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THE DENSITY OF THE RANGE OF $X \mapsto AX - XB$ WITH A, B^* M–HYPONORMALS

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Abstract. We extend a result of L. A. Fialkow concerning the density of the range and the injectivity of the operator $X \mapsto AX - XB$ with A, B^* M-hyponormal operators with no holes in the essential spectrum of negative Fredholm index.

1. Introduction

Let $\mathscr{H}$ be a complex, separable, infinite dimensional Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all linear bounded operators on $\mathscr{H}$. For operators $A, B \in \mathscr{L}(\mathscr{H})$, let Δ_{AB} denote the (generalized) derivation associated with A and B, that is the linear operator defined on the Banach space $\mathscr{L}(\mathscr{H})$ by $\Delta_{AB}(X) = AX - XB$. For an operator $T \in \mathscr{L}(\mathscr{H})$, let $\sigma_p(T)$, $\sigma_e(T)$, $\sigma_{re}(T)$, $\sigma_{le}(T)$ denote the *point spectrum, essential spectrum, right essential spectrum,* and *left essential spectrum,* respectively. We also denote by $(\mathscr{C}_p(\mathscr{H}), ||\cdot||_p)$, $p \geqslant 1$ the Schatten p-class and by $\mathscr{C}_{00}(\mathscr{H})$ the set of finite rank operators.

The operator Δ_{AB} has been extensively studied by many authors, but we mention only just a few papers such as, Davis-Rosenthal [2], Bhatia-Rosenthal [1], L. Fialkow [3, 4, 5] in which the injectivity and the density of the range of Δ_{AB} have been studied. In [5], Fialkow established amongst other results the following.

THEOREM A ([5], Prop. 4.2). *Let $A, B \in \mathscr{L}(\mathscr{H})$ be normal operators. The following are equivalent:*

(1) *Δ_{AB} has dense range;*

(2) *A and B satisfy* $[\mathrm{H}_1]$: $\sigma_{re}(A) \cap \sigma_{le}(B) = \emptyset$ & $[\mathrm{H}_2]$: $\sigma_p(A^*)^* \cap \sigma_p(B) = \emptyset$*;*

(3) *If $Y \in \mathscr{L}(\mathscr{H})$ and $\varepsilon > 0$, then there exists $X \in \mathscr{L}(\mathscr{H})$ such that $Y - \Delta_{AB}(X)$ belongs to $\mathscr{C}_{00}(\mathscr{H})$ and $||Y - \Delta_{AB}(X)||_1 < \varepsilon$.*

In this case Δ_{AB} and Δ_{BA} are injective. Moreover, if $(\mathscr{J}, ||\cdot||_{\mathscr{J}})$ is a normed ideal, $Y \in \mathscr{J}$, and $\varepsilon > 0$, then there exists $X \in \mathscr{J}$ such that $Y - \Delta_{AB}(X) \in \mathscr{C}_{00}(\mathscr{H})$ and $||Y - \Delta_{AB}(X)||_{\mathscr{J}} < \varepsilon$.

The above theorem was proved by making use of a general characterization of the density of the range of Δ_{AB}, result that we will need to apply later.

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This note is dedicated to the memory of my aunt, Leontina Mateiescu.

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THEOREM B ([5], Theorem 1.1). *Let $A, B \in \mathscr{L}(\mathscr{H})$. The following are equivalent:*

(1) Δ_{AB} *has dense range;*

(2) A, B *satisfy*

$$[\mathrm{H}_1]: \ \sigma_{re}(A) \cap \sigma_{le}(B) = \emptyset \quad \& \quad [\mathrm{F}_0]: \ Ker(\Delta_{BA}) \cap \mathscr{C}_1(\mathscr{H}) \setminus \{0\} = \emptyset.$$

Spectral conditions $[\mathrm{H}_1]$, $[\mathrm{H}_2]$ were first introduced by D. Herrero who raised the question whether they are necessary and sufficient for the density of the range of a generalized derivation Δ_{AB}. In [5], Fialkow showed that these conditions are not sufficient by providing examples of operators that satisfy conditions $[\mathrm{H}_1]$ and $[\mathrm{H}_2]$, but which do not satisfy $[\mathrm{F}_0]$. Since these conditions seem to have been used first time in [5], we denoted them by $[\mathrm{F}_0]$ (in Theorem B) and $[\mathrm{F}_1]$, $[\mathrm{F}_2]$ (see Theorem 1 below), respectively.

2. The density of the range

It is the purpose of this note to extend the above result to a larger class of derivations Δ_{AB} in which A, B^* are M-hyponormal operators.

For an operator $T \in \mathscr{L}(\mathscr{H})$ and non-negative integers $m, n,$ let

$$H_{m,n}(T) = \{\lambda \in \mathbb{C} \setminus \sigma_e(T) | \dim Ker(T-\lambda) = m \ \& \ \dim Ker(T^* - \overline{\lambda}) = n\},$$

and let $H_-(T) = \bigcup_{m<n} H_{m,n}(T)$.

An operator $T \in \mathscr{L}(\mathscr{H})$ is M-hyponormal if there exists a positive number m such that

$$||(T-\lambda)^*x|| \leqslant m||(T-\lambda)x||, \text{ for all } x \in \mathscr{H} \text{ and all } \lambda \in \mathbb{C}.$$

Let $H_M(\mathscr{H})$ denote the set of M-hyponormal operators in $\mathscr{L}(\mathscr{H})$ and let

$$H_M^0(\mathscr{H}) = \{T \in H_M(\mathscr{H})) | \ H_-(T) = \emptyset\}.$$

It is obvious that for $T \in H_M(\mathscr{H})$, $\dim Ker(T-\lambda) \leqslant \dim Ker(T^* - \overline{\lambda})$, and that the hyponormal operators (i.e., $T^*T \geqslant TT^*$) are M-hyponormal with $m = 1$. Thus, $H_M^0(\mathscr{H})$ denotes the class of those M-hyponormal operators whose holes of the essential spectrum (if any) are associated with equally dimensional kernel and co-kernel.

In proving the main result (Theorem 1), we need the following lemmas.

LEMMA 1. *If T is an M-hyponormal operator, then*

(1) $\sigma_{le}(T) \subseteq \sigma_{re}(T)$, *thus* $\sigma_e(T) = \sigma_{re}(T)$, *and*

(2) $\sigma(T) = \sigma_p(T^*)^* \cup \sigma_{re}(T)$.

Proof. Let T be an M-hyponormal operator and let $\lambda \in \sigma_{le}(T)$. It is well known that $\lambda \in \sigma_{le}(T)$ is equivalent to the existence of an orthonormal sequence $\{f_n\}_n$ so that $(T-\lambda)f_n \longrightarrow 0$ as $n \longrightarrow \infty$. Since T is M-hyponormal, then $(T-\lambda)^* f_n \longrightarrow 0$, that is $\overline{\lambda} \in \sigma_{le}(T^*)$, which is equivalent to $\lambda \in \sigma_{re}(T)$. To prove (2), first observe that the inclusion $\sigma(T) \supseteq \sigma_p(T^*)^* \cup \sigma_{re}(T)$ is obvious. If $\lambda \notin (\sigma_p(T^*)^* \cup \sigma_{re}(T))$, then according to the above $(T-\lambda)$ is a Fredholm operator. Since T is an M-hyponormal operator, $\dim Ker(T-\lambda) \leqslant \dim Ker(T^* - \overline{\lambda})$, and since $\lambda \notin \sigma_p(T^*)^*$, $Ker(T-\lambda) = Ker(T-\lambda)^* = (0)$ and thus $T-\lambda$ is invertible. □

LEMMA 2. *If $T \in H_M(\mathscr{H})$ and $\sigma_{le}(T)$ is an infinite set, then there is an orthonormal sequence $\{f_n\}_n$ so that $(T-\lambda_n)f_n \longrightarrow 0$ as $n \longrightarrow \infty$.*

Proof. Let $\lambda_n \longrightarrow \lambda_0$ be a sequence with $\lambda_n \in \sigma_{le}(T)$. Since $\sigma_{le}(T)$ is a closed set, then $\lambda_0 \in \sigma_{le}(T)$ and thus there exists an orthonormal sequence $\{f_n\}_n$ such that $(T-\lambda_0)f_n \longrightarrow 0$, and therefore $(T-\lambda_n)f_n \longrightarrow 0$. □

THEOREM 1. *Let $A, B^* \in H^0_M(\mathscr{H})$. Then the following are equivalent:*

(1) *Δ_{AB} has dense range;*

(2) *A, B satisfy the following four conditions:*

[H$_1$]: $\sigma_{re}(A) \cap \sigma_{le}(B) = \emptyset$, [H$_2$]: $\sigma_p(A^*)^* \cap \sigma_p(B) = \emptyset$,

[F$_1$]: $\sigma_p(A^*)^* \cap \sigma_{le}(B)$ *is a finite set,* & [F$_2$]: $\sigma_{re}(A) \cap \sigma_p(B)$ *is a finite set;*

(3) *If $Y \in \mathscr{L}(\mathscr{H})$ and $\varepsilon > 0$, then there exists $X \in \mathscr{L}(\mathscr{H})$ such that $Y - \Delta_{AB}(X) \in \mathscr{C}_{00}(\mathscr{H})$ and $||Y - \Delta_{AB}(X)||_1 < \varepsilon$.*

In this case Δ_{AB} and Δ_{BA} are injective. Moreover, if $(\mathscr{J}, ||\cdot||_{\mathscr{J}})$ is a normed ideal, $Y \in \mathscr{J}$, and $\varepsilon > 0$, then there exists $X \in \mathscr{J}$ such that $Y - \Delta_{AB}(X) \in \mathscr{C}_{00}(\mathscr{H})$ and $||Y - \Delta_{AB}(X)||_{\mathscr{J}} < \varepsilon$.

The proof of Theorem 1 uses the same circle of ideas developed in [5] adjusted to the class of M-hyponormal operators, but first some remarks are in order.

REMARK 1. *There exist operators $A, B^* \in H_M(\mathscr{H})$ that satisfy* [H$_1$], [H$_2$], [F$_1$], [F$_2$], *and the range of Δ_{AB} is not dense.*

Let $A = U$ and $B^* = UD$, where U is the unilateral shift and D is a diagonal operator with diagonal entries $\{\alpha_n\}_n$ such that $|\alpha_n| \searrow 0$. It is well known (and easy to verify) that A, B^* are hyponormal operators and $\sigma_p(A^*) = \mathbb{D}$, $\sigma_{re}(A) = \mathbb{T}$, $\sigma_p(B) = \emptyset$, $\sigma_{le}(B) = \{0\}$, and consequently [H$_1$], [H$_2$], [F$_1$], [F$_2$] are all satisfied. On the other hand, let X be the rank-one projection on the first vector of the canonical basis with respect to which the U and D have their standard matrix representation. Then $BX = XA = 0$ and thus $X \in Ker(\Delta_{BA}) \cap \mathscr{C}_1(\mathscr{H}) \setminus \{0\}$, which according to Theorem B implies that the range of Δ_{AB} is not dense.

A natural question that arises is whether the set $H^0_M(\mathscr{H})$ contains anything else besides normal operators. Let $N(\mathscr{H})$ denote the set of all normal operators in $\mathscr{L}(\mathscr{H})$.

REMARK 2. $H_M^0(\mathscr{H}) \setminus N(\mathscr{H}) \neq \emptyset$.

Proof. Indeed, if Σ is a closed set of positive planar Lebesgue density at each point, that is, any nonempty intersection of an open disc and the set Σ has nonzero planar Lebesgue measure, then according to a result of Pincus [7] there exists a pure hyponormal operator of rank one self-commutator whose spectrum is the set Σ. Furthermore, if the set Σ is a "swiss-cheese" type of set (that is a planar Cantor set), then each point of the spectrum is an accumulation point of boundary points of the spectrum itself. Consequently, (cf [6]), the spectrum and the essential spectrum are equal sets, and therefore the arising hyponormal operator belongs to $H_1^0(\mathscr{H}) \setminus N(\mathscr{H})$. $\square$

Proof Theorem 1. First, we will prove implication "(1)$\Rightarrow$(2)". The density of the range of Δ_{AB} implies [H_1] according to Theorem B. To prove that [H_2] holds, assume that $\lambda \in \sigma_p(A^*)^* \cap \sigma_p(B)$, thus there are vectors $e, f \in \mathscr{H}$ with $||e|| = ||f|| = 1$ so that $(A^* - \overline{\lambda})e = 0$ and $(B - \lambda)f = 0$. Let $x, y \in \mathscr{H}$ be such that $Ae = \alpha e + x$, $B^* f = \beta f + y$, $\langle e, x\rangle = 0$, and $\langle f, y\rangle = 0$. Then $\alpha = \lambda = \overline{\beta}$. Define an operator X in $\mathscr{L}(\mathscr{H})$ by $Xe = f$ and $Xg = 0$ for $\langle g, e\rangle = 0$. Then X is a nonzero trace-class operator that belongs to $Ker(\Delta_{BA})$, which, according to Theorem B, is a contradiction. Indeed, $XAe = X(\lambda e + x) = X(\lambda e) + 0 = \lambda Xe = \lambda f$ and $BXe = Bf = \lambda f$, On the other hand, if $\langle g, e\rangle = 0$, then $BXg = 0$ and $XAg = 0$ since $\langle Ag, e\rangle = \langle g, A^* e\rangle = \langle g, \overline{\lambda} e\rangle = 0$. To prove that [$\mathrm{F}_1$] holds, assume that there exist a sequence of distinct values $\{\lambda_n\}_n$ in $\sigma_p(A^*)^* \cap \sigma_{le}(B)$. According to Lemma 2, there exists and orthonormal sequence $\{f_n\}_n$ so that $\alpha_n := ||(B - \lambda_n)f_n|| \to 0$. Since A is an M-hyponormal operator, $\sigma(A) = \sigma_p(A^*)^* \cup \sigma_{re}(A)$ and since [H_1] holds, then $\{\lambda_n\} \subseteq \sigma_p(A^*)^* \setminus \sigma_{re}(A)$, that is $A - \lambda_n$ is a Fredholm operator. Since $A \in H_M^0(\mathscr{H})$, each λ_n belongs to a hole $H_{k_n,k_n}(A)$ with $k_n > 0$. Thus, for each n, we can choose e_n of norm one so that $(A - \lambda_n)e_n = 0$, and since A is M-hyponormal, then $(A^* - \overline{\lambda}_n)e_n = 0$, thus the sequence $\{e_n\}_n$ is orthonormal. We can now define an operator Y by $Yf_n = e_n$ for each n, and $Yg = 0$ when $\langle g, f_n\rangle = 0$ for all n's. The operator Y satisfies $||Y - \Delta_{AB}(X)|| \geqslant 1$, which contradicts (1). Indeed,

$$\begin{aligned} ||Y - \Delta_{AB}(X)|| &\geqslant \sup_n ||Yf_n - (A - \lambda_n)Xf_n + X(B - \lambda_n)f_n|| \geqslant \\ &\geqslant \sup_n (||e_n - (A - \lambda_n)Xf_n|| - \alpha_n||X||) \end{aligned}$$

and since

$$\begin{aligned} ||e_n - (A - \lambda_n)f_n||^2 &= 1 - 2\,\mathrm{Re}(\langle (A - \lambda_n)^* e_n, Xf_n\rangle) + ||(A - \lambda_n)Xf_n||^2 = \\ &= 1 + ||(A - \lambda_n)Xf_n||^2 \geqslant 1, \end{aligned}$$

the inequality above is proved. The implication of condition [F_2] is similar to the above proof with the role of A replaced by B^*.

To prove implication "(2)$\Rightarrow$(3)", we use again the fact that for M-hyponormal operators A and B^*, $\sigma(A) = \sigma_p(A^*)^* \cup \sigma_{re}(A)$, $\sigma(B) = \sigma_p(B) \cup \sigma_{le}(B)$. Assume that $\sigma_p(A^*)^* \cap \sigma_{le}(B)$ is $\{\lambda_1, \ldots, \lambda_m\}$ and $\sigma_p(B) \cap \sigma_{re}(A)$ is $\{\mu_1, \ldots, \mu_n\}$. According to the above spectral properties and since $\sigma_{re}(A) \cap \sigma_{le}(B) = \emptyset$ and $\sigma_p(A^*)^* \cap \sigma_p(B) = \emptyset$, we have

$$\{\lambda_1, \ldots, \lambda_m\} = (\sigma_p(A^*)^* \setminus \sigma_{re}(A)) \cap (\sigma_{le}(B) \setminus \sigma_p(B))$$

and

$$\{\mu_1, \ldots, \mu_n\} = (\sigma_p(B) \setminus \sigma_{le}(B)) \cap (\sigma_{re}(A) \setminus \sigma_p(A^*)^*).$$

Thus, $A - \lambda_i$ is a Fredholm operator, and since $A \in H_M^0(\mathscr{H})$, that is $H_-(A) = \emptyset$, $\dim Ker(A - \lambda_i) = \dim Ker(A - \lambda_i)^* > 0$. The case $Ker(A - \lambda_i) = Ker(A - \lambda_i)^* = (0)$ is excluded since otherwise $A - \lambda_i$ is invertible. Similarly, $\dim Ker(B^* - \overline{\mu}_j) = \dim Ker(B - \mu_j) > 0$, for all $j = 1, \ldots, n$. Let $\mathscr{H}_1 = \bigvee_{i=1}^m Ker(A - \lambda_i)$, $\mathscr{H}_2 = \mathscr{H}_1^\perp$ and $\mathscr{K}_1 = \bigvee_{j=1}^n Ker(B^* - \overline{\mu}_j)$, $\mathscr{K}_2 = \mathscr{K}_1^\perp$. Since a point $\lambda \in \sigma_p(A^*)^* \setminus \sigma_{re}(A)$ is an isolated point of $\sigma(A)$, then relative to the decomposition of $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, we can write $A = A_1 \oplus A_2$ with $\sigma(A_1) = \{\lambda_1, \ldots, \lambda_m\}$ and $\sigma(A_2) = \sigma(A) \setminus \sigma(A_1)$. Similarly, relative to the decomposition $\mathscr{H} = \mathscr{K}_1 \oplus \mathscr{K}_2$, $B = B_1 \oplus B_2$ with $\sigma(B_1) = \{\mu_1, \ldots, \mu_n\}$ and $\sigma(B_2) = \sigma(B) \setminus \sigma(B_1)$.

Let $Y \in \mathscr{L}(\mathscr{H})$ and $\varepsilon > 0$. The construction of an operator $X \in \mathscr{L}(\mathscr{H})$ so that $Y - \Delta_{AB}(X) \in \mathscr{C}_{00}(\mathscr{H})$ and $||Y - \Delta_{AB}(X)||_1 < \varepsilon$ is similar to that used in [5] and we include it here for reader's convenience. Let

$$\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$$

be the matrix representation of Y as an operator from $\mathscr{K}_1 \oplus \mathscr{K}_2$ into $\mathscr{H}_1 \oplus \mathscr{H}_2$. Since the pair of sets $\sigma(A_i)$, $\sigma(B_i)$ is disjoint, $\Delta_{A_iB_i}$ is an invertible operator, and thus there exists $X_{ii} \in \mathscr{L}(\mathscr{K}_i, \mathscr{H}_i)$ so that $Y_{ii} = \Delta_{A_iB_i}(X_{ii})$, $i = 1, 2$. To construct an operator X_{21}, let $\{f_1, \ldots, f_N\}$ be an orthonormal basis of $\mathscr{K}_1$ so that $B_1^* f_j = \overline{\mu}_j^k f_j$, $j = 1, \ldots, N$. Each $A_2 - \mu_j^k$ is invertible and thus it has dense range. Let $x_j \in \mathscr{H}_2$ so that

$$||(A - \mu_j^k)x_j - Y_{21}f_j|| < \frac{\varepsilon}{N}.$$

Define $X_{21} : \mathscr{K}_1 \to \mathscr{H}_2$ by $X_{21}f_j = x_j$, $j = 1, \ldots, N$. Thus

$$\begin{aligned} ||A_2X_{21} - X_{21}B_1 - Y_{21}||_1 &\leqslant \sum_{j=1}^N ||(A_2X_{21} - X_{21}B_1 - Y_{21})f_j|| \\ &= \sum_{j=1}^N ||(A_2 - \mu_j^k)X_{21}f_j - X_{21}(B_1 - \mu_j^k)f_j - Y_{21}f_j|| < \varepsilon. \end{aligned}$$

Similarly, using an orthonormal basis $\{e_1, \ldots, e_M\}$ of $\mathscr{H}_1$ such that $A_1e_i = \lambda_i^k e_i$, one can construct $X_{12} \in \mathscr{L}(\mathscr{K}_2, \mathscr{H}_1)$ so that $||X_{12}^*A_1^* - B_2^*X_{12}^* - Y_{12}^*||_1 < \varepsilon$, and thus, the operator $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ satisfies $Y - \Delta_{AB}X \in \mathscr{C}_{00}(\mathscr{H})$ and $||Y - \Delta_{AB}X||_1 < 2\varepsilon$.

Implication "(3)$\Rightarrow$(1)" is obvious since $||Q|| \leqslant ||Q||_1$.

Concerning the last part of the theorem, we only mention that the proof in [5, p 122-123] functions for any operators A, B that can be decomposed as $A = A_1 \oplus A_2$ and

$B = B_1 \oplus B_2$ relative to decompositions of $\mathscr{H}$ as $\mathscr{H}_1 \oplus \mathscr{H}_2$ and $\mathscr{K}_1 \oplus \mathscr{K}_2$ with $\mathscr{H}_1$ $\mathscr{K}_1$ the properties that were seen in the proof of implication "(2)⇒(3)". □

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