Operators
and
Matrices
Volume 5, Number 3 (2011), 455–467

UNIVERSAL SHIFTS AND COMPOSITION OPERATORS

JONATHAN R. PARTINGTON AND ELODIE POZZI

(*Communicated by S. McCullough*)

Abstract. It is shown that a large class of weighted shift operators T have the property that for every λ in the interior of the spectrum of T the operator $U = T - \lambda\mathrm{Id}$ is universal in the sense of Caradus; i.e., every Hilbert space operator has a non-zero multiple similar to the restriction of U to an invariant subspace. As an application, composition operators induced by power mappings on the L^2 and Sobolev spaces of the unit interval are shown to have the same property: thus a complete knowledge of their minimal invariant subspaces would imply a solution to the invariant subspace problem for Hilbert space. A new Müntz-like theorem is proved: this is used to show that generalized polynomials are cyclic vectors for these operators in the L^2 case but not in the Sobolev case.

1. Introduction

In the field of operator theory, one of the most prominent open problems is the invariant subspace problem, sometimes optimistically known as the invariant subspace conjecture. It is the question whether the following statement is true: Given a complex Hilbert space $\mathscr{H}$ of dimension > 1 and a bounded linear operator $T : \mathscr{H} \to \mathscr{H}$, then $\mathscr{H}$ has a non-trivial closed T-invariant subspace, i.e., there exists a closed linear subspace $\mathscr{M}$ of $\mathscr{H}$ which is different from $\{0\}$ and $\mathscr{H}$ such that $T\mathscr{M} \subset \mathscr{M}$.

While the general case of the invariant subspace problem is still open, many special cases have been settled (see, for example, [3, 6]). If the solution of the invariant subspace problem is positive, then at first sight it is necessary to prove a theorem that applies to all Hilbert space operators simultaneously. In fact, the situation is somewhat simplified by the existence of universal operators – these have the property that if we could describe their lattice of subspaces precisely enough, then we could solve the invariant subspace problem. Accordingly, we recall the following definition.

DEFINITION 1.1. Let $\mathscr{X}$ be a Banach space. Then an operator $U \in \mathscr{L}(\mathscr{X})$ is said to be *universal* for $\mathscr{X}$, if for each $T \in \mathscr{L}(\mathscr{X})$ there is a constant $\lambda \neq 0$ and an invariant subspace $\mathscr{M}$ for U such that $U_{|\mathscr{M}}$ is similar to λT, i.e., $\lambda JT = UJ$, where $J : \mathscr{X} \to \mathscr{M}$ is a linear isomorphism.

Mathematics subject classification (2010): 47A15, 47B33, 47A16.
Keywords and phrases: Invariant subspace, universal operator, weighted shift, composition operator, cyclic vector, Müntz theorem.

Paper OaM-05-33

This is most useful in the case of a separable Hilbert space, since all its closed infinite-dimensional subspaces are automatically isometric to the space itself (and we can then use the terminology "universal for Hilbert space"). Clearly, we have the fact that if $U \in \mathscr{L}(\mathscr{H})$ is a universal operator, then the invariant subspace problem for Hilbert spaces is equivalent to the assertion that every infinite-dimensional invariant subspace for U contains a nontrivial proper closed invariant subspace. This may be rewritten in the form "the minimal nontrivial invariant subspaces for U are one-dimensional".

We now recall a theorem due to Caradus [2] that will enable us, in the case of Hilbert space, to show the existence of many "natural" operators that are universal.

THEOREM 1.1. [2] *Let $\mathscr{H}$ be a separable infinite-dimensional Hilbert space and $U \in \mathscr{L}(\mathscr{H})$. Suppose that*

1. $\ker(U)$ *is infinite-dimensional; and*

2. $\mathrm{im}(U)$ *is* $\mathscr{H}$.

Then U is universal for $\mathscr{H}$.

An obvious example of such a universal operator is a backward shift of infinite multiplicity, but there are further examples where function theory can be use to obtain information on the invariant subspace lattice. For example, the Caradus result has been used by Nordgren, Rosenthal and Wintrobe [4] to show that for the composition operator C_ϕ on the Hardy space H^2, where ϕ is a hyperbolic automorphism of the disc, the operator $C_\phi - \lambda \mathrm{Id}$ is universal for any λ in the interior of the spectrum $\sigma(C_\phi)$.

The paper is organized as follows: in Section 2 we discuss bilateral weighted shifts T on $\ell^2(\mathbb{Z}, L^2((a_0,a_1)))$, providing sufficient conditions on the weight and the complex number λ so that $T - \lambda \mathrm{Id}$ is universal. Then, in Section 3, as an application of the previous section, we find some very simple new universal composition operators on $L^2((0,1))$ and the universality of the adjoints of composition operators on the Sobolev space $W_0((0,1))$. Finally, in Section 4, a new Müntz-type theorem is derived, and used to show the cyclicity of linear combinations of functions of the form $x \mapsto x^\alpha$ for the composition operator C_ϕ on $L^2((0,1))$, where ϕ is defined on $[0,1]$ by $\phi(x) = x^s$ with $s < 1$, and the non-cyclicity of all linear combinations of these functions for the composition operator C_ϕ on $W_0((0,1))$, where ϕ is defined on $[0,1]$ by $\phi(x) = x^s$ with $s > 1$.

2. Shift on $\ell^2(\mathbb{Z}, L^2((a_0,a_1)))$

Let $a_0 < a_1$ be real, and let $T : \ell^2(\mathbb{Z}, L^2((a_0,a_1))) \to \ell^2(\mathbb{Z}, L^2((a_0,a_1)))$ be the weighted right bilateral shift defined by $(Tx)_n = k_{n-1}x_{n-1}$ for $n \in \mathbb{Z}$ where $\{k_n\}$ is a sequence of positive and continuous functions on $[a_0,a_1]$ such that

$$k_n \xrightarrow{\text{uniformly}} \begin{cases} a \text{ as } n \to -\infty \\ b \text{ as } n \to +\infty \end{cases}$$

where $a < b$.

We denote by $\|.\|_2$ the norm $\|.\|_{L^2((a_0,a_1))}$.

THEOREM 2.1. *Let T be the bilateral weighted shift defined above. Then, $\sigma(T) = \sigma(T^*) = \{z \in \mathbb{C} : a \leqslant |z| \leqslant b\}$. If $a < |\lambda| < b$, then λ is an eigenvalue of T^* of infinite multiplicity, but $T - \lambda \mathrm{Id}$ is bounded below.*

Proof. Suppose that $a > 0$. Note that if $f = \sum_{m\in\mathbb{Z}} g_m e_m$ where $\{e_m\}$ is the standard orthonormal basis of $\ell^2(\mathbb{Z})$, then

$$\|Tf\|_2 \leqslant \sup_{m\in\mathbb{Z}} \|k_m\|_\infty \|f\|_2.$$

Let $\varepsilon > 0, m_0 \in \mathbb{Z}, A_\varepsilon \subseteq [a_0,a_1]$, $\mu(A_\varepsilon) > 0$ such that

$$\|k_{m_0}\|_\infty \geqslant \sup_{m\in\mathbb{Z}} \|k_m\|_\infty - \frac{\varepsilon}{2},$$

and for $x \in A_\varepsilon$,

$$|k_{m_0}(x)| > \|k_{m_0}\|_\infty - \frac{\varepsilon}{2}.$$

Also, if we take $f = \chi_{A_\varepsilon} e_{m_0}$, then we have

$$\|Tf\| > \left(\|k_{m_0}\|_\infty - \varepsilon\right) \sqrt{\mu(A_\varepsilon)}.$$

So, we have that $\|T\| = \sup_{m\in\mathbb{Z}} \|k_m\|_\infty$. By an inductive argument, we obtain that for $n \in \mathbb{N}^*$, $\|T^n\| = \sup_{m\in\mathbb{Z}} \|k_m k_{m+1} \dots k_{m+n-1}\|_\infty$. Since k_n converges uniformly to b as n tends to $+\infty$, it follows that

$$\sup_{m\in\mathbb{Z}} \|k_m k_{m+1} \dots k_{m+n-1}\|_\infty^{\frac{1}{n}} \underset{n\to\infty}{\longrightarrow} b.$$

Since T^{-1} is unitarily equivalent to a bilateral right shift with weights $\widetilde{k_n} = {k_{-n}}^{-1}$, in the same way, one can show that $\|T^{-n}\|^{\frac{1}{n}} \underset{n\to\infty}{\to} 1/a$. As 0 does not lie in the spectrum of T, $\sigma(T^{-1}) = \sigma(T)^{-1}$. Hence, we have that $\sigma(T) \subseteq \{z \in \mathbb{C} : a \leqslant |z| \leqslant b\}$; since $\sigma(T^*) = \overline{\sigma(T)}$, we obtain that

$$\sigma(T^*) \subseteq \{z \in \mathbb{C} : a \leqslant |z| \leqslant b\}.$$

Note that if $a = 0$, then, we have $\sigma(T^*) \subseteq \overline{\mathbb{D}}(0,b)$.

Let λ be such that $a < |\lambda| < b$. If $f = \sum_{n\in\mathbb{Z}} g_n e_n$, where for all $n \in \mathbb{Z}$, g_n is in $L^2((a_0,a_1))$, the equation $T^* f = \lambda f$ gives that

$$\lambda \sum_{n=-\infty}^{+\infty} g_n e_n = \sum_{n=-\infty}^{+\infty} g_n k_{n-1} e_{n-1} = \sum_{n=-\infty}^{+\infty} g_{n+1} k_n e_n$$

which implies that for all $n \in \mathbb{Z}$, $\lambda g_n = k_n g_{n+1}$. Setting g_0 to be any function of norm 1 in $L^2(a_0,a_1)$, and defining on (a_0,a_1),

$$g_n = \begin{cases} \lambda^n g_0/(k_0 k_1 \dots k_{n-1}) & \text{for} \quad n > 0, \\ \lambda^n g_0 k_n k_{n+1} \dots k_{-1} & \text{for} \quad n < 0, \end{cases}$$

we see easily that $\sum_{n\in\mathbb{Z}}\|g_n\|_2^2$ converges.

Hence, $f=\sum_{n=-\infty}^{\infty} g_n e_n$ is an eigenvector of T^* and $\lambda\in\sigma_p(T^*)$ with infinite multiplicity. We conclude that

$$\sigma(T)=\sigma(T^*)=\{z\in\mathbb{C}: a\leqslant |z|\leqslant b\}.$$

It remains to check that for $|\lambda|\in(a,b)$, $T-\lambda\mathrm{Id}$ is bounded below. Here our method generalizes some ideas due to Ridge [5]. Suppose towards a contradiction that λ is an approximate eigenvalue of T. So, for each $i\in\mathbb{N}^*$, there is a unit vector $f(i)=\{f_j(i)\}_{j\in\mathbb{Z}}$ such that

$$\|Tf(i)-\lambda f(i)\|<\frac{1}{i}.$$

- Suppose first that $\liminf_{i\to\infty}\|f_0(i)\|_2=0$. So, for all $\varepsilon>0$, there is an index i such that $\|f_0(i)\|_2<\varepsilon$ and $\|Tf(i)-\lambda f(i)\|<\varepsilon$. Denoting by $h(i)=f(i)-f_0(i)e_0$ (that is, setting to zero the component corresponding to $j=0$), we have

$$\begin{aligned}\|Th(i)-\lambda h(i)\| &\leqslant \|Tf(i)-\lambda f(i)\|+\|Tf_0(i)e_0-\lambda f_0(i)e_0\|\\ &<\varepsilon+(\|T\|+|\lambda|)\varepsilon\end{aligned}$$

and thus there is an approximate eigenvector $h(i)$ of norm 1 such that $h_0(i)=0$ and $\|Th(i)-\lambda h(i)\|<\varepsilon$.

We can write $h(i)=l(i)+r(i)$ where $l(i)$ is supported on the negative integers and $r(i)$ is supported on the positive integers. Since their supports are disjoint, $Th(i)-\lambda h(i)$ is the orthogonal sum of $Tl(i)-\lambda l(i)$ and $Tr(i)-\lambda r(i)$. Since $h(i)$ is of norm 1, one of $l(i)$ and $r(i)$ has norm greater than $\frac{1}{2}$ and thus, we may find a sequence of approximate eigenvectors supported entirely on either the positive or negative integers. We will denote it by $p(i)$, and may suppose without loss of generality that $\|p(i)\|=1$ and $\|Tp(i)-\lambda p(i)\|<1/i$. Now

$$T^np(i)-\lambda^np(i)=(T^{n-1}+\lambda T^{n-2}+\ldots+\lambda^{n-1}\mathrm{Id})(Tp(i)-\lambda p(i))$$

and so $\|T^np(i)-\lambda^np(i)\|<C_n/i$, where C_n depends on λ and the weights but not on i. If $p(i)=\{p_j(i)\}$ is supported on the positive integers, then

$$\begin{aligned}\|T^np(i)\|_2 &= \left(\sum_{j=1}^{\infty}\|k_jk_{j+1}\ldots k_{j+n-1}p_j(i)\|_2^2\right)^{1/2}\\ &\geqslant \left(\inf_{j>0}\left(\min_{u\in[a_0,a_1]}k_j(u)k_{j+1}(u)\ldots k_{j+n-1}(u)\right)^2\sum_{j=1}^{\infty}\|p_j(i)\|_2^2\right)^{1/2}\\ &= \left[\inf_{j>0}\left(\min_{u\in[a_0,a_1]}k_j(u)k_{j+1}(u)\ldots k_{j+n-1}(u)\right)\right].\end{aligned}$$

Without loss of generality, one can suppose that $b>1$. So, for n sufficiently large, we have

$$\inf_{j>0}\left(\min_{u\in[a_0,a_1]}k_j(u)k_{j+1}(u)\ldots k_{j+n-1}(u)\right)>|\lambda^n|+2.$$

Choosing i larger than C_n, we obtain a contradiction. Applying similar arguments to T^{-1}, we obtain a contradiction when T has an approximate eigenvector supported on the negative integers.

• Suppose that $\liminf_{i\to\infty}\|f_0(i)\|_2 = d > 0$. Since we have an approximate eigenvector $f(i) = \{f_n(i)\}$ of norm 1 such that $\|Tf(i) - \lambda f(i)\| < \frac{1}{i}$, then a simple inductive argument shows that there exist constants $\{D_n\}_{n\geqslant 0}$ independent of i such that

$$\left\| f_{n+1}(i) - \frac{k_0 k_1 \dots k_n}{\lambda^{n+1}} f_0(i) \right\|_2 \leqslant \frac{D_n}{i}, \qquad \text{for } n \in \mathbb{N}.$$

If $\|f_0(i)\|_2 \geqslant d/2$, then,

$$\begin{aligned}\|f_{n+1}(i)\|_2 &\geqslant \left\| \frac{k_0 k_1 \dots k_n}{\lambda^{n+1}} f_0(i) \right\|_2 - \frac{D_n}{i} \\ &\geqslant \frac{\min_{u\in[a_0,a_1]} k_j(u) k_{j+1}(u) \dots k_{j+n-1}(u)}{|\lambda^{n+1}|} \|f_0(i)\|_2 - \frac{D_n}{i}.\end{aligned}$$

Since

$$\frac{\min_{u\in[a_0,a_1]} k_j(u) k_{j+1}(u) \dots k_{j+n-1}(u)}{|\lambda^{n+1}|} \underset{n\to+\infty}{\longrightarrow} \infty,$$

we may find an index n such that $\|f_{n+1}(i)\|_2 \geqslant 2 - D_n/i$. But, as $f(i)$ is a vector of norm 1, if we choose i larger than D_n, we obtain a contradiction. $\square$

COROLLARY 2.1. *Let $T : \ell^2(\mathbb{Z}, L^2((a_0,a_1))) \to \ell^2(\mathbb{Z}, L^2((a_0,a_1)))$ be the weighted right bilateral shift defined by $(Tx)_n = k_{n-1}x_{n-1}$ for $n \in \mathbb{Z}$ where $\{k_n\}$ is a sequence of positive and continuous functions on $[a_0,a_1]$ such that k_n converges uniformly to b as $n \to -\infty$ and a as $n \to +\infty$ where $a < b$. Then, for any complex number $a < |\lambda| < b$, $T - \lambda\mathrm{Id}$ is a universal operator for Hilbert space. In particular, every operator has an invariant subspace if and only if the minimal nontrivial invariant subspaces of T are all one-dimensional.*

Proof. T^* is unitarily equivalent to a weighted right shift where the weights is a sequence of positive and continuous functions such that k_n tends uniformly to a at $-\infty$ and uniformly to b at $+\infty$. By Theorem 2.1, it follows that $T - \lambda\mathrm{Id}$ has an infinite dimensional kernel and, $T^* - \overline{\lambda}\mathrm{Id}$ is bounded below which implies that $T - \lambda\mathrm{Id}$ is surjective. So, we conclude that $T - \lambda\mathrm{Id}$ is universal for Hilbert space.

The last statement follows immediately from the remarks after Definition 1.1, noting that T and $T - \lambda\mathrm{Id}$ have the same lattice of invariant subspaces. $\square$

REMARK 2.1. Note that, in this case, the point spectrum of T^* is empty.

3. Applications to composition operators

3.1. Composition operators on $L^2((0,1))$

DEFINITION 3.1. A mapping $\phi : [0,1] \to [0,1]$ will be called *L^2-admissible*, if it is strictly increasing and differentiable, with $\phi(0)=0$ and $\phi(1)=1$, and has the properties that ϕ has no other fixed points, $\phi(x) \geqslant x$ on $[0,1]$, ϕ' is continuous on $(0,1)$ and $(\phi^{-1})'$ is bounded. We then define the composition operator C_ϕ by $C_\phi(f) = f \circ \phi$.

Such operators have been studied by Spalsbury [7] in the context of $C[0,1]$ and $C^1[0,1]$, for functions of the form $\phi(x) = x^s$, and generalizations of the Müntz theorem were obtained in order to obtain information on cyclic vectors. Here we work in a Hilbertian context, and with a more general range of symbols ϕ.

PROPOSITION 3.1. *Let ϕ be L^2-admissible. Then, C_ϕ is bounded on $L^2((0,1))$ and its adjoint is given by $C_\phi^*(f) = (\phi^{-1})'(f \circ \phi^{-1})$.*

Proof. Let f be in $L^2((0,1))$. Then

$$\begin{aligned}\int_0^1 |C_\phi(f)(x)|^2\,dx = \int_0^1 |f(\phi(x))|^2\,dx &= \int_0^1 |f(u)|^2(\phi^{-1})'(u)\,du \\ &\leqslant \sup_{u\in[0,1]} (\phi^{-1})'(u)\|f\|^2_{L^2((0,1))}.\end{aligned}$$

Thus, $\|C_\phi\| \leqslant \|(\phi^{-1})'\|_\infty^{1/2}$.

The computation of the adjoint is elementary, and we omit it. $\square$

Now, fix $a_0 \in (0,1)$. Let f be a function in $L^2((0,1))$. We denote by $\{a_n\}_{n\in\mathbb{Z}}$ the sequence $\{\phi^n(a_0)\}_{n\in\mathbb{Z}}$, where for $n<0$, $\phi^n = (\phi^{-1})^{-n}$. If we define $g_n(u)$ such that

$$g_n(u) = f(\phi^n(u)) \text{ for } n \in \mathbb{Z} \text{ and } u \in (a_0,a_1),$$

then each g_n lies in $L^2((a_0,a_1))$ and we have, writing $f_n = f_{|(a_n,a_{n+1})}$, that

$$\begin{aligned}\|f\|^2_{L^2((0,1))} &= \sum_{n\in\mathbb{Z}} \|f_n\|^2_{L^2((a_n,a_{n+1}))} \\ &= \sum_{n\in\mathbb{Z}} \int_{a_0}^{a_1} |f_n(\phi^n(u))|^2(\phi^n)'(u)\,du \\ &= \sum_{n\in\mathbb{Z}} \int_{a_0}^{a_1} |g_n(u)|^2(\phi^n)'(u)\,du \\ &= \sum_{n\in\mathbb{Z}} \left\| g_n\sqrt{(\phi^n)'}\right\|^2_{L^2((a_0,a_1))}.\end{aligned}$$

If we consider $V : L^2([0,1]) \longrightarrow \ell^2(\mathbb{Z}, L^2((a_0,a_1)))$ defined by $V(f) = \left(g_n\sqrt{(\phi^n)'}\right)_{n\in\mathbb{Z}}$, then V is a unitary operator.

Now, if we compose f by the operator C_ϕ, we have

$$\begin{aligned}\|C_\phi(f)\|^2_{L^2((0,1))} &= \sum_{n\in\mathbb{Z}} \int_{a_n}^{a_{n+1}} |f(\phi(x))|^2\,dx \\ &= \sum_{n\in\mathbb{Z}} \int_{a_0}^{a_1} |g_{n+1}(u)|^2 (\phi^n)'(u)\,du\end{aligned}$$

So, if $f = \sum_{n\in\mathbb{Z}} g_n \sqrt{(\phi^n)'} e_n$, where $(e_n)_{n\in\mathbb{Z}}$ is the standard orthonormal basis of $\ell^2(\mathbb{Z})$, then,

$$C_\phi(f) = \sum_{n\in\mathbb{Z}} g_{n+1} \sqrt{(\phi^n)'} e_n.$$

Thus, C_ϕ maps on $L^2((0,1))$ as a weighted left shift on $\ell^2(\mathbb{Z}, L^2((a_0,a_1)))$ with weights $k_n = \sqrt{\frac{(\phi^n)'}{(\phi^{n+1})'}} = \frac{1}{\sqrt{\phi'\circ\phi^n}}$, for $n\in\mathbb{Z}$.

THEOREM 3.1. *Let ϕ be an L^2-admissible function such that for some $a_0 \in (0,1)$, the sequence $\frac{1}{\sqrt{\phi'\circ\phi^n}}$ converges uniformly on $[a_0,\phi(a_0)]$ to a as $n\to-\infty$ and b as $n\to+\infty$ where $a<b$. Then, $\sigma(C_\phi) = \{z\in\mathbb{C} : a \leqslant |z| \leqslant b\}$ and $\{z\in\mathbb{C} : a<|z|<b\} \subseteq \sigma_p(C_\phi)$. Likewise, for any complex numbers $a<|\lambda|<b$, $C_\phi - \lambda\mathrm{Id}$ is a universal operator for Hilbert space.*

Proof. Since a weighted left shift with weights $\{k_n\}_{n\in\mathbb{Z}}$ is unitarily equivalent to a weighted right shift with weights $\{\widetilde{k_n}\}_{n\in\mathbb{Z}}$ where $\widetilde{k_n} = k_{-n}$ for $n\in\mathbb{Z}$, then, C_ϕ is unitarily equivalent to a weighted right shift on $\ell^2(\mathbb{Z}, L^2((a_0,a_1))$, denoted by S_ϕ with weights $\widetilde{k_n} = \frac{1}{\sqrt{(\phi^{-1})'\circ\phi^{-n}}}$, $n\in\mathbb{Z}$ such that

$$\widetilde{k_n} \overset{\text{uniformly}}{\longrightarrow} \begin{cases} b \text{ as } n\to-\infty \\ a \text{ as } n\to+\infty \end{cases}$$

Theorem 2.1 implies that $\sigma(S_\phi) = \{z\in\mathbb{C} : a\leqslant|z|\leqslant b\}$ and $\sigma_p(S_\phi) = \{z\in\mathbb{C} : a < |z| < b\}$. It follows from Corollary 2.1 that for any $a<|\lambda|<b$, $S_\phi - \lambda\mathrm{Id}$ is a universal operator for Hilbert space. So, we have the conclusion for $C_\phi - \lambda\mathrm{Id}$. $\square$

REMARK 3.1. If ϕ satisfies the hypothesis of Theorem 3.1, the point spectrum of C_ϕ^* is empty.

We now have an analogous version of Corollary 2.1 for C_ϕ.

COROLLARY 3.1. *Let ϕ and C_ϕ be as in Theorem* 3.1. *Then, every operator has an invariant subspace if and only if the minimal nontrivial invariant subspaces of C_ϕ are all one-dimensional.*

EXAMPLE 3.1. Fix $0 < s < 1$. Let ϕ be the function defined on $[0,1]$ by $\phi(x) = x^s$. Then C_ϕ is bounded on $L^2((0,1))$. Note that, for $n \in \mathbb{Z}$ and $x \in (0,1)$,

$$\frac{1}{\sqrt{\phi' \circ \phi^n(x)}} = \frac{1}{\sqrt{s}} x^{s^n(1-s)/2}.$$

Then we have

$$\frac{1}{\sqrt{\phi' \circ \phi^n}} \xrightarrow{\text{uniformly}} \begin{cases} 0 & \text{as } n \to -\infty \\ \frac{1}{\sqrt{s}} & \text{as } n \to +\infty \end{cases}$$

It follows from Theorem 3.1 that

$$\sigma(C_\phi) = \overline{\mathbb{D}}\left(0, 1/\sqrt{s}\right) \qquad \text{and} \qquad \left\{z \in \mathbb{C} : 0 < |z| < 1/\sqrt{s}\right\} \subseteq \sigma_p(C_\phi),$$

and for any complex number λ such that $0 < |\lambda| < 1/\sqrt{s}$, $C_\phi - \lambda \mathrm{Id}$ is a universal operator for Hilbert space. Thus every Hilbert space operator has an invariant subspace if and only if the minimal nontrivial invariant subspaces of C_ϕ are all one dimensional.

3.2. Composition operators on Sobolev spaces $W_0((0,1))$

Now let $W_0(0,1)$ be the space of absolutely continuous functions f defined on $[0,1]$ such that $f(0) = 0$ and $f' \in L^2((0,1))$, with norm

$$\|f\|_{W_0(0,1)} = \|f'\|_2.$$

where $\|.\|_2$ denotes the norm in $L^2((0,1))$.

DEFINITION 3.2. A mapping $\phi : [0,1] \to [0,1]$ will be called W_0*-admissible*, if it is a strictly increasing and continuously differentiable function, with $\phi(0) = 0$ and $\phi(1) = 1$, such that $\phi(x) \leqslant x$ for $x \in [0,1]$, and ϕ has no other fixed points. We denote by C_ϕ the composition operator induced by ϕ.

PROPOSITION 3.2. *Let ϕ be W_0-admissible. Then, C_ϕ is bounded on $W_0((0,1))$.*

Proof. Let f be in $W_0((0,1))$.

$$\begin{aligned} \|C_\phi(f)\|^2_{W(0,1)} &= \int_0^1 |f'(\phi(x))|^2 \phi'(x)^2\, dx = \int_0^1 |f(u)|^2 \phi'(\phi^{-1}(u))\, du \\ &\leqslant \|\phi'\|_\infty \|f\|^2_{W_0((0,1))}. \end{aligned}$$

Thus, $\|C_\phi\|_{W_0(0,1)} \leqslant \|\phi'\|_\infty^{1/2}$. □

Note that the operator C_ϕ is also bounded on $W(0,1)$, the space of absolutely continuous functions f on $[0,1]$ such that $f' \in L^2((0,1))$. As the mapping $f \mapsto f(0)$ is bounded on W, then $f = (f - f(0)) + f(0)$ and $C_\phi f = C_\phi(f - f(0)) + f(0)$.

LEMMA 3.1. *Let ϕ be W_0-admissible. Then C_ϕ on $W_0((0,1))$ is unitarily equivalent to a weighted composition operator D_ϕ on $L^2((0,1))$.*

Proof. Let f be in $W_0((0,1))$. We denote by F the derivative of f which is in $L^2((0,1))$. We have

$$\begin{aligned}\|C_\phi(f)\|^2_{W_0((0,1))} &= \int_0^1 |F(\phi(x))|^2 \phi'(x)^2\, dx \\ &= \|\phi'.(F\circ\phi)\|_2^2 \\ &= \|D_\phi(F)\|_2\end{aligned}$$

where D_ϕ is the weighted composition operator on $L^2((0,1))$ defined by

$$D_\phi(g) = \phi'(g\circ\phi)$$

for all $g \in L^2((0,1))$. If D is the mapping from $W_0((0,1))$ onto $L^2((0,1))$ defined by $D(f) = f'$ for all $f \in W_0((0,1))$, then we have the commutative diagram

$$\begin{tikzcd} f \arrow[r, "C_\phi"] \arrow[d, "D"'] & f\circ\phi \arrow[d, "D"'] \\ f' \arrow[r, "D_\phi"] & \phi'(f'\circ\phi) \end{tikzcd}$$

Since D is a unitary operator from $W_0((0,1))$ to $L^2((0,1))$, we have the conclusion. $\square$

PROPOSITION 3.3. *Let ϕ be W_0-admissible and suppose that for some $a_0 \in (0,1)$, the sequence $\frac{1}{\sqrt{(\phi^{-1})'\circ\phi^{-n}(x)}}$ converges uniformly on $[a_0,\phi^{-1}(a_0)]$ to a as $n\to-\infty$ and b as $n\to+\infty$ where $a<b$. Then $\sigma(C_\phi)=\{z\in\mathbb{C} : a\leqslant|z|\leqslant b\}$ and $\sigma_p(C_\phi)=\varnothing$. Likewise, for any complex numbers $a<|\lambda|<b$, $C^*_\phi-\lambda\mathrm{Id}$ is a universal operator for Hilbert space.*

Proof. Note that if $F,G\in L^2((0,1))$, then

$$\begin{aligned}\langle D_\phi(F),G\rangle_2 &= \int_0^1 \phi'(x)F(\phi(x))\overline{G(x)}\,dx \\ &= \int_0^1 F(u)\overline{G(\phi^{-1}(u))}\,du \\ &= \langle F, S_{\phi^{-1}}(G)\rangle_2,\end{aligned}$$

where $S_{\phi^{-1}}$ denotes the composition operator on $L^2((0,1))$. So, we have $D^*_\phi = S_{\phi^{-1}}$. Thus, we deduce by Theorem 3.1 that $\sigma(D^*_\phi)=\{z\in\mathbb{C} : a\leqslant|z|\leqslant b\}$ and $\sigma_p(D^*_\phi)=\{z\in\mathbb{C} : a<|z|<b\}$. Likewise, for any complex numbers $a<|\lambda|<b$, $D^*_\phi-\lambda\mathrm{Id}$ is a universal operator for Hilbert space. By unitary equivalence from Lemma 3.1 and by Remark 3.1, we have the conclusion. $\square$

EXAMPLE 3.2. Fix $s>1$. Let ψ be the function defined on $[0,1]$ by $\psi(x)=x^s$. Then, C_ψ is bounded on $W_0((0,1))$. Note that for all $x\in[0,1]$, we have $\psi^{-1}(x)=x^{1/s}$; using similar arguments as in Example 3.1, one can prove that ψ satisfies the hypothesis of Proposition 3.3.

It follows that $\sigma(C_\psi)=\overline{\mathbb{D}}\left(0,\sqrt{s}\right)$, $\sigma_p(C_\psi)=\varnothing$ and for any complex number λ such that $0<|\lambda|<\sqrt{s}$, $C_\psi^*-\lambda\mathrm{Id}$ is a universal operator for Hilbert space. Thus, every operator has an invariant subspace if and only if the minimal nontrivial invariant subspaces of C_ψ^* are all one-dimensional.

4. Cyclic vectors for C_ϕ

In the sequel, we will denote by ϕ the function $x\mapsto x^s$ where $0<s<1$. Here, we are interested in cyclic vectors of composition operators induced by ϕ: in view of the universality results proved in Section 3, these are of particular significance in their application to the study of invariant subspaces.

PROPOSITION 4.1. *Let p be the function defined on $[0,1]$ by $p(x)=cx^\alpha$, where $c\in\mathbb{C}\setminus\{0\}$ and $0\neq\alpha>-\frac{1}{2}$. Then p is a cyclic vector for C_ϕ in $L^2((0,1))$.*

Proof. Note that $\mathrm{span}\{p(x^{s^k}) : k\geqslant 0\}=\mathrm{span}\{x^{\lambda_k} : k\geqslant 0\}$, where $(\lambda_k)_{k\geqslant 0}$ is the infinite sequence of positive and distinct real numbers defined by $\lambda_k=\alpha s^k$, for $k=0,1,2,\ldots$. By the full Müntz theorem in $L^2((0,1))$, see [1], we have that

$$\overline{\mathrm{span}}\{x^{\lambda_k},k\geqslant 0\}=L^2((0,1))\Longleftrightarrow\sum_k\frac{2\lambda_k+1}{(2\lambda_k+1)^2+4}=\infty.$$

Now

$$\frac{2\alpha s^k}{(2\alpha s^k+1)^2+4}\underset{k\to+\infty}{\sim}\frac{2}{5}\alpha s^k$$
and
$$\frac{1}{(2\alpha s^k+1)^2+4}\underset{k\to+\infty}{\sim}\frac{1}{5}.$$

So, $\displaystyle\sum_{k=0}^{\infty}\frac{2\alpha s^k+1}{(2\alpha s^k+1)^2+4}$ is divergent, and we conclude that

$$\overline{\mathrm{span}}\{p(x^{s^k}) : k\geqslant 0\}=L^2((0,1)).$$

□

Using a more complicated method, we now prove that every "generalized polynomial" which maps zero to zero is also cyclic for C_ϕ.

THEOREM 4.1. *Let P be the "generalized polynomial" on $[0,1]$ defined by $P(x)=\sum_{k=0}^n a_kx^{r_k}$ with $0<r_1<\ldots<r_n$ and the a_k non-zero. Then P is cyclic in $L^2(0,1)$ for C_ϕ.*

Proof. To see this, we consider the isometry between $L^2(0,1)$ and $L^2(0,\infty)$ defined by $(Jf)(t) = f(e^{-t})e^{-t/2}$, which takes a function $x \mapsto x^r$ to $t \mapsto e^{-rt-t/2}$.

Then a function f is orthogonal to $C_\phi^j P$ if and only if the corresponding Jf is orthogonal to

$$P_j : t \mapsto \sum_{k=1}^n a_k \exp(-s^j r_k t - t/2).$$

Taking the Fourier–Laplace transform and using the Paley–Wiener theorem, we arrive at a function $F \in H^2(\mathbb{C}_+)$, where $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, such that

$$\sum_{k=1}^n a_k F\left(is^j r_k + \frac{i}{2}\right) = 0 \qquad (j = 0,1,2,\ldots).$$

We have the completeness of the iterates $C_\phi^j P$ if we can show that F is identically zero.

Note that, in the well-known case $n = 1$, this now implies that F is identically zero since $(is^j r_1 + \frac{i}{2})_j$ is not a Blaschke sequence, given that $r_1 > 0$ by assumption.

In general, since $s^j r_k \to 0$ as $j \to \infty$ for all k, we have, writing $G(z) = F(i(z + 1/2))$ for z in a neighbourhood of 0, that

$$\sum_{k=1}^n a_k G(zr_k) = 0,$$

by the principle of isolated zeroes, and G is holomorphic near 0.

Now G is not a polynomial, unless it is identically zero (since $F \in H^2(\mathbb{C}_+)$) and so if $(g_m)_m$ are its Taylor coefficients, then infinitely often we have $g_m \neq 0$ and

$$\sum_{k=1}^n a_k r_k^m = 0,$$

which cannot be true since this is asymptotic to $a_n r_n^m$ as $m \to \infty$. Hence G must be identically zero and so the system of iterates is complete. $\square$

For a contrasting result, we now denote by ϕ the function $x \mapsto x^s$ where $s > 1$ and by D_ϕ the weighted composition operator in $L^2((0,1))$ introduced in Lemma 3.1.

PROPOSITION 4.2. *Let P be the function defined on $[0,1]$ by $P(x) = \sum_{i=0}^r c_i x^{\alpha_i}$, where each $c_i \in \mathbb{C}$ and $-\frac{1}{2} < \alpha_0 < \alpha_1 < \ldots < \alpha_r$. Then, P is not a cyclic vector for D_ϕ in $L^2((0,1))$. Thus, if $Q(x) = \sum_{i=0}^r b_i x^{\beta_i}$, with $b_i \in \mathbb{C}$ and $\frac{1}{2} < \beta_0 < \beta_1 < \ldots < \beta_r$, then the function Q is not a cyclic vector for C_ϕ in $W_0((0,1))$.*

Proof. Note that

$$D_\phi^k(P) = \sum_{j=0}^r s^k c_j x^{s^k \alpha_j + s^k - 1}.$$

Now, we have $\operatorname{span}\{D_\phi^k, k \geqslant 0\} \subseteq \operatorname{span}\{x^{\lambda_k}, k \geqslant 0\}$ where $\{\lambda_k\}_k$ is a sequence of positive and distinct numbers such that there is $(i,j) \in \mathbb{N} \times \{0,\dots,r\}$, $\lambda_k = s^i\alpha_j + s^i - 1$. So, it follows from the full Müntz theorem in $L^2((0,1))$ that P is not a cyclic vector for D_ϕ if and only if $\sum_k \sum_{j=0}^r \frac{2(s^k\alpha_j + s^k) - 1}{(2s^k(\alpha_j+1)-1)^2+4}$ converges. Now, for each j, we see that

$$\frac{s^k\alpha_j + s^k}{(2s^k(\alpha_j+1)-1)^2+4} \underset{k\to+\infty}{\sim} \frac{\alpha_j+1}{4(\alpha_j+1)^2 s^k}$$
and
$$\frac{1}{(2s^k(\alpha_j+1)-1)^2+4} \underset{k\to+\infty}{\sim} \frac{1}{4(\alpha_j+1)^2 s^{2k}}.$$

We conclude that $\sum_k \sum_{j=0}^r \frac{2(s^k\alpha_j + s^k) - 1}{(2s^k(\alpha_j+1)-1)^2+4}$ converges and thus P is not a cyclic vector for D_ϕ.

To see that Q is not cyclic for C_ϕ in $W_0((0,1))$, we now set $P = Q'$ and use Lemma 3.1. □

Let $\mathscr{C}_1([0,1]) := \{f \in \mathscr{C}^1([0,1]), f(0) = 0\}$.

We then obtain a non-cyclicity result in $\mathscr{C}_1([0,1])$ for a special class of functions which can be written as $f(x) = bx + r(x)$, where $\frac{r(x)}{x} \to 0$ as x tends to 0 (see Theorem 3 in [7]). Recall that $\phi(x) = x^s$ for $s > 1$.

COROLLARY 4.1. *Let P be the function defined on $[0,1]$ by $P(x) = \sum_{i=0}^r c_i x^{\alpha_i}$, where each $c_i \in \mathbb{C}$ and $1 \leqslant \alpha_0 < \alpha_1 < \ldots < \alpha_r$. Then, P is not a cyclic vector for C_ϕ in $\mathscr{C}_1((0,1))$.*

Proof. The non-cyclicity of P follows easily from Proposition 4.2 and the following inequalities:

$$\|f\|_{W_0((0,1))} := \|f'\|_{L^2((0,1))} \leqslant \|f\|_\infty + \|f'\|_\infty =: \|f\|_{\mathscr{C}_1([0,1])},$$

for all $f \in \mathscr{C}_1([0,1])$.

□

Acknowledgements

The authors are indebted to Isabelle Chalendar for many helpful discussions during the preparation of this paper.

REFERENCES

[1] P. BORWEIN AND T. ERDÉLYI, *The full Müntz theorem in $C[0,1]$ and $L_1[0,1]$*, J. London Math. Soc. (2), **54**, 1 (1996), 102–110.

[2] S. R. CARADUS, *Universal operators and invariant subspaces*, Proc. Amer. Math. Soc., **23**, 1 (1969), 526–527.

[3] I. CHALENDAR AND J. ESTERLE, *Le problème du sous-espace invariant*, Development of mathematics 1950–2000, pages 235–267, Birkhäuser, Basel, 2000.

[4] E. NORDGREN, P. ROSENTHAL, AND F. S. WINTROBE, *Invertible composition operators on H^p*, J. Funct. Anal., **73**, 2 (1987), 324–344.

[5] W. C. RIDGE, *Approximate point spectrum of a weighted shift*, Trans. Amer. Math. Soc., **147** (1970), 349–356.

[6] H. RADJAVI AND P. ROSENTHAL, *Invariant subspaces*, Dover Publications Inc., Mineola, NY, second edition, 2003.

[7] A. SPALSBURY, *Perturbations in Müntz's theorem*, J. Approx. Theory, **150**, 1 (2008), 48–68.

(Received April 30, 2010)

Jonathan R. Partington
School of Mathematics, University of Leeds
Leeds LS2 9JT, U.K.
e-mail: J.R.Partington@leeds.ac.uk

Elodie Pozzi
Université Lyon, CNRS
Université Lyon 1, Institut Camille Jordan
43 boulevard du 11 novembre 1918
69622 Villeurbanne Cedex
France
e-mail: or-pozzi@math.univ-lyon1.fr