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THE HAUSDORFF MEASURE OF NONCOMPACTNESS OF MATRIX OPERATORS ON SOME BK SPACES

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Abstract. In the present paper, we establish some identities or estimates for the operator norms and the Hausdorff measures of noncompactness of certain matrix operators on the spaces c_0^λ and ℓ_∞^λ which have recently been introduced in [On the spaces of λ-convergent and bounded sequences, Thai J. Math. 8(2) (2010) 311–329]. Further, by using the Hausdorff measure of noncompactness, we characterize some classes of compact operators on these spaces.

1. Background, notations and preliminaries

It seems to be quite natural, in view of the fact that matrix mappings between BK spaces are continuous, to find necessary and sufficient conditions for the entries of an infinite matrix to define a compact operator between BK spaces. This can be achieved, in many cases, by applying the Hausdorff measure of noncompactness (see for example [3, 5, 6] with references given there). In this section, we give some related definitions, notations and preliminary results.

1.1. Compact operators and matrix transformations

Let X and Y be *Banach spaces*. Then, we write $\mathscr{B}(X,Y)$ for the set of all bounded (continuous) linear operators $L: X \to Y$, which is a Banach space with the operator norm given by $\|L\| = \sup_{x\in S_X} \|L(x)\|_Y$ for all $L \in \mathscr{B}(X,Y)$, where S_X denotes the unit sphere in X, that is $S_X = \{x \in X : \|x\| = 1\}$. A linear operator $L : X \to Y$ is said to be *compact* if the domain of L is all of X and for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a subsequence which converges in Y. By $\mathscr{C}(X,Y)$, we denote the class of all compact operators in $\mathscr{B}(X,Y)$. An operator $L \in \mathscr{B}(X,Y)$ is said to be *of finite rank* if $\dim R(L) < \infty$, where $R(L)$ is the range space of L. An operator of finite rank is clearly compact.

By w, we shall denote the space of all complex sequences. If $x \in w$, then we write $x = (x_k)$ instead of $x = (x_k)_{k=0}^\infty$. Also, we write ϕ for the set of all finite sequences that terminate in zeros. Further, we shall use the conventions that $e = (1,1,1,\ldots)$ and $e^{(n)}$

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is the sequence whose only non-zero term is 1 in the n^{th} place for each $n \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Any vector subspace of w is called a *sequence space*. We shall write ℓ_∞, c and c_0 for the sequence spaces of all bounded, convergent and null sequences, respectively. Further, by cs and ℓ_1 we denote the spaces of all sequences associated with convergent and absolutely convergent series, respectively.

The β-dual of a subset X of w is defined by

$$X^\beta = \{a = (a_k) \in w : \ ax = (a_k x_k) \in cs \text{ for all } x = (x_k) \in X\}.$$

Throughout this paper, the matrices are infinite matrices of complex numbers. If A is an infinite matrix with complex entries a_{nk} $(n, k \in \mathbb{N})$, then we write $A = (a_{nk})$ instead of $A = (a_{nk})_{n,k=0}^\infty$. Also, we write A_n for the sequence in the n^{th} row of A, that is $A_n = (a_{nk})_{k=0}^\infty$ for every $n \in \mathbb{N}$. In addition, if $x = (x_k) \in w$ then we define the A-*transform* of x as the sequence $Ax = (A_n(x))_{n=0}^\infty$, where

$$A_n(x) = \sum_{k=0}^\infty a_{nk} x_k; \quad (n \in \mathbb{N}) \tag{1.1}$$

provided the series on the right converges for each $n \in \mathbb{N}$.

For arbitrary subsets X and Y of w, we write (X, Y) for the class of all infinite matrices that map X into Y. Thus $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for all $n \in \mathbb{N}$ and $Ax \in Y$ for all $x \in X$. Moreover, the *matrix domain* of an infinite matrix A in X is defined by

$$X_A = \{x \in w : Ax \in X\}.$$

The theory of BK spaces is the most powerful tool in the characterization of matrix transformations between sequence spaces.

A sequence space X is called a BK *space* if it is a Banach space with continuous coordinates $p_n : X \to \mathbb{C}$ $(n \in \mathbb{N})$, where $\mathbb{C}$ denotes the complex field and $p_n(x) = x_n$ for all $x = (x_k) \in X$ and every $n \in \mathbb{N}$ [1, p.255].

The sequence spaces ℓ_∞, c and c_0 are BK spaces with the same sup-norm given by $\|x\|_{\ell_\infty} = \sup_k |x_k|$, where the supremum is taken over all $k \in \mathbb{N}$. Also, the space ℓ_1 is a BK space with the usual ℓ_1-norm defined by $\|x\|_{\ell_1} = \sum_{k=0}^\infty |x_k|$ [11, p.55].

If $X \supset \phi$ is a BK space and $a = (a_k) \in w$, then we write

$$\|a\|_X^* = \sup_{x \in S_X} \left| \sum_{k=0}^\infty a_k x_k \right| \tag{1.2}$$

provided the expression on the right is defined and finite [1, p.257]; which is the case whenever $a \in X^\beta$ [11, Theorem 7.2.9].

An infinite matrix $T = (t_{nk})$ is called a *triangle* if $t_{nn} \neq 0$ and $t_{nk} = 0$ for all $k > n$ $(n \in \mathbb{N})$. The study of matrix domains of triangles in sequence spaces has a special importance due to the various properties which they have. For example, if X is a BK space then X_T is also a BK space with the norm given by $\|x\|_{X_T} = \|Tx\|_X$ for all $x \in X_T$ [4, Lemma 3 (i)].

The following known results are fundamental for our investigation.

LEMMA 1.1. [4, Lemma 6] *Let X denote any of the spaces c_0, c or ℓ_∞. Then, we have $X^\beta = \ell_1$ and $\|a\|_X^* = \|a\|_{\ell_1}$ for all $a \in \ell_1$.*

LEMMA 1.2. [7, Lemma 15 (a)] *Let X and Y be BK spaces. Then, we have $(X,Y) \subset \mathscr{B}(X,Y)$, that is, every matrix $A \in (X,Y)$ defines an operator $L_A \in \mathscr{B}(X,Y)$ by $L_A(x) = Ax$ for all $x \in X$.*

LEMMA 1.3. [3, Lemma 2.2] *Let $X \supset \phi$ be a BK space and Y be any of the spaces c_0, c or ℓ_∞. If $A \in (X,Y)$, then we have*

$$\|L_A\| = \|A\|_{(X,\ell_\infty)} = \sup_n \|A_n\|_X^* < \infty.$$

LEMMA 1.4. [9, Theorem 1.6] *Let T be a triangle. Then, we have*

(a) *For arbitrary subsets X and Y of w, $A \in (X,Y_T)$ if and only if $B = TA \in (X,Y)$.*

(b) *Further, if X and Y are BK spaces and $A \in (X,Y_T)$, then $\|L_A\| = \|L_B\|$.*

1.2. The Hausdorff measure of noncompactness

Let S and M be subsets of a metric space (X,d) and $\varepsilon > 0$. Then S is called an ε*-net* of M in X if for every $x \in M$ there exists $s \in S$ such that $d(x,s) < \varepsilon$. Further, if the set S is finite, then the ε-net S of M is called a *finite ε-net* of M, and we say that M has a finite ε-net in X. A subset of a metric space is said to be *totally bounded* if it has a finite ε-net for every $\varepsilon > 0$.

By $\mathscr{M}_X$, we denote the collection of all bounded subsets of a metric space (X,d). If $Q \in \mathscr{M}_X$, then the *Hausdorff measure of noncompactness* of the set Q, denoted by $\chi(Q)$, is defined by

$$\chi(Q) = \inf\{\varepsilon > 0 : Q \text{ has a finite } \varepsilon\text{-net in } X\}.$$

The function $\chi : \mathscr{M}_X \to [0,\infty)$ is called the *Hausdorff measure of noncompactness* [9, p.2543].

The basic properties of the Hausdorff measure of noncompactness can be found in [3, p.46] and [7, Lemma 2]. For example, if Q, Q_1 and Q_2 are bounded subsets of a metric space X, then we have

$$\chi(Q) = 0 \text{ if and only if } Q \text{ is totally bounded,}$$

$$Q_1 \subset Q_2 \text{ implies } \chi(Q_1) \leqslant \chi(Q_2).$$

Further, if X is a normed space then the function χ has some additional properties connected with the linear structure, e.g.

$$\chi(Q_1 + Q_2) \leqslant \chi(Q_1) + \chi(Q_2),$$

$$\chi(\alpha Q) = |\alpha|\chi(Q) \text{ for all } \alpha \in \mathbb{C}.$$

The following result shows how to compute the Hausdorff measure of noncompactness in the BK space c_0.

LEMMA 1.5. [3, Lemma 3.5] *Let* $Q \in \mathscr{M}_{c_0}$ *and* $P_r : c_0 \to c_0$ $(r \in \mathbb{N})$ *be the operator defined by* $P_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots)$ *for all* $x = (x_k) \in c_0$. *Then, we have*

$$\chi(Q) = \lim_{r\to\infty}\left(\sup_{x\in Q}\|(I-P_r)(x)\|_{\ell_\infty}\right),$$

where I *is the identity operator on* c_0.

Further, it is known that every sequence $z = (z_n) \in c$ has a unique representation $z = \bar{z}e + \sum_{n=0}^{\infty}(z_n - \bar{z})e^{(n)}$, where $\bar{z} = \lim_{n\to\infty} z_n$ [7, Lemma 10]. Thus, the operator $P_r : c \to c$ $(r \in \mathbb{N})$, defined by

$$P_r(z) = \bar{z}e + \sum_{n=0}^{r}(z_n - \bar{z})e^{(n)}; \quad (r \in \mathbb{N}) \tag{1.3}$$

for all $z = (z_n) \in c$ with $\bar{z} = \lim_{n\to\infty} z_n$, is called the projector onto the linear span of $\{e, e^{(0)}, e^{(1)}, \ldots, e^{(r)}\}$. In this situation, the following result gives an estimate for the Hausdorff measure of noncompactness in the BK space c.

LEMMA 1.6. [7, Theorem 5 (b)] *Let* $Q \in \mathscr{M}_c$ *and* $P_r : c \to c$ $(r \in \mathbb{N})$ *be the projector onto the linear span of* $\{e, e^{(0)}, e^{(1)}, \ldots, e^{(r)}\}$. *Then, we have*

$$\frac{1}{2}\cdot\lim_{r\to\infty}\left(\sup_{x\in Q}\|(I-P_r)(x)\|_{\ell_\infty}\right) \leqslant \chi(Q) \leqslant \lim_{r\to\infty}\left(\sup_{x\in Q}\|(I-P_r)(x)\|_{\ell_\infty}\right),$$

where I *is the identity operator on* c.

Moreover, we have the following result concerning with the Hausdorff measure of noncompactness in the matrix domains of triangles in normed sequence spaces.

LEMMA 1.7. [2, Theorem 2.6] *Let* X *be a normed sequence space,* T *a triangle and* χ_T *and* χ *denote the Hausdorff measures of noncompactness on* $\mathscr{M}_{X_T}$ *and* $\mathscr{M}_X$, *the collections of all bounded sets in* X_T *and* X, *respectively. Then* $\chi_T(Q) = \chi(T(Q))$ *for all* $Q \in \mathscr{M}_{X_T}$.

The most effective way in the characterization of compact operators between the Banach spaces is by applying the Hausdorff measure of noncompactness. This can be achieved as follows (see [3, Lemma 3.3]):

Let X and Y be Banach spaces and $L \in \mathscr{B}(X,Y)$. Then, the *Hausdorff measure of noncompactness* of L, denoted by $\|L\|_\chi$, is defined by

$$\|L\|_\chi = \chi(L(S_X)) \tag{1.4}$$

and we have

$$L \text{ is compact if and only if } \|L\|_\chi = 0. \tag{1.5}$$

REMARK 1.8. Let X and Y be BK spaces, T a triangle, $A \in (X, Y_T)$ and $B = TA$. Then, we have $Bx = (TA)x = T(Ax)$ for all $x \in X$. Thus, by combining (1.4) with Lemmas 1.4 and 1.7, we can reduce the evaluation of $\|L_A\|_\chi$ to that of $\|L_B\|_\chi$ (see [9, Remark 2.4]).

2. The sequence spaces c_0^λ and ℓ_∞^λ

Throughout this paper, let $\lambda = (\lambda_k)_{k=0}^\infty$ be a strictly increasing sequence of positive reals tending to infinity, that is $0 < \lambda_0 < \lambda_1 < \cdots$ and $\lambda_k \to \infty$ as $k \to \infty$.

By using the convention $\lambda_{-1} = 0$, we define the infinite matrix $\Lambda = (\lambda_{nk})_{n,k=0}^\infty$ by

$$\lambda_{nk} = \begin{cases} \dfrac{\lambda_k - \lambda_{k-1}}{\lambda_n}; & (0 \leqslant k \leqslant n), \\ 0; & (k > n). \end{cases} \tag{2.1}$$

Recently, the sequence spaces c_0^λ and ℓ_∞^λ have been introduced in [8] as the matrix domains of the triangle Λ in the spaces c_0 and ℓ_∞, respectively.

It is obvious that c_0^λ and ℓ_∞^λ are BK spaces with the same norm given by

$$\|x\|_{\ell_\infty^\lambda} = \|\Lambda(x)\|_{\ell_\infty} = \sup_n |\Lambda_n(x)|. \tag{2.2}$$

Throughout, for any sequence $x = (x_k) \in w$, we define the *associated sequence* $y = (y_k)$, which will frequently be used, as the Λ-transform of x, i.e., $y = \Lambda(x)$ and so

$$y_k = \sum_{j=0}^{k} \left(\frac{\lambda_j - \lambda_{j-1}}{\lambda_k}\right) x_j; \quad (k \in \mathbb{N}). \tag{2.3}$$

Further, let X be any of the spaces c_0^λ or ℓ_∞^λ and Y be the respective one of the spaces c_0 or ℓ_∞. If the sequences x and y are connected by the relation (2.3), then $x \in X$ if and only if $y \in Y$, furthermore if $x \in X$, then $\|x\|_{\ell_\infty^\lambda} = \|y\|_{\ell_\infty}$. In fact, the linear operator $L_\Lambda : X \to Y$, which maps every sequence in X to its associated sequence in Y, is bijective and norm preserving.

The β-duals of the spaces c_0^λ and ℓ_∞^λ have been determined and some related matrix mappings characterized. We refer the reader to [8] for relevant terminology.

Moreover, the following results are essential for our study and we may begin with the following lemma which is immediate by [2, Theorem 1.6; Remark 1.7] and [8, Theorem 5.5].

LEMMA 2.1. *Let X be any of the spaces c_0^λ or ℓ_∞^λ. If $a = (a_k) \in X^\beta$, then $\tilde{a} = (\tilde{a}_k) \in \ell_1$ and the equality*

$$\sum_{k=0}^\infty a_k x_k = \sum_{k=0}^\infty \tilde{a}_k y_k \tag{2.4}$$

holds for every $x = (x_k) \in X$, where $y = \Lambda(x)$ is the associated sequence defined by (2.3) *and*

$$\tilde{a}_k = \left(\frac{a_k}{\lambda_k - \lambda_{k-1}} - \frac{a_{k+1}}{\lambda_{k+1} - \lambda_k}\right)\lambda_k; \quad (k \in \mathbb{N}).$$

LEMMA 2.2. *Let X be any of the spaces c_0^λ or ℓ_∞^λ. Then, we have*

$$\|a\|_X^* = \|\tilde{a}\|_{\ell_1} = \sum_{k=0}^\infty |\tilde{a}_k| < \infty$$

for all $a=(a_k)\in X^\beta$, *where* $\tilde{a}=(\tilde{a}_k)$ *is as in Lemma* 2.1.

Proof. Let Y be the respective one of the spaces c_0 or ℓ_∞, and take any $a=(a_k)\in X^\beta$. Then, we have from Lemma 2.1 that $\tilde{a}=(\tilde{a}_k)\in\ell_1$ and the equality (2.4) holds for all sequences $x=(x_k)\in X$ and $y=(y_k)\in Y$ which are connected by the relation (2.3). Also, it follows by (2.2) that $x\in S_X$ if and only if $y\in S_Y$. Therefore, we derive from (1.2) and (2.4) that

$$\|a\|_X^* = \sup_{x\in S_X}\left|\sum_{k=0}^{\infty} a_k x_k\right| = \sup_{y\in S_Y}\left|\sum_{k=0}^{\infty}\tilde{a}_k y_k\right| = \|\tilde{a}\|_Y^*.$$

Further, since $\tilde{a}\in\ell_1$, we obtain from Lemma 1.1 that

$$\|a\|_X^* = \|\tilde{a}\|_Y^* = \|\tilde{a}\|_{\ell_1} < \infty$$

which concludes the proof. □

Throughout this paper, we shall use the following notation:
For an infinite matrix $A=(a_{nk})$, we define the *associated matrix* $\tilde{A}=(\tilde{a}_{nk})$ by

$$\tilde{a}_{nk} = \left(\frac{a_{nk}}{\lambda_k-\lambda_{k-1}} - \frac{a_{n,k+1}}{\lambda_{k+1}-\lambda_k}\right)\lambda_k; \quad (n,k\in\mathbb{N}). \tag{2.5}$$

Then, we have

LEMMA 2.3. *Let* X *be any of the spaces* c_0^λ *or* ℓ_∞^λ, Y *the respective one of the spaces* c_0 *or* ℓ_∞, Z *a sequence space and* $A=(a_{nk})$ *an infinite matrix. If* $A\in(X,Z)$, *then* $\tilde{A}\in(Y,Z)$ *such that* $Ax=\tilde{A}y$ *for all sequences* $x\in X$ *and* $y\in Y$ *which are connected by the relation* (2.3), *where* $\tilde{A}=(\tilde{a}_{nk})$ *is the associated matrix defined by* (2.5).

Proof. Let $x\in X$ and $y\in Y$ be connected by the relation (2.3) and suppose that $A\in(X,Z)$. Then $A_n\in X^\beta$ for all $n\in\mathbb{N}$. Thus, it follows by Lemma 2.1 that $\tilde{A}_n\in\ell_1=Y^\beta$ for all $n\in\mathbb{N}$ and the equality $Ax=\tilde{A}y$ holds, hence $\tilde{A}y\in Z$. Since every $y\in Y$ is the associated sequence of some $x\in X$, we deduce that $\tilde{A}\in(Y,Z)$. This completes the proof. □

Finally, we conclude this section with the following result on operator norms.

LEMMA 2.4. *Let* X *be any of the spaces* c_0^λ *or* ℓ_∞^λ, $A=(a_{nk})$ *an infinite matrix and* $\tilde{A}=(\tilde{a}_{nk})$ *the associated matrix. If* A *is in any of the classes* (X,c_0), (X,c) *or* (X,ℓ_∞), *then*

$$\|L_A\| = \|A\|_{(X,\ell_\infty)} = \sup_n\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}|\right) < \infty.$$

Proof. This is immediate by combining Lemmas 1.3 and 2.2. □

REMARK 2.5. The characterization of matrix classes, considered in this paper, can be found in [8]. Thus, we shall omit it and only deal with the operator norms and the Hausdorff measures of noncompactness of some operators which are given by infinite matrices in such classes.

3. Compact operators on the spaces c_0^λ and ℓ_∞^λ

In the present section, we establish some identities or estimates for the Hausdorff measures of noncompactness of certain matrix operators on the spaces c_0^λ and ℓ_∞^λ. Further, we apply our results to characterize some classes of compact operators on those spaces.

We may begin with quoting the following lemma which is immediate by [9, Lemma 3.1].

LEMMA 3.1. *Let X denote any of the spaces c_0 or ℓ_∞. If $A \in (X,c)$, then we have*

$$\alpha_k = \lim_{n\to\infty} a_{nk} \text{ exists for every } k \in \mathbb{N},$$

$$\alpha = (\alpha_k) \in \ell_1,$$

$$\sup_n \left(\sum_{k=0}^{\infty} |a_{nk} - \alpha_k| \right) < \infty,$$

$$\lim_{n\to\infty} A_n(x) = \sum_{k=0}^{\infty} \alpha_k x_k \text{ for all } x = (x_k) \in X.$$

Now, let $A = (a_{nk})$ be an infinite matrix and $\tilde{A} = (\tilde{a}_{nk})$ the associated matrix defined by (2.5). Then, we have the following result on Hausdorff measures of noncompactness.

THEOREM 3.2. *Let X denote any of the spaces c_0^λ or ℓ_∞^λ. Then, we have*
(a) *If $A \in (X, c_0)$, then*

$$\|L_A\|_\chi = \limsup_{n\to\infty} \left(\sum_{k=0}^{\infty} |\tilde{a}_{nk}| \right). \tag{3.1}$$

(b) *If $A \in (X,c)$, then*

$$\frac{1}{2} \cdot \limsup_{n\to\infty} \left(\sum_{k=0}^{\infty} |\tilde{a}_{nk} - \tilde{\alpha}_k| \right) \leqslant \|L_A\|_\chi \leqslant \limsup_{n\to\infty} \left(\sum_{k=0}^{\infty} |\tilde{a}_{nk} - \tilde{\alpha}_k| \right), \tag{3.2}$$

where $\tilde{\alpha}_k = \lim_{n\to\infty} \tilde{a}_{nk}$ for all $k \in \mathbb{N}$.
(c) *If $A \in (X, \ell_\infty)$, then*

$$0 \leqslant \|L_A\|_\chi \leqslant \limsup_{n\to\infty} \left(\sum_{k=0}^{\infty} |\tilde{a}_{nk}| \right). \tag{3.3}$$

Proof. Let us remark that the expressions in (3.1) and (3.3) exist by Lemma 2.4. Also, by combining Lemmas 2.3 and 3.1, we deduce that the expression in (3.2) exists.

We write $S = S_X$, for short. Then, we obtain by (1.4) and Lemma 1.2 that

$$\|L_A\|_\chi = \chi(AS). \tag{3.4}$$

For (a), we have $AS \in \mathscr{M}_{c_0}$. Thus, it follows by applying Lemma 1.5 that

$$\chi(AS) = \lim_{r\to\infty}\left(\sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty}\right), \tag{3.5}$$

where $P_r : c_0 \to c_0$ $(r \in \mathbb{N})$ is the operator defined by $P_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots)$ for all $x = (x_k) \in c_0$. This yields that $\|(I-P_r)(Ax)\|_{\ell_\infty} = \sup_{n>r}|A_n(x)|$ for all $x \in X$ and every $r \in \mathbb{N}$. Therefore, by using (1.1), (1.2) and Lemma 2.2, we have for every $r \in \mathbb{N}$ that

$$\sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty} = \sup_{n>r}\|A_n\|_X^* = \sup_{n>r}\|\tilde{A}_n\|_{\ell_1}.$$

This and (3.5) imply that

$$\chi(AS) = \lim_{r\to\infty}\left(\sup_{n>r}\|\tilde{A}_n\|_{\ell_1}\right) = \limsup_{n\to\infty}\|\tilde{A}_n\|_{\ell_1}.$$

Hence, we get (3.1) by (3.4).

To prove (b), we have $AS \in \mathscr{M}_c$. Thus, we are going to apply Lemma 1.6 to get an estimate for the value of $\chi(AS)$ in (3.4). For this, let $P_r : c \to c$ $(r \in \mathbb{N})$ be the projectors defined by (1.3). Then, we have for every $r \in \mathbb{N}$ that $(I-P_r)(z) = \sum_{n=r+1}^{\infty}(z_n - \bar{z})e^{(n)}$ and hence

$$\|(I-P_r)(z)\|_{\ell_\infty} = \sup_{n>r}|z_n - \bar{z}| \tag{3.6}$$

for all $z = (z_n) \in c$ and every $r \in \mathbb{N}$, where $\bar{z} = \lim_{n\to\infty} z_n$ and I is the identity operator on c.

Now, by using (3.4), we obtain by applying Lemma 1.6 that

$$\frac{1}{2}\cdot\lim_{r\to\infty}\left(\sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty}\right) \leqslant \|L_A\|_\chi \leqslant \lim_{r\to\infty}\left(\sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty}\right). \tag{3.7}$$

On the other hand, it is given that $X = c_0^\lambda$ or $X = \ell_\infty^\lambda$, and let Y be the respective one of the spaces c_0 or ℓ_∞. Also, for every given $x \in X$, let $y \in Y$ be the associated sequence defined by (2.3). Since $A \in (X,c)$, we have by Lemma 2.3 that $\tilde{A} \in (Y,c)$ and $Ax = \tilde{A}y$. Further, it follows from Lemma 3.1 that the limits $\tilde{\alpha}_k = \lim_{n\to\infty}\tilde{a}_{nk}$ exist for all k, $\tilde{\alpha} = (\tilde{\alpha}_k) \in \ell_1 = Y^\beta$ and $\lim_{n\to\infty}\tilde{A}_n(y) = \sum_{k=0}^{\infty}\tilde{\alpha}_k y_k$. Consequently, we derive from (3.6) that

$$\begin{aligned}\|(I-P_r)(Ax)\|_{\ell_\infty} &= \|(I-P_r)(\tilde{A}y)\|_{\ell_\infty}\\ &= \sup_{n>r}\left|\tilde{A}_n(y) - \sum_{k=0}^{\infty}\tilde{\alpha}_k y_k\right|\\ &= \sup_{n>r}\left|\sum_{k=0}^{\infty}(\tilde{a}_{nk} - \tilde{\alpha}_k)y_k\right|\end{aligned}$$

for all $r \in \mathbb{N}$. Moreover, since $x \in S = S_X$ if and only if $y \in S_Y$, we obtain by (1.2) and

Lemma 1.1 that

$$\begin{aligned}\sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty} &= \sup_{n>r}\left(\sup_{y\in S_Y}\left|\sum_{k=0}^{\infty}(\tilde{a}_{nk}-\tilde{\alpha}_k)y_k\right|\right)\\ &= \sup_{n>r}\|\tilde{A}_n-\tilde{\alpha}\|_Y^*\\ &= \sup_{n>r}\|\tilde{A}_n-\tilde{\alpha}\|_{\ell_1}\end{aligned}$$

for all $r\in\mathbb{N}$. Hence, from (3.7) we get (3.2).

Finally, to prove (c) we define the operators $P_r:\ell_\infty\to\ell_\infty$ $(r\in\mathbb{N})$ as in the proof of part (a) for all $x=(x_k)\in\ell_\infty$. Then, we have

$$AS\subset P_r(AS)+(I-P_r)(AS);\quad (r\in\mathbb{N}).$$

Thus, it follows by the elementary properties of the function χ that

$$\begin{aligned}0\leqslant\chi(AS)&\leqslant\chi(P_r(AS))+\chi((I-P_r)(AS))\\ &=\chi((I-P_r)(AS))\\ &\leqslant\sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty}\\ &=\sup_{n>r}\|\tilde{A}_n\|_{\ell_1}\end{aligned}$$

for all $r\in\mathbb{N}$ and hence

$$\begin{aligned}0\leqslant\chi(AS)&\leqslant\lim_{r\to\infty}\left(\sup_{n>r}\|\tilde{A}_n\|_{\ell_1}\right)\\ &=\limsup_{n\to\infty}\|\tilde{A}_n\|_{\ell_1}.\end{aligned}$$

This and (3.4) together imply (3.3) and complete the proof. □

COROLLARY 3.3. *Let X denote any of the spaces c_0^λ or ℓ_∞^λ. Then, we have*
(a) *If $A\in(X,c_0)$, then*

$$L_A \text{ is compact if and only if } \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}|\right)=0.$$

(b) *If $A\in(X,c)$, then*

$$L_A \text{ is compact if and only if } \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}-\tilde{\alpha}_k|\right)=0,$$

where $\tilde{\alpha}_k=\lim_{n\to\infty}\tilde{a}_{nk}$ for all $k\in\mathbb{N}$.
(c) *If $A\in(X,\ell_\infty)$, then*

$$L_A \text{ is compact if } \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}|\right)=0. \tag{3.8}$$

Proof. This result follows from Theorem 3.2 by using (1.5). □

It is worth mentioning that the condition in (3.8) is only a sufficient condition for the operator L_A to be compact, where $A \in (X, \ell_\infty)$ and X is any of the spaces c_0^λ or ℓ_∞^λ. More precisely, the following example will show that it is possible for L_A to be compact while $\lim_{n\to\infty}(\sum_{k=0}^{\infty}|\tilde{a}_{nk}|) \neq 0$. Hence, in general, we have just 'if' in (3.8) of Corollary 3.3 (c).

EXAMPLE 3.4. Let X be any of the spaces c_0^λ or ℓ_∞^λ and define the matrix $A = (a_{nk})$ by $a_{n0} = 1$ and $a_{nk} = 0$ for $k \geqslant 1$ $(n \in \mathbb{N})$. Then, we have $Ax = x_0 e$ for all $x = (x_k) \in X$, hence $A \in (X, \ell_\infty)$. Also, it is obvious that L_A is of finite rank which yields that L_A is compact. On the other hand, by using (2.5), it can easily be seen that $\tilde{A} = A$. Thus $\tilde{A}_n = e^{(0)}$ and so $\|\tilde{A}_n\|_{\ell_1} = 1$ for all $n \in \mathbb{N}$. This implies that $\lim_{n\to\infty}\|\tilde{A}_n\|_{\ell_1} = 1$.

Finally, by using the notations of Lemma 1.2, we end this section with the following corollary:

COROLLARY 3.5. *We have $(\ell_\infty^\lambda, c_0) \subset \mathscr{C}(\ell_\infty^\lambda, c_0)$ and $(\ell_\infty^\lambda, c) \subset \mathscr{C}(\ell_\infty^\lambda, c)$, that is, for every matrix $A \in (\ell_\infty^\lambda, c_0)$ or $A \in (\ell_\infty^\lambda, c)$, the operator L_A is compact.*

Proof. Let $A \in (\ell_\infty^\lambda, c_0)$. Then, we have by Lemma 2.3 that $\tilde{A} \in (\ell_\infty, c_0)$ which implies that $\lim_{n\to\infty}(\sum_{k=0}^{\infty}|\tilde{a}_{nk}|) = 0$ [10, p.4]. This leads us with Corollary 3.3 (a) to the consequence that L_A is compact. Similarly, if $A \in (\ell_\infty^\lambda, c)$ then $\tilde{A} \in (\ell_\infty, c)$ and so $\lim_{n\to\infty}(\sum_{k=0}^{\infty}|\tilde{a}_{nk} - \tilde{\alpha}_k|) = 0$, where $\tilde{\alpha}_k = \lim_{n\to\infty}\tilde{a}_{nk}$ for all k. Hence, we deduce from Corollary 3.3 (b) that $L_A \in \mathscr{C}(\ell_\infty^\lambda, c)$. □

4. Some applications

In this last section, we apply our previous results to derive some identities or estimates for the operator norms and the Hausdorff measures of noncompactness of certain matrix operators that map any of the spaces c_0^λ or ℓ_∞^λ into the matrix domains of triangles in the spaces c_0, c and ℓ_∞. Further, we deduce the necessary and sufficient (or only sufficient) conditions for such operators to be compact.

We assume throughout that $A = (a_{nk})$ is an infinite matrix and $T = (t_{nk})$ is a triangle, and we define the matrix $B = (b_{nk})$ by

$$b_{nk} = \sum_{m=0}^{n} t_{nm} a_{mk}; \quad (n, k \in \mathbb{N}), \tag{4.1}$$

that is $B = TA$ and hence

$$B_n = \sum_{m=0}^{n} t_{nm} A_m = \left(\sum_{m=0}^{n} t_{nm} a_{mk}\right)_{k=0}^{\infty}; \quad (n \in \mathbb{N}).$$

Further, let $\tilde{A} = (\tilde{a}_{nk})$ and $\tilde{B} = (\tilde{b}_{nk})$ be the associated matrices of A and B, respectively. Then, it can easily be seen that

$$\tilde{b}_{nk} = \sum_{m=0}^{n} t_{nm}\tilde{a}_{mk}; \quad (n,k \in \mathbb{N}), \tag{4.2}$$

hence

$$\tilde{B}_n = \sum_{m=0}^{n} t_{nm}\tilde{A}_m = \left(\sum_{m=0}^{n} t_{nm}\tilde{a}_{mk}\right)_{k=0}^{\infty}; \quad (n \in \mathbb{N}).$$

Moreover, we define the sequence $\tilde{\beta} = (\tilde{\beta}_k)_{k=0}^{\infty}$ by

$$\tilde{\beta}_k = \lim_{n\to\infty}\left(\sum_{m=0}^{n} t_{nm}\tilde{a}_{mk}\right); \quad (k \in \mathbb{N}) \tag{4.3}$$

provided the limits on the right exist for all $k \in \mathbb{N}$ which is the case whenever $A \in (c_0^\lambda, c_T)$ or $A \in (\ell_\infty^\lambda, c_T)$ by Lemmas 1.4 (a), 2.3 and 3.1.

Now, by using (4.1), (4.2) and (4.3), we have the following results:

THEOREM 4.1. *Let X be any of the spaces c_0^λ or ℓ_∞^λ, T a triangle and A an infinite matrix. If A is in any of the classes $(X,(c_0)_T)$, (X,c_T) or $(X,(\ell_\infty)_T)$, then*

$$\|L_A\| = \|A\|_{(X,(\ell_\infty)_T)} = \sup_n\left(\sum_{k=0}^{\infty}\left|\sum_{m=0}^{n} t_{nm}\tilde{a}_{mk}\right|\right) < \infty.$$

Proof. This is immediate by combining Lemmas 1.4 (b) and 2.4. □

THEOREM 4.2. *Let T be a triangle. If either $A \in (\ell_\infty^\lambda,(c_0)_T)$ or $A \in (\ell_\infty^\lambda,c_T)$, then L_A is compact.*

Proof. This result can similarly be proved as the proof of Corollary 3.5 by means of Lemmas 1.4 (a) and 1.7 and by using Remark 1.8. □

THEOREM 4.3. *Let T be a triangle. Then, we have*

(a) *If $A \in (c_0^\lambda,(c_0)_T)$, then*

$$\|L_A\|_\chi = \limsup_{n\to\infty}\left(\sum_{k=0}^{\infty}\left|\sum_{m=0}^{n} t_{nm}\tilde{a}_{mk}\right|\right)$$

and

$$L_A \text{ is compact if and only if } \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}\left|\sum_{m=0}^{n} t_{nm}\tilde{a}_{mk}\right|\right) = 0.$$

(b) *If* $A \in (c_0^\lambda, c_T)$, *then*

$$\frac{1}{2}\cdot\limsup_{n\to\infty}\left(\sum_{k=0}^{\infty}\left|\sum_{m=0}^{n} t_{nm}\tilde{a}_{mk}-\tilde{\beta}_k\right|\right)\leqslant\|L_A\|_\chi\leqslant\limsup_{n\to\infty}\left(\sum_{k=0}^{\infty}\left|\sum_{m=0}^{n} t_{nm}\tilde{a}_{mk}-\tilde{\beta}_k\right|\right)$$

and

$$L_A \text{ is compact if and only if } \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}\left|\sum_{m=0}^{n} t_{nm}\tilde{a}_{mk}-\tilde{\beta}_k\right|\right)=0.$$

(c) *If either* $A \in (c_0^\lambda, (\ell_\infty)_T)$ *or* $A \in (\ell_\infty^\lambda, (\ell_\infty)_T)$, *then*

$$0\leqslant\|L_A\|_\chi\leqslant\limsup_{n\to\infty}\left(\sum_{k=0}^{\infty}\left|\sum_{m=0}^{n} t_{nm}\tilde{a}_{mk}\right|\right)$$

and

$$L_A \text{ is compact if } \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}\left|\sum_{m=0}^{n} t_{nm}\tilde{a}_{mk}\right|\right)=0. \tag{4.4}$$

Proof. This is obtained from Theorem 3.2 and Corollary 3.3 by using Lemmas 1.4 (a) and 1.7. □

REMARK 4.4. As we have seen in Example 3.4, it can be shown that the equivalence in (4.4) of Theorem 4.3 (c) does not hold.

Now, it is obvious that Theorems 4.1, 4.2 and 4.3 have several consequences with any particular triangle T. For instance, let $\lambda' = (\lambda'_k)_{k=0}^\infty$ be a strictly increasing sequence of positive reals tending to infinity and $\Lambda' = (\lambda'_{nk})$ be the triangle defined by (2.1) with the sequence λ' instead of λ. Also, let $c_0^{\lambda'}$, $c^{\lambda'}$ and $\ell_\infty^{\lambda'}$ be the matrix domains of the triangle Λ' in the spaces c_0, c and ℓ_∞, respectively. Then, the following corollaries are immediate by Theorems 4.1 and 4.2.

COROLLARY 4.5. *Let X be any of the spaces c_0^λ or ℓ_∞^λ and A an infinite matrix. If A is in any of the classes $(X, c_0^{\lambda'})$, $(X, c^{\lambda'})$ or $(X, \ell_\infty^{\lambda'})$, then*

$$\|L_A\| = \|A\|_{(X,\ell_\infty^{\lambda'})} = \sup_n\left(\sum_{k=0}^{\infty}\left|\sum_{m=0}^{n}\lambda'_{nm}\tilde{a}_{mk}\right|\right)<\infty.$$

COROLLARY 4.6. *If either $A \in (\ell_\infty^\lambda, c_0^{\lambda'})$ or $A \in (\ell_\infty^\lambda, c^{\lambda'})$, then L_A is compact.*

Similarly, by using Theorem 4.3, we get some identities or estimates for the Hausdorff measures of noncompactness of operators given by matrices in the classes $(c_0^\lambda, c_0^{\lambda'})$, $(c_0^\lambda, c^{\lambda'})$, $(c_0^\lambda, \ell_\infty^{\lambda'})$ and $(\ell_\infty^\lambda, \ell_\infty^{\lambda'})$, and deduce the necessary and sufficient (or only sufficient) conditions for such operators to be compact.

Finally, let bs, cs and cs_0 be the spaces of all sequences associated with bounded, convergent and null series, respectively. Then, we conclude our work with the following consequences of Theorems 4.1, 4.2 and 4.3.

COROLLARY 4.7. *Let X be any of the spaces c_0^λ or ℓ_∞^λ and A an infinite matrix. If A is in any of the classes (X,cs_0), (X,cs) or (X,bs), then*

$$\|L_A\| = \|A\|_{(X,bs)} = \sup_n \left(\sum_{k=0}^{\infty} \left| \sum_{m=0}^{n} \tilde{a}_{mk} \right| \right) < \infty.$$

COROLLARY 4.8. *If either $A \in (\ell_\infty^\lambda, cs_0)$ or $A \in (\ell_\infty^\lambda, cs)$, then L_A is compact.*

COROLLARY 4.9. *We have*

(a) *If $A \in (c_0^\lambda, cs_0)$, then*

$$\|L_A\|_\chi = \limsup_{n\to\infty} \left(\sum_{k=0}^{\infty} \left| \sum_{m=0}^{n} \tilde{a}_{mk} \right| \right)$$

and

$$L_A \text{ is compact if and only if } \lim_{n\to\infty} \left(\sum_{k=0}^{\infty} \left| \sum_{m=0}^{n} \tilde{a}_{mk} \right| \right) = 0.$$

(b) *If $A \in (c_0^\lambda, cs)$, then*

$$\frac{1}{2} \cdot \limsup_{n\to\infty} \left(\sum_{k=0}^{\infty} \left| \sum_{m=0}^{n} \tilde{a}_{mk} - \tilde{b}_k \right| \right) \leqslant \|L_A\|_\chi \leqslant \limsup_{n\to\infty} \left(\sum_{k=0}^{\infty} \left| \sum_{m=0}^{n} \tilde{a}_{mk} - \tilde{b}_k \right| \right)$$

and

$$L_A \text{ is compact if and only if } \lim_{n\to\infty} \left(\sum_{k=0}^{\infty} \left| \sum_{m=0}^{n} \tilde{a}_{mk} - \tilde{b}_k \right| \right) = 0,$$

where $\tilde{b}_k = \lim_{n\to\infty} (\sum_{m=0}^{n} \tilde{a}_{mk})$ for all $k \in \mathbb{N}$.

(c) *If either $A \in (c_0^\lambda, bs)$ or $A \in (\ell_\infty^\lambda, bs)$, then*

$$0 \leqslant \|L_A\|_\chi \leqslant \limsup_{n\to\infty} \left(\sum_{k=0}^{\infty} \left| \sum_{m=0}^{n} \tilde{a}_{mk} \right| \right)$$

and

$$L_A \text{ is compact if } \lim_{n\to\infty} \left(\sum_{k=0}^{\infty} \left| \sum_{m=0}^{n} \tilde{a}_{mk} \right| \right) = 0.$$

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