb{P}\text{-}a.e.\omega\,,$$

where $\mathscr{B} := \{\lambda : g_{(m,m)}(\lambda) = g_{(m,n)}(\lambda) = 0\}$.

Observing that $g_{(m,n)}(\lambda) = 0$ implies

$$\boldsymbol{R}_m(\lambda) \begin{pmatrix} c_{m+1}(\lambda, n) \\ s_{m+1}(\lambda, n) \end{pmatrix} = 0\,,$$

we obtain

$$b_m(\lambda)c_{m+1}(\lambda,n)s_{m+1}(\lambda,n) = -a_m(\lambda)c_{m+1}^2(\lambda,n) \tag{4.4}$$

for any $n,m \in \mathbb{Z}$. On the other hand, (4.2) and (2.6), (2.7) imply $g_{(m,m)}(\lambda) = a_m(\lambda)$. From (4.2) and (4.4), it follows that

$$g_{(m,n)}(\lambda) = s_{m+1}^2(\lambda,n) - a_m(\lambda)(s_{m+1}^2\left(\lambda,n) + c_{m+1}^2(\lambda,n)\right) .$$

So, assuming that $g_{(m,m)}(\lambda) = g_{(m,n)}(\lambda) = 0$ one obtains $g_{(m,n)}(\lambda) = s_{m+1}^2(\lambda,n)$. This implies that the set $\mathscr{B}$ is finite and its elements satisfy the conditions imposed on the elements of the sequence $\{r_k(\omega)\}_k$ used in Theorem 3.1. Theorems 3.1 and 3.2 yield that $\mu_m^\omega + \mu_n^\omega \sim \mu_m^\omega + \mu_{m+1}^\omega$. Now, the claim of the theorem follows from Remark 3. □

REMARK 7. In the case of absolutely continuous distributions, it is proven in [9] the stronger statement $\mu_m^\omega \sim \mu_n^\omega$ for $\mathbb{P}$-a. e. ω and any $m,n \in \mathbb{Z}$.

THEOREM 4.3. *Consider an interval $\widetilde{I}$ such that $\widetilde{I} \subset I \setminus \{m,m+1\}$, where $n_1 + 1 \leqslant m \leqslant n_2 - 2$. Assume that p_m, p_{m+1} are continuous measures. Let H_ω be the operator in $l^2(I,\mathbb{C})$ defined in Section 2 and $\widetilde{H}_\omega$ the operator defined analogously in $l^2(\widetilde{I},\mathbb{C})$. Then,*

$$\mathbb{P}(\{\omega \in \Omega : \sigma_p(H_\omega) \cap \sigma_p(\widetilde{H}_\omega) \neq \emptyset\}) = 0. \tag{4.5}$$

Proof. Observe that $\sigma_p(\widetilde{H}_\omega)$ does not depend on $\omega(m), \omega(m+1)$. Thus, it follows from Theorems 3.1, 3.2 and Corollary 3.1 that

$$\mu_\phi^\omega(\sigma_p(\widetilde{H}_\omega)) = 0 \tag{4.6}$$

for $\mathbb{P}$-a. e. ω.

If n_1 (or n_2) is finite, take $\phi = \delta_{n_1+1}$ ($\phi = \delta_{n_2-1}$), and, taking into account that $\lambda \in \sigma_p(H_\omega)$ if and only if $\mu_{n_1+1}^\omega(\{\lambda\}) > 0$ ($\mu_{n_2-1}^\omega(\{\lambda\}) > 0$), the theorem follows from (4.6).

Now, assume that both n_1, n_2 are infinite and choose consecutively $\phi = \delta_0$ and $\phi = \delta_1$. Then

$$(\mu_0^\omega + \mu_1^\omega)(\sigma_p(\widetilde{H}_\omega)) = 0$$

for $\mathbb{P}$-a. e. ω. Since

$$\sigma_p(H_\omega) = \{\lambda \in \mathbb{R} : (\mu_0^\omega + \mu_1^\omega)(\{\lambda\}) > 0\}$$

(see [15, Eq. 2.141]), the result follows. □

COROLLARY 4.1. *Take an arbitrary interval $\widetilde{I} \subsetneq I$ and assume that at least one of the numbers n_1, n_2 is infinite. Whenever the set $I \setminus \widetilde{I}$ has at least two elements, additionally require that p_n is continuous for two consecutive values of $n \in I \setminus \widetilde{I}$. Then* (4.5) *holds.*

Proof. Assume for example that $n_2 = +\infty$ and n_1 finite. Choose $\widetilde{I} = I \setminus \{n_1 + 1\}$. It is known that $\sigma_p(H_\omega) \cap \sigma_p(\widetilde{H}_\omega) = \emptyset$ for every ω [6]. If we take any other $\widetilde{I} \subsetneq I$, Theorems 3.1 and 3.2 can be applied. The other cases are handled analogously. □

REMARK 8. A more general situation could be considered along the same lines. Indeed, assume the same conditions as in Theorem 4.3 and let $r_k(\widetilde{H}_\omega)$ be a measurable real valued function of ω determined by $\widetilde{H}_\omega$. Then

$$\mathbb{P}(\{\omega \in \Omega : \sigma_p(H_\omega) \cap \cup_k r_k(\widetilde{H}_\omega) \neq \emptyset\}) = 0.$$

For example each r_k could be a matrix entry, a moment or any other quantity associated to $\widetilde{H}_\omega$.

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