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HYPERINVARIANT SUBSPACES FOR OPERATORS HAVING A NORMAL PART

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Abstract. Let T be a nonscalar operator of the form $\left(\begin{smallmatrix} A & * \\ 0 & B \end{smallmatrix}\right)$. It is well known ([5], [6]) that if both A and B are normal operators, then T has a nontrivial hyperinvariant subspace. In this paper, it is shown that if A is a nonscalar normal operator, then either $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} A & 0 \\ D & B \end{smallmatrix}\right)$ has a nontrivial hyperinvariant subspace.

1. Introduction

Let H be a separable infinite dimensional complex Hilbert space and $\mathscr{L}(H)$ be the algebra of all bounded linear operators acting on H. The commutant of T, denoted by $\{T\}'$, is the algebra of all operators X in $\mathscr{L}(H)$ such that $XT = TX$. A subspace $M \subset H$ is called a *nontrivial hyperinvariant subspace* for T if $\{0\} \neq M \neq H$ and $XM \subseteq M$ for each $X \in \{T\}'$. In particular, if $TM \subseteq M$, then the subspace M is called a *nontrivial invariant subspace* for T. The *hyperinvariant subspace problem* is the question of whether every operator in $\mathscr{L}(H)\backslash\mathbb{C}$ has a nontrivial hyperinvariant subspace. An operator $T \in \mathscr{L}(H)$ is called *normal* if $T^*T = TT^*$. It is well known that every normal operator in a Hilbert space has a nontrivial hyperinvariant subspace. Moreover, [3, Theorem 1.4] says that if $T = A \oplus B$, where A is normal, then T has a nontrivial hyperinvariant subspace.

Now, let $T \in \mathscr{L}(H)$ be an operator which has a normal part, that is, T is an operator of the form $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, where A is a normal operator. In this paper we examine the following question.

$$\textit{Does } T \textit{ have a nontrivial hyperinvariant subspace if } C \textit{ is nonzero}? \qquad (1)$$

In 1971, H. Radjavi and P. Rosenthal [5], [6] showed that the answer of the question (1) is true if B is also normal. Moreover, in 1972, R.G. Douglas and C. Pearcy [3] showed that if A and B are similar, then T has a nontrivial hyperinvariant subspace.

In Section 2, we show that if the spectrum of B does not contain the spectrum of A, then the answer of the above question (1) is affirmative and that consider the notion of extremal vectors and introduce some lemmas. In Section 3, we show that if

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A is normal, then either $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ or $\begin{pmatrix} A & 0 \\ D & B \end{pmatrix}$ has a nontrivial hyperinvariant subspace and then provide a sufficient condition and a nontrivial example that the answer of the above question (1) is affirmative.

2. Normal operators and extremal vectors

We first introduce a result due to H. Radjavi and P. Rosenthal.

THEOREM 2.1. ([5], [6, Theorem 6.22]) *Let T be an operator in the upper triangular form*

$$T := \begin{pmatrix} A_{11} & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & & & \\ 0 & & 0 & A_{nn} \end{pmatrix},$$

where the spectra of A_{11} and A_{nn} are disjoint, then T has a nontrivial hyperinvariant subspace.

Let $T \in \mathscr{L}(H)$ be an operator of the form $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{matrix} M \\ M^{\perp} \end{matrix}$, where A is a normal operator. To find a nontrivial hyperinvariant subspace, we can assume that A is nonscalar, since every eigenspace of A is a nontrivial hyperinvariant subspace. Denote by $\sigma(T)$ the spectrum of T. By Theorem 2.1, if $\sigma(A) \cap \sigma(B) = \emptyset$, then T has a nontrivial hyperinvariant subspace. The following corollary is a sufficient condition of the question (1) introduced by H. Radjavi and P. Rosenthal ([6]).

COROLLARY 2.2. ([6, Corollary 6.23]) *With the above notation, if B is normal then T has a nontrivial hyperinvariant subspace.*

Let $A \in \mathscr{L}(M)$ be a normal operator. Then there exists a unique spectral measure E on the Borel subsets of $\sigma(A)$ such that

$$A = \int z dE(z).$$

If G is a nonempty relatively open subset of $\sigma(A)$, then $N := E(G)M$ is a nontrivial reducing subspace for A. Let $A' := A|_N$. Then A' is also normal and the spectrum of A' is contained in $\overline{G}$. Therefore we have:

PROPOSITION 2.3. *With the above notation, if $\sigma(A) \nsubseteq \sigma(B)$, then T has a nontrivial hyperinvariant subspace.*

Proof. Choose a vector x_0 in $\sigma(A) \setminus \sigma(B)$. Then since $\sigma(B)$ is closed, there exists an open set S containing x_0 such that $\overline{S}$ and $\sigma(B)$ are disjoint. Since $G := S \cap \sigma(A)$ is relatively open and nonempty, $N := E(G)M$ is a nontrivial reducing subspace for A.

Write $A_1 := A|_N$ and $A_2 := A|_{N^\perp}$. Then we can write $A = A_1 \oplus A_2$ satisfying $\sigma(A_1) = \overline{G}$ and $\sigma(B)$ are disjoint. Therefore T has a nontrivial hyperinvariant subspace by Theorem 2.1. □

An operator T is called a *normaloid operator* if $r(T) = \|T\|$, where $r(T)$ is the spectral radius of T. The typical example of normaloid operators is a normal operator. Proposition 2.3 gives the following corollary.

COROLLARY 2.4. *With the above notation, if either B is compact or $\|B\| < \|A\|$, then T has a nontrivial hyperinvariant subspace.*

Proof. First, suppose B is compact. Since every eigenspace of B is a hyperinvariant subspace, we can assume that B is a quasinilpotent operator, i.e., $\sigma(B) = \{0\}$. Since A is nonzero and normal, we have $r(A) = \|A\| > 0 = r(B)$. If instead $\|B\| < \|A\|$, then $r(B) \leqslant \|B\| < \|A\| = r(A)$. Since the fact $r(B) < r(A)$ implies $\sigma(A) \nsubseteq \sigma(B)$, T has a nontrivial hyperinvariant subspace by Proposition 2.3. □

Before we proceed, we introduce the notion of extremal vectors by P. Enflo [1]. Assume that T has dense range. Choose a unit vector $x_0 \in H$ and $0 < \varepsilon < 1$. If $\mathscr{F} = \{y \in H : \|Ty - x_0\| \leqslant \varepsilon\}$, then $\mathscr{F}$ is a nonempty, norm closed and convex set. So there exists a unique minimal vector $y_0 = y_0(x_0, \varepsilon) \in \mathscr{F}$. We say that y_0 is the *extremal (minimal) vector* for T, x_0, ε. In this case, $\|Ty_0 - x_0\| = \varepsilon$. In [1], Enflo established an important equation on extremal vectors called "Orthogonality Equation"

LEMMA 2.5. (Orthogonality equation) *If y_0 is the extremal vector for T, x_0, ε, then*

$$T^*(x_0 - Ty_0) = \delta y_0, \quad \textit{for some } \delta > 0.$$

Moreover, by the minimality of extremal vectors, it is easy to show that

$$\|x_0\|(\|x_0\| - \varepsilon) < \langle Ty_0, x_0\rangle \leqslant \|x_0\|^2 - \varepsilon^2 \tag{2}$$

Let y_n be the extremal vector for T^n, x_0, ε. Then since the sequence $\{T^n y_n\}$ is uniformly bounded, it follows from (2) that there exists a subsequence of $\{T^n y_n\}$, which converges to a nonzero vector weakly. In particular, if T is a normal operator, then the sequence $\{T^n y_n\}$ converges in norm.

LEMMA 2.6. ([2, Proposition 2.1]) *Let $T \in \mathscr{L}(H)$ be a normal operator with dense range. For each $x_0 \in H$ and ε, let y_n be the extremal vector for T^n, x_0, ε. Then the sequence $\{T^n y_n\}$ converges in norm.*

The following lemmas are needed to show the main result in next section.

LEMMA 2.7. ([4, Lemma 3.3]) *Let $x_0 \in M \subseteq H$ and y_0 be the extremal vector for T, x_0, ε. If M reduces T, then $y_0 \in M$.*

LEMMA 2.8. *Let y_0 be the extremal vector for T, x_0, ε. Then*

$$\|y_0\| > \frac{\|x_0\|(\|x_0\| - \varepsilon)}{\|T^* x_0\|} \geqslant \frac{\|x_0\| - \varepsilon}{\|T\|}$$

Proof. Immediate from the first inequality of (2). □

For given $r > 0$, we will denote D_r by an open disk of radius r centered at zero throughout this paper. Then we have:

LEMMA 2.9. *Let $T \in \mathscr{L}(H)$ be a normal operator. If $\sigma(T) \cap D_r = \emptyset$ for some $r > 0$, then $\|T^{-1}\| \leqslant \frac{1}{r}$, so that $\|Tx\| \geqslant r\|x\|$ for all $x \in H$.*

Proof. By the spectral mapping theorem, $\sigma(T^{-1}) = \{\frac{1}{\lambda} : \lambda \in \sigma(T)\}$, so that $r(T^{-1}) \leqslant \frac{1}{r}$. Moreover since T^{-1} is also normal, it follows that $\|T^{-1}\| = r(T^{-1}) \leqslant \frac{1}{r}$. Therefore for each $x \in H$, $\|Tx\| \geqslant r\|T^{-1}\|\|Tx\| \geqslant r\|T^{-1}Tx\| = r\|x\|$. □

3. Operators having a normal part

Our main result is as follows:

THEOREM 3.1. *Suppose M is a nontrivial subspace of H. Let $A \in \mathscr{L}(M)$ and $B \in \mathscr{L}(M^{\perp})$. If A is a nonscalar normal operator, then either $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ or $\begin{pmatrix} A & 0 \\ D & B \end{pmatrix}$ has a nontrivial hyperinvariant subspace.*

Proof. Since every eigenspace of an operator is clearly a nontrivial hyperinvariant subspace, we can assume that both A and B have dense ranges. Since the spectrum of A is not singleton, by translation and scalar multiplication, we can assume that A is not invertible and $\|A\| = r(A) > 1$. Choose a unit vector $x_1 \in M^{\perp}$, and $0 < \varepsilon < 1$. Let z_n be the extremal vector for B^n, x_1, ε. There are two cases to consider.

(Case 1) *There exists $c > 0$ such that $\|z_n\| \leqslant c$ for all n.*
Since A is not invertible, it follows that $G := D_r \cap \sigma(A)$ for some $0 < r < 1$ is a nonempty proper subset of $\sigma(A)$. Hence $N := E(G)M$ is a nontrivial reducing subspace for A by the spectral theorem of normal operators. Write $A_1 := A|_N$ and $A_2 := A|_{N^{\perp}}$. Then we can write $A = A_1 \oplus A_2$, where $\sigma(A_1) \subseteq \overline{D}_r$. Moreover, since A_1 is also normal, it follows that $\|A_1\| = r(A_1) \leqslant r$. Choose a unit vector $x_0 \in N$, and $0 < \varepsilon < 1$. Let y_n be the extremal vector for A^n, x_0, ε. By Lemma 2.7 we have $y_n \in N$ for each n. Indeed, y_n is the minimal vector satisfying

$$\|A_1^n y_n - x_0\| = \varepsilon,$$

so that y_n is also the extremal vector for A_1^n, x_0, ε by the uniqueness of the extremal vector. We now claim

$$\lim_{n \to \infty} \frac{\|z_n\|}{\|y_n\|} = 0. \tag{3}$$

Indeed, by Lemma 2.8, we have

$$\|y_n\| > \frac{1-\varepsilon}{\|A_1^n\|} \geqslant \frac{1-\varepsilon}{r^n}.$$

Therefore

$$\frac{\|z_n\|}{\|y_n\|} < Kr^n, \quad K = \frac{c}{1-\varepsilon},$$

so that the sequence $\{\frac{\|z_n\|}{\|y_n\|}\}$ converges to zero as $n \to \infty$. By Lemma 2.6 the sequence $\{A^n y_n\}$ converges to t_0 in norm. Choose a subsequence $\{n_k\}$ such that $\{B^{n_k} z_{n_k}\}$ converges to s_0 weakly. Then by the inequality (2), we can easily show that s_0 and t_0 are nonzero. Write $s_k := B^{n_k} z_{n_k} \in M^\perp$, $t_k := A^{n_k} y_{n_k} \in M$ and $T := \begin{pmatrix} A & 0 \\ D & B \end{pmatrix}$. We now claim that

$$\langle X s_k, x_0 - t_k \rangle \to 0 \quad \text{for each contraction } X \in \{T\}'. \tag{4}$$

Let

$$X z_{n_k} := \alpha_k y_{n_k} + \omega_k, \quad \text{where} \quad \omega_k \perp y_{n_k}.$$

Then

$$\|z_{n_k}\|^2 \geqslant |\alpha_k|^2 \|y_{n_k}\|^2 + \|\omega_k\|^2,$$

which gives

$$|\alpha_k| \leqslant \frac{\|z_{n_k}\|}{\|y_{n_k}\|} \to 0 \tag{5}$$

by (3). On the other hand,

$$\langle X s_k, x_0 - t_k \rangle = \langle \alpha_k y_{n_k}, T^{*n_k}(x_0 - t_k) \rangle + \langle \omega_k, T^{*n_k}(x_0 - t_k) \rangle.$$

By the orthogonality equation in Lemma 2.5, we have $T^{*n_k}(x_0 - t_k) = A^{*n_k}(x_0 - t_k) = \delta_{n_k} y_{n_k}$ for some $\delta_{n_k} > 0$, and hence $\langle \omega_k, T^{*n_k}(x_0 - t_k) \rangle = 0$. Therefore

$$\langle X s_k, x_0 - t_k \rangle = \langle \alpha_k y_{n_k}, T^{*n_k}(x_0 - t_k) \rangle = \alpha_k \langle A^{n_k} y_{n_k}, x_0 - t_k \rangle.$$

But since $\|A^{n_k} y_{n_k}\| < 1$ and $\|x_0 - t_k\| = \varepsilon$, it follows from (5) that

$$|\langle X s_k, x_0 - t_k \rangle| \leqslant \varepsilon |\alpha| \to 0$$

which proves (4). Moreover, since $t_k = A^{n_k} y_{n_k} \to t_0$ in norm, the sequence $\{x_0 - t_k\}$ converges to $x_0 - t_0$ in norm. Then by (4) we have

$$\langle X s_0, x_0 - t_0 \rangle = 0 \quad \text{for all } X \in \{T\}'.$$

Note that $x_0 - t_0$ is a nonzero vector. Indeed, we obtain

$$\varepsilon^2 = \|x_0 - t_k\|^2 = \langle x_0, x_0 - t_k \rangle - \langle t_k, x_0 - t_k \rangle,$$

so that $\langle x_0, x_0 - t_k \rangle = \varepsilon^2 + \delta_{n_k} \|y_{n_k}\|^2 > 0$ for each k. Also, since s_0 is a nonzero vector, $L \equiv cl\{T\}' s_0$ is a nontrivial hyperinvariant subspace for $T = \begin{pmatrix} A & 0 \\ D & B \end{pmatrix}$.

(Case 2) *There exists a subsequence* $\{n_j\}$ *such that*

$$\|z_{n_j}\| \to \infty. \tag{6}$$

Since A is a normaloid operator, that is, $r(A) = \|A\| > 1$, it follows that $\sigma(A)\setminus\overline{\mathbb{D}}$ is nonempty. For given $z \in \sigma(A)\setminus\overline{\mathbb{D}}$, let Δ be an open disk of radius $r = |z| - 1$ centered at z. Then $S := \Delta \cap \sigma(A)$ is a nonempty proper subset of $\sigma(A)$ such that $\overline{S} \cap \mathbb{D} = \emptyset$. Since $N' := E(S)M$ is also a nontrivial reducing subspace for A, we can write $A_3 := A|_{N'}$ and $A_4 := A|_{(N')^\perp}$. Then A_3 is also normal and $\sigma(A_3) \cap \mathbb{D} = \emptyset$. Choose a unit vector $x_0 \in N'$, and $0 < \varepsilon < 1$. Let y_n be the extremal vector for A^n, x_0, ε. By the same argument in (Case 1), y_n is also the extremal vector for A_3^n, x_0, ε. Since $\mathbb{D} = D_1$, it follows from Lemma 2.9 that

$$\|y_n\| \leqslant \|A_3 y_n\| \leqslant \cdots \leqslant \|A_3^n y_n\| < 1.$$

Therefore by (6) we have

$$\lim_{j\to\infty} \frac{\|y_{n_j}\|}{\|z_{n_j}\|} = 0.$$

Choose a subsequence $\{n_{j_k}\}$ of $\{n_j\}$ such that $\{B^{n_{j_k}} z_{n_{j_k}}\}$ converges to nonzero s_0 weakly. Write $s_k := A^{n_{j_k}} y_{n_{j_k}} \in M$, $t_k := B^{n_{j_k}} z_{n_{j_k}} \in M^\perp$ and $T := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. Then by Theorem 2.6 the sequence $\{s_k\}$ converges to nonzero t_0 in norm. By the same argument in (Case 1) we have

$$\langle X s_k, x_1 - t_k \rangle \to 0 \quad \text{for each contraction } X \in \{T\}'. \tag{7}$$

Since the sequence $\{s_k\}$ converges in norm and $x_1 - t_0$ is nonzero, it follows from (7) that

$$\langle X s_0, x_1 - t_0 \rangle = 0.$$

Since s_0 is a nonzero vector, $L \equiv cl\{T\}' s_0$ is a nontrivial hyperinvariant subspace for $T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. This completes the proof. □

In Theorem 3.1, the "nonscalar" of A is a condition only to avoid a trivial case. Indeed, if $A = B = \alpha I$ for some $\alpha > 0$ and $C = D = 0$, then the operator matices are a scalar operator αI which has no nontrivial hyperinvariant subspace. Now, the following corollaries give partial solutions of the question (1).

COROLLARY 3.2. *Let* T *be a nonscalar operator of the form* $\begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$*, where* A *is a normal operator with* $\|A\| > 1$*. If there exists a subsequence* $\{n_k\}$ *such that the sequence* $\{\|B^{*n_k} x_0\|\}$ *is uniformly bounded for some unit vector* $x_0 \in M^\perp$*, then* T *has a nontrivial hyperinvariant subspace.*

Proof. To avoid the trivial case, we assume that T is injective and has dense range. Then both A and B have dense ranges since A is normal. Since $r(A) = \|A\| > 1$, it follows that $\sigma(A)\backslash D_r$ is nonempty for some $1 < r < \|A\|$. For given $\lambda \in \sigma(A)\backslash \overline{D}_r$, let Δ be an open disk of radius $|\lambda| - r$ centered at λ. Then $S := \Delta \cap \sigma(A)$ is a nonempty proper subset of $\sigma(A)$ such that $\overline{S} \cap D_r = \emptyset$. Write $N := E(S)M$ and $A' := A|_N$. Then A' is also normal and $\sigma(A') \cap D_r = \emptyset$. Choose a unit vector $x_1 \in N$, and $0 < \varepsilon < 1$. Let y_n be the extremal vector for A^n, x_1, ε. By the same argument as (Case 2) of proof in Theorem 3.1, it follows from Lemma 2.9 that

$$\|y_n\| \leqslant \frac{1}{r}\|A'y_n\| \leqslant \frac{1}{r^2}\|A'^2 y_n\| \leqslant \cdots \leqslant \frac{1}{r^n}\|A'^n y_n\| < \frac{1}{r^n}, \quad r > 1. \tag{8}$$

On the other hand, let z_n be the extremal vector for B^n, x_0, ε. Since $\{\|B^{*n_k} x_0\|\}$ is uniformly bounded, it follows from Lemma 2.8 that

$$\|z_{n_k}\| > \frac{1-\varepsilon}{\|B^{*n_k} x_0\|} \geqslant (1-\varepsilon)K \tag{9}$$

for some $K > 0$. Therefore by (8) and (9) we have

$$\lim_{k\to\infty} \frac{\|y_{n_k}\|}{\|z_{n_k}\|} = 0.$$

Then by the same argument as (Case 2) of proof in Theorem 3.1, $L \equiv cl\{T\}' s_0$ is a nontrivial hyperinvariant subspace for $T = \begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$. □

Here we note that if $\|B\| \leqslant 1$, then T has a nontrivial hyperinvariant subspace, but $\sigma(A) \nsubseteq \sigma(B)$ so that this condition satisfies the hypothesis of Proposition 2.3. In this viewpoint, it is interesting to find an example of the case of $\sigma(A) \subseteq \sigma(B)$. We conclude the paper with giving a nontrivial example which Corollary 3.2 can be applied.

EXAMPLE 3.3. Let A be a normal operator with $\|A\| = \frac{3}{2}$. Define a bilateral sequence $\{\alpha_n\}$ by

$$\alpha_n := \begin{cases} \frac{1}{n} & \text{if } n > 0 \\ 2 & \text{if } n \leqslant 0. \end{cases}$$

Let B be a bilateral weighted shift defined by the equation $Be_n = \alpha_n e_{n-1}$ $(n \in \mathbb{Z})$, where $\{e_n\}$ is the orthonormal basis of $H := \ell^2(\mathbb{Z})$. We now claim $\sigma(A) \subseteq \sigma(B)$. Indeed, for each n,

$$\|B^n\| = \sup_l \left| \prod_{i=1}^{n} \alpha_{l+i} \right| = 2^n, \tag{10}$$

so that B is bounded and $r(B) = \lim \|B^n\|^{\frac{1}{n}} = 2$. Moreover, since the bilateral sequence $\{\alpha_n\}$ converges to 0 as $n \to \infty$ and 2 as $n \to -\infty$, thus $\sigma(B) = \overline{D}_2$. On the other hand, $r(A) = \|A\| = \frac{3}{2}$ which implies $\sigma(A) \subseteq \overline{D}_{\frac{3}{2}}$, and so $\sigma(A) \subseteq \sigma(B)$. By a straightforward calculation, we have for each n,

$$\|B^n e_0\| = \frac{1}{n!} \leqslant 1.$$

This implies operators A and B satisfy the hypothesis in Corollary 3.2, and hence every nonscalar operator of this form $\begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$ has a nontrivial hyperinvariant subspace.

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REFERENCES

[1] S. ANSARI AND P. ENFLO, *Extremal vectors and invariant subspaces*, Trans. Amer. Math. Soc., **350** (1998), 539–558.

[2] I. CHALENDAR AND J. PARTINGTON, *Convergence properties of minimal vectors for normal operators and weighted shifts*, Proc. Amer. Math. Soc., **133** (2005), 501–510.

[3] R. G. DOUGLAS AND C. PEARCY, *Hyperinvariant subspaces and transitive algebras*, Michigan Math. J., **19** (1972), 1–12.

[4] H. J. KIM, *Hyperinvariant subspace problem for quasinilpotent operators*, Integral Equations Operator Theory, **61**, 1 (2008), 103–120.

[5] H. RADJAVI AND P. ROSENTHAL, *Hyperinvariant subspaces for spectral and n-normal operators*, Acta Sci. Math. (Szeged), **32** (1971), 121–126.

[6] H. RADJAVI AND P. ROSENTHAL, *Invariant Subspaces*, second ed., Dover Publications, Mineda, New York, 2003.

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