Operators
and
Matrices
Volume 5, Number 3 (2011), 469–472

SVEP AND BISHOP'S PROPERTY FOR $k*$–PARANORMAL OPERATORS

N. L. BRAHA AND K. TANAHASHI

(*Communicated by T. Furuta*)

Abstract. A bounded linear operator T on a complex Hilbert space $\mathscr{H}$ is said to be $k*$-paranormal if $\|T^*x\|^k \leqslant \|T^k x\|$ for every unit vector $x \in \mathscr{H}$ where k is a natural number with $2 \leqslant k$. This class of operators is an extension of hyponormal operators and have many interesting properties. We show that $k*$-paranormal operators have Bishop's property (β), i.e., if $f_n(\lambda)$ is an analytic function on some open set $\mathscr{D} \subset \mathbb{C}$ such that $(T-z)f_n(z) \to 0$ uniformly on every compact subset $\mathscr{K} \subset \mathscr{D}$, then $f_n(z) \to 0$ uniformly on $\mathscr{K}$. In case of $k=2$, this means that $*$-paranormal operators have Bishop's property (β).

1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space and $B(\mathscr{H})$ the Banach algebra of all bounded linear operators on $\mathscr{H}$. Let $T \in B(\mathscr{H})$.

Bishop's property (β) is an important property in the operator theory and it is known that many operators have this property. There are several important Hilbert space operator classes as follows:

(1) hyponormal : $TT^* \leqslant T^*T$.
(2) paranormal : $\|Tx\|^2 \leqslant \|T^2x\|$ for all unit vectors $x \in \mathscr{H}$.
(3) $*$-paranormal : $\|T^*x\|^2 \leqslant \|T^2x\|$ for all unit vectors $x \in \mathscr{H}$
where k is a natural number with $2 \leqslant k$.
(4) $k*$-paranormal : $\|T^*x\|^k \leqslant \|T^kx\|$ for all unit vectors $x \in \mathscr{H}$.
(5) normaloid : $\|T\| = r(T)$ (spectral radius).

Hyponormal operators are paranormal and $*$-paranormal. There are no relation for paranormal operators and $*$-paranormal operators. Paranormal operators are normaloid and $*$-paranormal operators are normaloid. If $k=2$, then $k*$-paranormal operators means $*$-paranormal ([1], [2], [4], [8]).

The class of paranormal operators was defined by Istrăţescu, Saitō and Yoshino [4] as class (N) and they proved that class (N) operators are normaloid. Furuta [2] renamed this class from class (N) to paranormal. The class of $*$-paranormal operators was defined by S.M. Patel [8]. S.C. Arora and J.K. Thukral [1] proved that if T is $*$-paranormal, then $\ker(T-\lambda) \subset \ker(T-\lambda)^*$ for all $\lambda \in \mathbb{C}$. The class of $k*$-paranormal operators was defined by M.Y. Lee, S.H. Lee and C.S. Ryoo [6] and they proved that if T is $k*$-paranormal, then $\ker(T-\lambda) \subset \ker(T-\lambda)^*$ for all $\lambda \in \mathbb{C}$.

Mathematics subject classification (2010): Primary 47A10, 47B20.
Keywords and phrases: $*$-paranormal, $k*$-paranormal, SVEP, Bishop's property (β).

To study non-normal operator T, it is important to know that T has single valued extension property (SVEP) and Bishop's property (β).

T is said to have SVEP if $f(z)$ is an analytic vector valued function on some open set $\mathscr{D} \subset \mathbb{C}$ such that $(T-z)f(z)=0$ for all $z \in \mathscr{D}$, then $f(z)=0$ for all $z \in \mathscr{D}$.

T is said to have Bishop's property (β) if $f_n(z)$ is an analytic vector valued function on some open set $\mathscr{D} \subset \mathbb{C}$ such that $(T-z)f_n(z) \to 0$ uniformly on each compact subset $\mathscr{K} \subset \mathscr{D}$, then $f_n(z) \to 0$ uniformly on $\mathscr{K}$.

Hence if T has Bishop's property (β), then T has SVEP. K.B. Laursen [5] proved that if T is totally paranormal, i.e., $T-\lambda$ is paranormal for all $\lambda \in \mathbb{C}$, then T has SVEP. Recently, A. Uchiyama and K. Tanahashi [9] proved that paranormal operators have Bishop's property (β). Y.M. Han and A.H. Kim [3] proved that if T is $*$-paranormal, then $\ker(T-\lambda) = \ker(T-\lambda)^2$ for all $\lambda \in \mathbb{C}$ and T has SVEP. It is known that if T is the unilateral shift on ℓ^2, then T^* is normaloid and does not have SVEP.

In this paper, we show that $k*$-paranormal operators have Bishop's property (β). In case of $k=2$, this means that $*$-paranormal operators have Bishop's property (β).

2. Main Results

THEOREM 1. *$k*$-paranormal operators have Bishop's property (β).*

Proof. Let $\sigma_a(T)$ be the approximate point spectrum of T. Uchiyama and Tanahashi [9] defined the spectral properties (I), (I′) and (II) as follows: T has the property

(I) if $\lambda \in \sigma_a(T)$ and $(T-\lambda)x_n \to 0$ for some sequence of bounded vectors $\{x_n\} \subset \mathscr{H}$, then $(T-\lambda)^* x_n \to 0$,

(I′) if $\lambda \in \sigma_a(T) \setminus \{0\}$ and $(T-\lambda)x_n \to 0$ for some sequence of bounded vectors $\{x_n\} \subset \mathscr{H}$, then $(T-\lambda)^* x_n \to 0$,

(II) if $\lambda,\ \mu \in \sigma_a(T)$ $(\lambda \neq \mu)$ and $(T-\lambda)x_n \to 0, (T-\mu)y_n \to 0$ for some sequence of bounded vectors $\{x_n, y_n\} \subset \mathscr{H}$, then $\langle x_n, y_n \rangle \to 0$,

and they proved that

(1) If T is paranormal, then T has the property (II).

(2) If T satisfies (II), then T has Bishop's proprty (β).

Hence paranormal operators have Bishop's property (β) by (1) and (2).

Let T be a $k*$-paranormal operator. We show that T has the property (I). Since (I) implies (II) by Lemma 2.1 of [9], we have that T has Bishop's property (β).

Let $(T-\lambda)x_n \to 0$. We may assume that $\|x_n\| = 1$. Since T is $k*$-paranormal,

$$\|T^* x_n\|^k \leqslant \|T^k x_n\|.$$

Since

$$T^k = (T-\lambda)^k + {}_kC_1 \lambda (T-\lambda)^{k-1} + \cdots + {}_kC_{k-1} \lambda^{k-1} (T-\lambda) + \lambda^k,$$

we have

$$\begin{aligned}\|T^*x_n\|^k &\leqslant \|T^kx_n\| \\ &\leqslant \|(T-\lambda)^kx_n\| + {}_kC_1|\lambda|\|(T-\lambda)^{k-1}x_n\| + \cdots \\ &\quad + {}_kC_{k-1}|\lambda|^{k-1}\|(T-\lambda)x_n\| + |\lambda|^k\end{aligned}$$

and

$$\limsup_{n\to\infty}\|T^*x_n\| \leqslant |\lambda|.$$

Hence

$$\begin{aligned}\|(T-\lambda)^*x_n\|^2 &= \langle T^*x_n, T^*x_n\rangle - \overline{\lambda}\langle x_n, T^*x_n\rangle - \lambda\langle T^*x_n, x_n\rangle + |\lambda|^2 \\ &= \|T^*x_n\|^2 - \overline{\lambda}\langle Tx_n, x_n\rangle - \lambda\langle x_n, Tx_n\rangle + |\lambda|^2 \\ &= \|T^*x_n\|^2 - \overline{\lambda}\langle (T-\lambda)x_n, x_n\rangle - \lambda\langle x_n, (T-\lambda)x_n\rangle - |\lambda|^2\end{aligned}$$

and

$$\limsup_{n\to\infty}\|(T-\lambda)^*x_n\|^2 \leqslant |\lambda|^2 - |\lambda|^2 = 0.$$

This implies $(T-\lambda)^*x_n \to 0$ and thus T has the property (I). $\square$

COROLLARY 2. *$*$-paranormal operators have Bishop's property (β).*

REMARK 3. *S. H. Lee, C. S. Ryoo* [7] *proved that $*$-paranormal operators have the property (I).*

Acknowledgment

The authors would like to express their sincere thanks to the referee and the editor for their helpful comments.

This research was supported by Grant-in-Aid Scientific Research No. 20540184.

REFERENCES

[1] S. C. ARORA AND J. K. THUKRAL, *On the class of operators*, Glasnik Math., **21** (1986), 381–386.
[2] T. FURUTA, *On the class of paranormal operators*, Proc. Japan Acad., **43** (1967), 594–598.
[3] Y. M. HAN AND A. H. KIM, *A note on $*$-paranormal operators*, Integral Equations and Operator Theory, **49** (2004), 435–444.
[4] V. ISTRĂŢESCU, T. SAITŌ AND T. YOSHINO, *On a class of operators*, Tôhoku Math. J., (2), **18** (1966), 410-413.
[5] K. B. LAURSEN, *Operators with finite ascent*, Pacific J. Math., **152** (1992), 323–336.
[6] M. Y. LEE, S. H. LEE AND C. S. RHOO, *Some remarks on the structure of $k*$-paranormal operators*, Kyungpook Math. J., **35** (1995), 205–211.

[7] S. H. LEE AND C. S. RHOO, *Some properties of certain nonhyponormal operators*, Bull. Korean Math. Soc., **31** (1994), 133–141.
[8] S. M. PATEL, *Contributions to the study of spectraloid operators,* Ph. D. Thesis, Delhi Univ., 1974.
[9] A. UCHIYAMA AND K. TANAHASHI, *Bishop's property (β) for paranormal operators*, Operators and Matrices, **3** (2009), 517–524.

(Received July 28, 2010)

N.L. Braha
Department of Mathematics and Computer Sciences
Avenue Mother Theresa 5
Prishtine 100
Kosovo
e-mail: nbraha@yahoo.com

Kotaro Tanahashi
Department of Mathematics
Tohoku Pharmaceutical University
Sendai 981-8558
Japan
e-mail: tanahasi@tohoku-pharm.ac.jp