ALGEBRAIC ELEMENTS AND INVARIANT SUBSPACES

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Abstract. We prove that if a completely non-unitary contraction T in $L(H)$ has a non-trivial algebraic element h, then T has a non-trivial invariant subspace.

Introduction

One of the most interesting open problems is the invariant subspace problem. The invariant subspace problem is the question whether the following statement is true or not:

Every bounded linear operator T on a separable Hilbert space H of dimension $\geqslant 2$ over $\mathbb{C}$ has a non-trivial invariant subspace.

We know that the invariant subspace problem is solved for all finite dimensional complex vector spaces of dimension at least 2. Thus, in this note, H denotes a separable Hilbert space whose dimension is infinite. Since it is enough to consider a contraction T, i.e., $\|T\| \leqslant 1$ on H, in this note, T denotes a contraction.

First, we focus on completely non-unitary contractions to use a property of a multiplicity-free operator of class C_0, and we consider *algebraic elements with respect to* a completely non-unitary contraction T introduced in [2].

If T is a contraction, then

(Case 1) T is a completely non-unitary contraction with a *non-trivial algebraic element*, or

(Case 2) T is a completely non-unitary contraction without a non-trivial algebraic element, that is, every non-zero element in H is *transcendental with respect to T*, or

(Case 3) T is not completely non-unitary.

In this note, we discuss the invariant subspace problem for operators of (Case 1). By using a classification of the invariant subspaces of a multiplicity-free operator of class C_0 ([1]), in Theorem 2.6, we prove that, for a completely non-unitary contraction T, if T has a non-trivial algebraic element h, then T has a non-trivial invariant subspace. By Theorem 2.6, we conclude that every C_0-operator has a non-trivial invariant subspace (Corollary 2.7).

It is well known that every contraction which is not completely non-unitary, has a non-trivial invariant subspace. Therefore, to answer the invariant subspace problem, it suffices to answer for (Case 2). The operators of (Case 2) are called *transcendental operators*.

We do not consider operators of (Case 2) in this note, and leave as a question;

QUESTION. Does every transcendental operator have a non-trivial invariant subspace?

1. Preliminaries and Notation

In this note, $\mathbb{C}$, $\overline{M}$ and $L(H)$ denote the set of complex numbers, the (norm) closure of a set M, and the set of bounded linear operators from H to H where H is a separable Hilbert space whose dimension is infinite, respectively.

If $T \in L(H)$ and M is an invariant subspace for T, then $T|M$ is used to denote the restriction of T to M.

1.1. A Functional Calculus.

Let H^∞ be the Banach space of all (complex-valued) bounded analytic functions on the open unit disk **D** with supremum norm [3]. A contraction T in $L(H)$ is said to be *completely non-unitary* provided its restriction to any non-zero reducing subspace is never unitary.

B. Sz.-Nagy and C. Foias introduced an important functional calculus for completely non-unitary contractions.

PROPOSITION 1.1. *Let $T \in L(H)$ be a completely non-unitary contraction. Then there is a unique algebra representation Φ_T from H^∞ into $L(H)$ such that :*

(i) $\Phi_T(1) = I_H$, where $I_H \in L(H)$ is the identity operator;
(ii) $\Phi_T(g) = T$, if $g(z) = z$ for all $z \in \mathbf{D}$;
(iii) Φ_T is continuous when H^∞ and $L(H)$ are given the weak*-topology.
(iv) Φ_T is contractive, i.e. $\|\Phi_T(u)\| \leqslant \|u\|$ for all $u \in H^\infty$.

We simply denote by $u(T)$ the operator $\Phi_T(u)$.

B. Sz.- Nagy and C. Foias [3] defined the *class C_0* relative to the open unit disk **D** consisting of completely non-unitary contractions T on H such that the kernel of Φ_T is not trivial. If $T \in L(H)$ is an operator of class C_0, then

$$\ker \Phi_T = \{u \in H^\infty : u(T) = 0\}$$

is a weak*-closed ideal of H^∞, and hence there is an inner function generating $\ker \Phi_T$. The *minimal function m_T* of an operator T of class C_0 is the generator of $\ker \Phi_T$, that is,

$$\ker \Phi_T = m_T H^\infty. \tag{1.1}$$

1.2. Algebraic Elements

To provide a sufficient condition for (non-trivial) invariant subspaces, we will use the notion of *algebraic elements* for a completely non-unitary contraction T in $L(H)$.

DEFINITION 1.2. [2] Let $T \in L(H)$ be a completely non-unitary contraction. An element h of H is said to be *algebraic with respect to T* provided that $\theta(T)h = 0$ for some $\theta \in H^{\infty} \setminus \{0\}$. If $h \neq 0$, then h is said to be a *non-trivial algebraic element* with respect to T.

If h is not algebraic with respect to T, then h is said to be *transcendental with respect to T*.

If B is a closed subspace of H generated by $\{b_i \in H : i = 1, 2, 3, \cdots\}$, then B will be denoted by $\bigvee_{n=1}^{\infty} b_i$.

1.3. Blaschke Products

For each $\alpha \in \mathbf{D}$, the *Blaschke factor b_α* is a function defined by

$$b_\alpha(z) = \frac{|\alpha|}{\alpha} \frac{\alpha - z}{1 - \overline{\alpha}z}, \ z \in \mathbf{D},$$

if $\alpha \neq 0$, and $b_0(z) = z$ for $z \in \mathbf{D}$. We recall that a *Blaschke Product* is a function of the form

$$b(z) = \prod_j b_{\alpha_j}(z), \ z \in \mathbf{D},$$

where $\{\alpha_j\}_{j=0}^{\infty}$ is a sequence in $\mathbf{D}$ such that $\sum_j (1 - |\alpha_j|) < \infty$.

A function $\mu : \mathbf{D} \to \{0, 1, 2, \cdots\}$ is said to be a *Blaschke function* if

$$\sum_{\alpha \in \mathbf{D}} \mu(\alpha)(1 - |\alpha|) < \infty.$$

If b is a Blaschke Product, then we have a Blaschke function μ such that

$$b(z) = \prod_{\alpha \in \mathbf{D}} (b_\alpha(z))^{\mu(\alpha)},$$

where $\mu(\alpha)$ represents the multiplicity of α as a zero of b, and this Blaschke Product b will be denoted by b_μ. Recall that a *singular inner function* is determined by a positive finite measure ν on $\partial \mathbf{D}$, singular with respect to Lebesgue measure, via the formula

$$s_\nu(\lambda) = \exp\left(-\int_{\partial \mathbf{D}} \frac{\xi + \lambda}{\xi - \lambda} d\nu(\xi) \right),$$

for $\lambda \in \mathbf{D}$ ([1]).

If θ is an inner function, then there exist a Blaschke product b, a singular inner function s, and a constant γ, $|\gamma| = 1$, such that

$$\theta = \gamma b s.$$

Let us recall that the set of positive finite measures on $\partial \mathbf{D}$ has a lattice structure with respect to the following relation; $v \leqslant v' \Leftrightarrow v(A) \leqslant v'(A)$ for every Borel subset A of $\partial \mathbf{D}$, and the set of Blaschke functions can also be organized as a lattice with respect to the following relation; $\mu \leqslant \mu' \Leftrightarrow \mu(z) \leqslant \mu'(z)$ for every $z \in \mathbf{D}$ [1].

Let θ and θ' be two functions in H^∞. We say that θ *divides* θ'(or $\theta \mid \theta'$) if θ' can be written as $\theta' = \theta \cdot \phi$ for some $\phi \in H^\infty$. We will use the notation $\theta \equiv \theta'$ if θ and θ' are two inner functions that differ only by a constant scalar factor of absolute value one. Thus, the relations $\theta \mid \theta'$ and $\theta' \mid \theta$ imply that $\theta \equiv \theta'$.

PROPOSITION 1.3. *([1]) Let μ and μ' be Blaschke functions, v and v' singular measures on $\partial \mathbf{D}$, γ and γ' complex numbers of absolute value one, and set $\theta = \gamma b_\mu s_v$, $\theta' = \gamma' b_{\mu'} s_{v'}$.*
Then, $\theta \mid \theta'$ if and only if $\mu \leqslant \mu'$ and $v \leqslant v'$.

2. The Main Results

2.1. Transcendental Operators

In this section, we provide fundamental properties of transcendental operators.

LEMMA 2.1. *If $T : H \to H$ is a transcendental operator, then, for any $\theta \in H^\infty \setminus \{0\}$, $\theta(T)$ is one-to-one.*

Proof. Suppose that $\theta(T)$ is not one-to-one for a function $\theta \in H^\infty \setminus \{0\}$. Then, there is a non-zero element h in H such that

$$\theta(T)h = 0;$$

that is, h is a non-trivial algebraic element with respect to T. This, however, is a contradiction, since T is a transcendental operator. Thus, $\theta(T)$ is one-to-one for any $\theta \in H^\infty \setminus \{0\}$. $\square$

Recall that an arbitrary subset M of H is said to be *linearly independent* if every nonempty finite subset of M is linearly independent.

PROPOSITION 2.2. *If $T : H \to H$ is a transcendental operator, then, for any non-zero element h in H, $M = \{T^n h : n = 0, 1, 2, \cdots\}$ is linearly independent.*

Proof. Let $h \in H \setminus \{0\}$ be given. Suppose that $M = \{T^n h : n = 0, 1, 2, \cdots\}$ is not linearly independent. Then, there is a polynomial $p \in H^\infty \setminus \{0\}$ such that $p(T)h = 0$. Thus h is a non-trivial algebraic element with respect to T. This, however, is a contradiction, since T is a transcendental operator. Therefore, M is linearly independent. $\square$

COROLLARY 2.3. *Under the same assumption as Proposition 2.2, for a given function $\theta \in H^\infty \setminus \{0\}$, $M' = \{\theta(T)^n h : n = 0, 1, 2, \cdots\}$ is linearly independent.*

Proof. In the same way as Proposition 2.2, it is proven. $\square$

2.2. Algebraic Elements and Invariant Subspaces

We recall that an operator T ia said to be *multiplicity-free* if T has a cyclic vector. The invariant subspaces of a multiplicity-free operator of class C_0 have a classification as following ;

PROPOSITION 2.4. ([1], Theorem 3.2.13) *For every operator T of class C_0, the following assertions are equivalent;*
(i) T is multiplicity-free.
(ii) For every inner divisor ϕ of m_T (that is, ϕ is an inner function such that $\phi | m_T$), there exists a unique invariant subspace K for T such that $m_{T|K} \equiv \phi$.

If T is multiplicity-free, then the unique invariant subspace in (ii) is given by $K = \ker \phi(T)$.

Note that m_T always has two *trivial inner divisors* $\phi_1 \equiv 1$ and $\phi_2 \equiv m_T$.

PROPOSITION 2.5. *Let $T : H \to H$ be a multiplicity-free operator of class C_0 and ϕ be an inner divisor of m_T.*
If ϕ is not a trivial inner divisor, then $\ker \phi(T)$ is a non-trivial invariant subspace for T.

Proof. Suppose $\ker \phi(T) = H$. Then, by equation (1.1), $m_T | \phi$. Since ϕ is an inner divisor of m_T, $\phi \equiv m_T$ which is a contradiction.
Suppose that $\ker \phi(T) = \{0\}$. Since ϕ is an inner divisor of m_T,

$$m_T = \phi\varphi, \tag{2.1}$$

for a function $\varphi \in H^\infty$. Since $m_T(T) = \phi(T)\varphi(T) = 0$ and $\ker \phi(T) = \{0\}$, $\varphi(T) = 0$. Thus, by equation (1.1),

$$m_T | \varphi. \tag{2.2}$$

By equation (2.1) and (2.2), we have $\varphi \equiv m_T$, and equation (2.1) implies that $\phi \equiv 1$ which is a contradiction. $\square$

THEOREM 2.6. *Let $T \in L(H)$ be a completely non-unitary contraction.*
If T has a non-trivial algebraic element h, then T has a non-trivial invariant subspace.

Proof. Suppose that T has a non-trivial algebraic element h. Then, there is a non-zero function $\theta \in H^\infty$ such that $\theta(T)h = 0$.
Let $M = \bigvee_{n=0}^{\infty} T^n h$, and $T_1 = T|M$.
Since $\theta(T)(T^n h) = T^n(\theta(T)h) = 0$ for any $n = 0, 1, 2, \cdots$, $\theta(T_1) = 0$. Thus, the operator $T_1 : M \to M$ is a multiplicity-free operator of class C_0. If $m_{T_1} \equiv 1$, then, by Proposition 1.1,

$$m_{T_1}(T_1)h = h. \tag{2.3}$$

Since $m_{T_1}(T_1) \equiv 0$, equation (2.3) implies that $h = 0$ which is a contradiction. Thus,

$$m_{T_1} \neq 1. \tag{2.4}$$

By Proposition 2.4, for every inner divisor ϕ of m_{T_1}, there exists a unique invariant subspace $K(= \ker \phi(T_1))$ for T_1.

By Proposition 2.5, if ϕ is not a trivial inner divisor of m_{T_1}, then $\ker \phi(T_1)$ is a non-trivial invariant subspace for T_1. Note that $\ker \phi(T_1)$ is also a non-trivial invariant subspace for T.

If m_{T_1} has no non-trivial inner devisors, then, by Proposition 1.3 and (2.4), m_{T_1} must be a Blaschke factor. Then, $m_{T_1} \equiv \frac{a-z}{1-\bar{a}z}$ for $z \in \mathbf{D}$ where a is a complex number in $\mathbf{D}$, and in this case we clearly have $m_{T_1} | (z - a)$. Therefore,

$$(z - a) \in \ker \Phi_{T_1} = \{u \in H^\infty : u(T_1) = 0\}.$$

Thus, $(T - aI_H)h = (T_1 - aI_H)h = 0$ and so h is an eigenvector of T. Since $h \neq 0$ and $\dim H \geqslant 2$, the closed subspace $M(= \bigvee_{n=0}^\infty T^n h)$ generated by the eigenvector h is a non-trivial invariant subspace for T. $\quad\square$

COROLLARY 2.7. *Every C_0-operator has a non-trivial invariant subspace.*

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