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THE ORTHOGONALITY STRUCTURE DETERMINES A C^*–ALGEBRA WITH CONTINUOUS TRACE

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Abstract. There are four versions of disjointness structures of a C*-algebra: zero product, range orthogonality, domain orthogonality and doubly orthogonality. Recently, Leung and Wong show that the linear and zero product structures are sufficient to determine the CCR C*-algebras with Hausdorff spectrums. In this paper, we investigate the orthogonality structures of the C*-algebras. More precisely, let θ be a bijective linear map between two C*-algebras with continuous traces. We prove that θ is automatically continuous whenever it preserves range (respectively, domain) orthogonal elements in both senses.

1. Introduction

It is a well-known fact that two C*-algebras are isomorphic as C*-algebras if and only if they are isomorphic as *-algebras. That is, the norm structure of a C*-algebra can be recovered from its *-algebraic structure. It is further showed by Gardner [14] (See, e.g., Sakai [20, Theorem 4.1.20]) that two C*-algebras are *-algebraic isomorphic if and only if they are algebraically isomorphic. Therefore, C*-algebras are completely determined by their algebraic structures.

C*-algebras also carry the so-called *disjointness structures*. There are four versions of disjointness: zero product ($ab = 0$), range orthogonality ($a^*b = 0$), domain orthogonality ($ab^* = 0$), and doubly orthogonality ($a^*b = ab^* = 0$). Note that all the four versions of disjointness structures coincide whenever the C*-algebras are abelian. A linear map $\theta\colon \mathscr{A} \to \mathscr{B}$ between two C*-algebras is said to be

(a) zero product preserving if $\theta(a)\theta(b) = 0$ whenever $ab = 0$,

(b) range orthogonality preserving if $\theta(a)^*\theta(b) = 0$ whenever $a^*b = 0$,

(c) domain orthogonality preserving if $\theta(a)\theta(b)^* = 0$ whenever $ab^* = 0$,

(d) doubly orthogonality preserving if $\theta(a)^*\theta(b) = \theta(a)\theta(b)^* = 0$ whenever $a^*b = ab^* = 0$.

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The starting point of the study in disjointness preserving operators between C*-algebras is the paper [5], where Arendt established a complete description of all bounded linear disjointness preservers between $C(K)$ spaces. In [15], Jarosz extended the study to the setting of linear disjointness preservers (not necessarily continuous) between abelian C*-algebras. Among the consequences derived from Jarosz result, it follows that every bijective linear disjointness preserver between $C(K)$ spaces is automatically continuous.

In the setting of general C*-algebras, the four versions of disjointness preserving operators need not coincide. In 1994, Wolff [23] classified the symmetric bounded linear doubly orthogonality preserving operators form a unital C*-algebra into another C*-algebra. In general, bounded (but not necessarily symmetric) zero-product preservers between (not necessarily unital) C*-algebras were studied in [9, 25], and range (or domain) orthogonality preservers were studied in [21].

Every C*-algebra A admits a triple product defined by $2\{a,b,c\} := ab^*c + cb^*a$. This triple product characterizes orthogonal elements in A. Wong established, in [24], that a bounded linear operator θ between two C*-algebras is a triple homomorphism (i.e., it preserves triple products) if and only if θ is doubly orthogonality preserving and $\theta^{**}(1)$ is a partial isometry in B^{**}. A complete description of all doubly orthogonality preserving bounded linear operators between C*-algebras (also between JB*-algebras and JB*-triples) was obtained by Burgos, Fernández-Polo, Garcés, Martınez-Moreno and Peralta in [6, 7]. Among the consequences of this description, it follows that a bounded linear operators between C*-algebras is doubly orthogonality preserving if and only if it preserves zero-triple-products.

There exists a vast list of contributions to the study of zero-products and (range, domain, or doubly) orthogonality preserving operators between Banach algebras and C*-algebras, see for example [1, 3, 8, 9, 10, 13, 15, 16, 18].

Let X be a Banach space. A subalgebra $\mathscr{A}$ of the space $B(X)$ of all bounded linear operators on X is said to be a *standard algebra* if it contains all finite-dimensional operators and the identity operator on X. Araujo and Jarosz proved in [4] that every bijective linear zero product preserver between two standard operator algebras, is automatically continuous and a scalar multiple of an algebra isomorphism. They conjectured that every bijective linear zero product preserver between C*-algebras is automatically continuous; a conjecture which remains open even for the other three orthogonality preservers between C*-algebras. See [22] for more discussions.

The Araujo-Jarosz conjecture was recently treated by Leung and Wong [19] in the setting of CCR C*-algebras with Hausdorff spectrum (for the definition, see Section 2). They proved that this conjecture is true in this setting. We state their result here.

THEOREM 1. ([19, Theorem 3.3]) *Let $\mathscr{A}$ and $\mathscr{B}$ be CCR C*-algebras with Hausdorff spectrum. Let $\theta : \mathscr{A} \to \mathscr{B}$ be a bijective linear map such that*

$$ab = 0 \text{ in } \mathscr{A} \quad \text{if and only if} \quad \theta(a)\theta(b) = 0 \text{ in } \mathscr{B}.$$

Then θ is automatically bounded. Indeed, $\theta = m\Psi$ where $m = \theta^{**}(1)$ is an invertible central multiplier of $\mathscr{B}$ and Ψ is an algebra isomorphism from $\mathscr{A}$ onto $\mathscr{B}$.

In this paper, we investigate the structure of the linear (range or domain) orthogonality preservers between C*-algebras with continuous traces. More precisely, we will prove that linear and orthogonality structures are sufficient to determine C*-algebras with continuous traces.

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2. Preliminaries

Let X be a locally compact Hausdorff space, called the *base space*. Suppose that for each x in X there is a C*-algebra A_x. A *vector field* f is an element in the product space $\prod_{x\in X} A_x$, that is, $f(x) \in A_x, \forall x \in X$.

DEFINITION 2. ([11, 12]) A *continuous field* $\mathscr{E} = (X, \{A_x\}, \mathscr{A})$ *of non trivial C*-algebras* over a locally compact Hausdorff space X is a family $\{A_x\}_{x\in X}$ of C*-algebras, with a set $\mathscr{A}$ of vector fields, satisfying the following conditions.

1. $\mathscr{A}$ is a *-subalgebra of $\prod_{x\in X} A_x$.
2. For every x in X, the set of all $f(x)$ with f in $\mathscr{A}$ is dense in A_x.
3. For every f in $\mathscr{A}$, the function $x \mapsto \|f(x)\|$ is continuous on X and vanishes at infinity.
4. Let f be a vector field. Suppose for every x_0 in X and every $\varepsilon > 0$, there is a neighborhood U of x_0 and a g in $\mathscr{A}$ such that $\|f(x) - g(x)\| < \varepsilon$ for all x in U. Then $f \in \mathscr{A}$.

Elements in $\mathscr{A}$ are called *continuous vector fields*. Let f be a vector field, we define the *support*, $\operatorname{supp} f$, of f to be the closure of the set $\{x \in X : f(x) \neq 0\}$ in X.

If g is a bounded continuous function on X, and $f \in \mathscr{A}$, then $x \mapsto g(x)f(x)$ defines a continuous vector field gf on X. The set of all $f(x)$ with f in $\mathscr{A}$ coincides with A_x for every x in X. Moreover, for any distinct points x, y in X and any α in A_x and β in A_y, there is a continuous vector field f such that $f(x) = \alpha$ and $f(y) = \beta$ (see, e.g., [12, 17]).

When all A_x equal to a fixed C*-algebra A, and $\mathscr{A}$ consists of all continuous functions from X into A vanishing at infinity, we call $\mathscr{E}$ a *constant field*. In this case, we write $\mathscr{A} = C_0(X,A)$, or $\mathscr{A} = C(X,A)$ when X is compact, as usual. In general, $\mathscr{A}$ becomes a C*-algebra under the supremum norm $\|f\| = \sup_{x\in X} \|f(x)\|$.

Conversely, a C*-algebra $\mathscr{A}$ is called a CCR C*-algebra if every irreducible representation of $\mathscr{A}$ consists of compact operators. The spectrum $\hat{\mathscr{A}}$ of $\mathscr{A}$ is the family of unitary equivalence classes of nonzero irreducible representations under the hull-kernel topology. This topology is always locally compact, and the spectrum of a CCR C*-algebra is T_1 (compare [11, §3]). Suppose that $X = \hat{\mathscr{A}}$ is Hausdorff. According to [11, Theorem 10.5.4], we can represent $\mathscr{A}$ as a continuous field of C*-algebras $(X, \{A_x\}, \mathscr{A})$, where for each x in X, A_x is the C*-algebra of all compact operators on a complex Hilbert space H_x (i.e. an elementary C*-algebra).

Every C*-algebra with a continuous trace is a CCR C*-algebra with Hausdorff spectrum and satisfies *Fell's condition*, that is, if $(X,\{A_x\},\mathscr{A})$ is a representation satisfying the conditions of the above paragraph, for every x_0 in X, there exists a neighborhood U of x_0 and a vector field p of $(X,\{A_x\},\mathscr{A})$, defined and continuous in U, such that, $p(x)$ is a projection of rank one for every x in U (compare [11, Proposition 4.5.3]). Actually, a CCR C*-algebra with Hausdorff spectrum admits a continuous trace if and only if its representation as continuous field of C*-algebras satisfies Fell's condition (cf. [11, Theorem 10.5.8]).

Throughout this paper, we use the following conventions. For a locally compact Hausdorff space X, we write

$$X_\infty = X \cup \{\infty\},$$

for its one-point compactification. If X is already compact, then the point ∞ at infinity is an isolated point in X_∞. Moreover, we identify

$$C_0(X) = \{f \in C(X_\infty) : f(\infty) = 0\},$$

and other similar spaces for those of continuous functions on X vanishing at infinity. For a continuous field $(X,\{A_x\},\mathscr{A})$ of C*-algebras, associate to each x in X the sets

$$\begin{aligned} I_x &= \{f \in \mathscr{A} : f \text{ vanishes in a neighborhood in } X_\infty \text{ of } x\}, \\ M_x &= \{f \in \mathscr{A} : f(x) = 0\}. \end{aligned}$$

In particular,

$$\begin{aligned} I_\infty &= \{f \in \mathscr{A} : f \text{ has a compact support}\}, \\ M_\infty &= \mathscr{A}. \end{aligned}$$

Furthermore, denote by δ_x the evaluation map at x in X, i.e.,

$$\delta_x(f) = f(x), \quad \forall f \in \mathscr{A}.$$

3. Results

THEOREM 3. *Let $\mathscr{A}$, $\mathscr{B}$ be two C*-algebras with continuous traces. Let $\theta : \mathscr{A} \to \mathscr{B}$ be a bijective linear map. The the following hold.*

(a) θ sends elements having orthogonal ranges to elements having orthogonal ranges, i.e.,

$$a^*b = 0 \textit{ in } \mathscr{A} \quad \Leftrightarrow \quad \theta(a)^*\theta(b) = 0 \textit{ in } \mathscr{B},$$

*if and only if $\theta = \Psi\theta^{**}(1)$, where $\theta^{**}(1)$ is an invertible right multiplier of $\mathscr{B}$, and Ψ is a *-algebra isomorphism from $\mathscr{A}$ onto $\mathscr{B}$.*

(b) θ sends elements having orthogonal domains to elements having orthogonal domains, i.e.,

$$ab^* = 0 \textit{ in } \mathscr{A} \quad \Leftrightarrow \quad \theta(a)\theta(b)^* = 0 \textit{ in } \mathscr{B},$$

*if and only if $\theta = \theta^{**}(1)\Psi$, where $\theta^{**}(1)$ is an invertible left multiplier of $\mathscr{B}$, and Ψ is a *-algebra isomorphism from $\mathscr{A}$ onto $\mathscr{B}$.*

(c) *θ sends elements having orthogonal ranges to elements having orthogonal domains, i.e.,*

$$a^*b = 0 \text{ in } \mathscr{A} \quad \Leftrightarrow \quad \theta(a)\theta(b)^* = 0 \text{ in } \mathscr{B},$$

if and only if $\theta = \theta^{**}(1)\Psi$, where $\theta^{**}(1)$ is an invertible left multiplier of $\mathscr{B}$, and Ψ is an anti--algebra isomorphism from $\mathscr{A}$ onto $\mathscr{B}$.*

(d) *θ sends elements having orthogonal domains to elements having orthogonal ranges, i.e.,*

$$ab^* = 0 \text{ in } \mathscr{A} \quad \Leftrightarrow \quad \theta(a)^*\theta(b) = 0 \text{ in } \mathscr{B},$$

if and only if $\theta = \Psi\theta^{**}(1)$, where $\theta^{**}(1)$ is an invertible right multiplier of $\mathscr{B}$, and Ψ is an anti--algebra isomorphism from $\mathscr{A}$ onto $\mathscr{B}$.*

In the following, only the first case will be proved. The proofs of the other cases are similar. We divide the proof into several lemmas as in [19], in which the zero product preservers are studied. We will complete the proof in the setting of continuous fields of C*-algebras.

Hereafter, we will regard $\mathscr{A}$ and $\mathscr{B}$ as the C*-algebras of continuous operator fields $(X, \{A_x\}, \mathscr{A})$ and $(Y, \{B_y\}, \mathscr{B})$, where X and Y are (Hausdorff) spectrums of $\mathscr{A}$ and $\mathscr{B}$, respectively. Note that A_x and B_y are elementary C*-algebras for every x in X and y in Y.

LEMMA 4. $\operatorname{supp}\theta^{-1}(\theta(f)g) \subseteq \operatorname{supp} f$ *for any f in $\mathscr{A}$ and g in $\mathscr{B}$.*

Proof. For any f in $\mathscr{A}$. Suppose that x_0 is not in $\operatorname{supp} f$. Since every locally compact Hausdorff space is completely regular, there is an open neighborhood U of x_0 such that $U \cap \operatorname{supp} f = \emptyset$. Fix $x \in U$ and a rank one projection $u_x \otimes u_x$ in A_x, there is a continuous operator field h such that $h(x) = u_x \otimes u_x$ and $\operatorname{supp} h \subseteq U$, that is, $h^* f = 0$. Since θ preserves range orthogonality, we have $\theta(h)^*\theta(f) = 0$. It is clear that for any g in $\mathscr{B}$, we have $\theta(h)^*\theta(f)g = 0$. The bijectivity and range orthogonality preserving property of θ^{-1} ensures that $h^*\theta^{-1}(\theta(f)g) = 0$. Hence, $h(x)^*\theta^{-1}(\theta(f)g)(x) = 0$ or u_x is not in the range of $\theta^{-1}(\theta(f)g)(x)$. Since u_x is arbitrary, $\theta^{-1}(\theta(f)g)(x) = 0$. This concludes that $U \cap \operatorname{supp}\theta^{-1}(\theta(f)g) = \emptyset$. $\square$

For every open subset U of X, denote by $\mathscr{A}_U$ the subalgebra

$$\mathscr{A}_U = \left\{ f \in \mathscr{A} : f \text{ vanishing outside a compact subset of } U \right\}.$$

For each y in Y, denote by

$$S_y = \Big\{ x \in X_\infty : \text{for every open neighborhood } U \text{ of } x,$$
$$\text{there is an } f \text{ in } \mathscr{A}_U \text{ such that } \theta(f)(y) \neq 0 \Big\}.$$

LEMMA 5. *S_y is a singleton, for all y in Y.*

Proof. First, we prove that S_y is not empty, for all y in Y. Suppose on the contrary that $S_y = \emptyset$ for some y in Y. Then for each x in X_∞ there is an open neighborhood U_x of x in X_∞ such that $\theta(f)(y) = 0$ for all f in $\mathscr{A}_{U_x}$. Let V_x be an open neighborhood of x with compact closure $\overline{V_x} \subseteq U_x$. By the compactness of X_∞,

$$X_\infty = V_{x_0} \cup V_{x_1} \cup \cdots \cup V_{x_n}$$

for some points $x_0 = \infty$, $x_1, \ldots, x_n$ in X_∞. Let

$$1 = h_0 + h_1 + \cdots + h_n$$

be a continuous partition of unity such that h_i vanishes outside V_{x_i} for $i = 0, 1, \ldots, n$. For any g in $\mathscr{A}$, observe that

$$h_i g \in \mathscr{A}_{U_{x_i}} \quad \text{implies} \quad \theta(h_i g)(y) = 0$$

and then

$$\theta(g)(y) = \theta(\sum_{i=0}^{n} h_i g)(y) = \sum_{i=0}^{n} \theta(h_i g)(y) = 0, \quad \forall g \in \mathscr{A}.$$

This gives a contradiction that $y = \infty$.

Before we show that S_y consists of exactly one point for all y in Y, we prove the following claim.

CLAIM. The cozero sets of $\theta(f_1)$ and $\theta(f_2)$ are disjoint whenever f_1, $f_2 \in \mathscr{A}$ have disjoint supports.

Suppose on the contrary that there exist f_1, $f_2 \in \mathscr{A}$ and $y \in Y$ such that $\operatorname{supp} f_1 \cap \operatorname{supp} f_2 = \emptyset$ but $\theta(f_1)(y) \neq 0$ and $\theta(f_2)(y) \neq 0$. Note that $g_1 = \theta(f_1)$ and $g_2 = \theta(f_2)$ have orthogonal ranges since f_1 and f_2 have. Because $\mathscr{B}$ has continuous trace, there is a neighborhood W of y, and two continuous rank one operator fields $u \otimes u$ and $v \otimes u$ defined on W, where $u(y)$ and $v(y)$ are eigenvectors of $g_1(y)^* g_1(y)$ and $g_2(y)^* g_2(y)$ associated with nonzero eigenvalues, respectively. Pick any $\lambda \in C_0(Y)$ such that $\operatorname{supp} \lambda \subseteq W$ and $\lambda(y) = 1$. Define $h_1, h_2 \in \mathscr{B} \setminus \{0\}$ by

$$h_1(\mu) := \begin{cases} \lambda(\mu) \|g_2(\mu) v(\mu)\| g_1(\mu)(u(\mu) \otimes u(\mu)) & \mu \in W, \\ 0 & \mu \notin W, \end{cases}$$

and

$$h_2(\mu) := \begin{cases} \lambda(\mu) \|g_1(\mu) u(\mu)\| g_2(\mu)(v(\mu) \otimes u(\mu)) & \mu \in W, \\ 0 & \mu \notin W. \end{cases}$$

Note that h_1 and h_2 have orthogonal ranges as g_1 and g_2 have. Observe that

$$\begin{aligned}
(h_1 + h_2)^*(h_1 - h_2) &= h_1^* h_1 - h_1^* h_2 + h_2^* h_1 - h_2^* h_2 \\
&= h_1^* h_1 - h_2^* h_2 \\
&= \lambda^2 \|g_2 v\|^2 (u \otimes u) g_1^* g_1 (u \otimes u) - \lambda^2 \|g_1 u\|^2 (u \otimes v) g_2^* g_2 (v \otimes u) \\
&= \lambda^2 \|g_2 v\|^2 (u \otimes g_1 u)(g_1 u \otimes u) - \lambda^2 \|g_1 u\|^2 (u \otimes g_2 v)(g_2 v \otimes u) \\
&= \lambda^2 \|g_2 v\|^2 \|g_1 u\|^2 (u \otimes u) - \lambda^2 \|g_1 u\|^2 \|g_2 v\|^2 (u \otimes u) \\
&= 0
\end{aligned}$$

on W. It is clear that $(h_1+h_2)^*(h_1-h_2)=0$ outside W. This shows that h_1+h_2 and h_1-h_2 have range orthogonality. Set $f_1' := \theta^{-1}(h_1) \neq 0$ and $f_2' := \theta^{-1}(h_2) \neq 0$. Because θ^{-1} is linear and preserves range orthogonality, this ensures the range orthogonality of f_1' and f_2' as well as that of $f_1'+f_2'$ and $f_1'-f_2'$. It follows that

$$\begin{aligned} 0 = (f_1'+f_2')^*(f_1'-f_2') &= f_1'^*f_1' - f_1'^*f_2' + f_2'^*f_1' - f_2'^*f_2' \\ &= f_1'^*f_1' - f_2'^*f_2', \end{aligned}$$

that is $f_1'^*f_1' = f_2'^*f_2' \neq 0$ or $\operatorname{supp} f_1'^*f_1' = \operatorname{supp} f_2'^*f_2' \neq \emptyset$. Note that $\operatorname{supp} f_1' \subseteq \operatorname{supp} f_1$ and $\operatorname{supp} f_2' \subseteq \operatorname{supp} f_2$ by Lemma 4, we have $\operatorname{supp} f_1' \cap \operatorname{supp} f_2' = \emptyset$. Thus $\operatorname{supp} f_1'^*f_1' \cap \operatorname{supp} f_2'^*f_2' = \emptyset$, which is a contradiction.

Now we are ready to prove that S_y consists of exactly one point for all y in Y. Suppose that x_1, $x_2 \in S_y$ and $x_1 \neq x_2$. Let U_1 and U_2 be disjoint open neighborhoods of x_1 and x_2, respectively. By the definition of S_y, there exist f_1, $f_2 \in \mathscr{A}$ with $\operatorname{supp} f_i \subset U_i$ and $\theta(f_i)(y) \neq 0$ $(i=1,2)$ which contradicts the claim. $\square$

Define a map φ from Y into X_∞ by $S_y = \{\varphi(y)\}$.

LEMMA 6. *The point $\varphi(y)$ is the unique point in X_∞ satisfying the condition that*

$$\theta(I_{\varphi(y)}) \subseteq M_y, \quad \forall y \in Y. \tag{1}$$

Proof. Let $f \in I_{\varphi(y)}$ vanish in an open neighborhood U of $\varphi(y)$. For all $x \notin U$, by the definition of S_y there is an open neighborhood V_x of x such that $\theta(\mathscr{A}_{V_x})(y) = \{0\}$. By compactness, we can write $X_\infty = U \cup V_{x_1} \cup \cdots \cup V_{x_n}$ for some $x_1, \ldots, x_n$ in $X_\infty \setminus U$. Let $1 = h + h_1 + \cdots + h_n$ be a corresponding continuous partition of unity. Note that $\theta(h_i g)(y) = 0$ for all g in $\mathscr{A}$ and $i = 1, \ldots, n$. Hence, $\theta(g)(y) = \theta(hg)(y)$ for all g in $\mathscr{A}$. As $f^*(hg) = 0$, we see that $\theta(f)(y)^*\theta(g)(y) = \theta(f)(y)^*\theta(hg)(y) = 0$. Since θ is bijective, $\theta(f)(y)^* = 0$ or $\theta(f)^* \in M_y$. Therefore, $\theta(f)(y) = 0$ or $\theta(f) \in M_y$. Finally, the uniqueness assertion follows from the definition of S_y. $\square$

The proofs of the following two lemmas are similar to those in [19], and thus omitted. It should be noticed that, since θ is a bijection and A_y is non-zero, $\delta_y \circ \theta \neq 0$ for every $y \in Y$.

LEMMA 7. *$\varphi : Y \to X_\infty$ is continuous.*

LEMMA 8. *Let $\{y_n\}$ be a sequence in Y such that $\varphi(y_n)$ are distinct points in X_∞. Then*

$$\limsup \|\delta_{y_n} \circ \theta\| < +\infty.$$

Proof of Theorem 3. With the above lemmas, we have already constructed a continuous function φ from Y into X_∞, which satisfies the condition

$$\theta(I_{\varphi(y)}) \subseteq M_y, \quad \forall y \in Y.$$

Set

$$Y_1 = \{y \in Y : \theta(M_{\varphi(y)}) \subseteq M_y\},$$
$$Y_2 = \{y \in Y : \theta(M_{\varphi(y)}) \nsubseteq M_y\}.$$

Then $Y_\infty = \{\infty\} \cup Y_1 \cup Y_2$ is a disjoint union. If $y \in Y_1$, we have $\theta(M_{\varphi(y)}) \subseteq M_y$ and hence there is a linear operator $H_y : A_{\varphi(y)} \to B_y$ such that

$$\theta(f)(y) = H_y(f(\varphi(y))), \quad \forall f \in \mathscr{A}. \tag{2}$$

To see Y_2 is empty, we first prove that $\varphi(Y_2)$ is a finite set of non-isolated points in X_∞. Let $x = \varphi(y)$ with y in Y_2. Then we have

$$\theta(I_x) \subseteq M_y \quad \text{but} \quad \theta(M_x) \nsubseteq M_y.$$

This implies the linear operator $\delta_y \circ \theta$ is unbounded, since I_x is dense in M_x by Uryshon's Lemma. By Lemma 8, we can have only finitely many of such x's. So $\varphi(Y_2)$ is a finite set. Moreover, if x is an isolated point in X_∞ then $I_x = M_x$ and thus $x \notin \varphi(Y_2)$.

Next we prove that Y_2 is open, or equivalently, $\{\infty\} \cup Y_1$ is closed in Y_∞. Let $y_\lambda \to y$ with y_λ in $\{\infty\} \cup Y_1$. We want to show that $y \in \{\infty\} \cup Y_1$. We might assume $y_\lambda \in Y_1$ for all λ and $y \neq \infty$. By Lemma 7, we see that $\varphi(y_\lambda) \to \varphi(y)$. In case there is any subnet of $\{\varphi(y_\lambda)\}$ consisting of only finitely many points, we can assume $\varphi(y_\lambda) = \varphi(y)$ for all λ. Then for all f in $\mathscr{A}$, $f(\varphi(y)) = 0$ implies $f(\varphi(y_\lambda)) = 0$, and thus $\theta(f)(y_\lambda) = 0$ for all λ by (2). By continuity, $\theta(f)(y) = 0$. Consequently, $\theta(M_{\varphi(y)}) \subseteq M_y$, and thus $y \in Y_1$. In the other case, every subnet of $\{\varphi(y_\lambda)\}$ contains infinitely many points. Lemma 8 asserts that $M = \limsup \|H_{y_\lambda}\| < +\infty$. This gives

$$\|\theta(f)(y)\| = \lim \|\theta(f)(y_\lambda)\| = \lim \|H_{y_\lambda}(f(\varphi(y_\lambda)))\| \leqslant M\|f(\varphi(y))\|.$$

Thus, if $f(\varphi(y)) = 0$ we have $\theta(f)(y) = 0$. Consequently, $y \in Y_1$.

Now, we are ready to show that Y_2 is empty. It follows from (1) that $\varphi(Y) = \varphi(Y_1) \cup \varphi(Y_2)$ is dense in X. Since $\varphi(Y_2)$ is a finite set of non-isolated points in X, we see that $\varphi(Y_1)$ alone is dense in X. On the other hand, let $y \in Y_1$ with $\varphi(y) = x$ in X, and $\psi(x) = z$ in Y_∞. Here, the map $\psi : X \to Y_\infty$, and the decomposition $X = X_1 \cup X_2$ is induced by θ^{-1} in an analogous way. In particular, we have

$$\theta(M_x) \subseteq M_y \quad \text{and} \quad \theta^{-1}(I_z) \subseteq M_x.$$

Consequently, $I_z \subseteq \theta(M_x) \subseteq M_y$ gives $y = z \in \psi(X)$. In case $y \in \psi(X_1)$, we have $\theta(M_x) = M_y$. Since $\psi(X_2)$ is a finite set of non-isolated points in Y, we have $\theta(M_{\varphi(y)}) = M_y$ for all but at most finitely many y in Y_1. Therefore, the linear map H_y is bijective for all but at most finitely many y in Y_1, which are non-isolated points in Y. Hence, if $\theta(f)$ vanishes in Y_1 then f vanishes on the dense set $\varphi(Y_1)$ by (2), and thus $f = 0$. Therefore, Y_1 is dense in Y by the surjectivity of φ. The openness of Y_2 forces itself to be empty.

Now, $Y = Y_1$ and $X = X_1$ imply that $\psi = \varphi^{-1}$ and thus φ is a homeomorphism from Y onto X. In addition to this, both θ and θ^{-1} can be written as weighted composition operators:

$$\theta(f)(y) = H_y(f(\varphi(y))), \quad \forall f \in \mathscr{A}, \forall y \in Y,$$

$$\theta^{-1}(g)(x) = T_x(g(\psi(x))), \quad \forall g \in \mathscr{B}, \forall x \in X.$$

It is easy to see that the linear map $H_y \colon A_{\varphi(y)} \to B_y$ has an inverse T_y for every y in Y, and thus it is bijective.

Suppose $\alpha^*\beta = 0$ in A_x for some $x = \varphi(y)$. Consider the closed two-sided ideal $I = \{c \in \mathscr{A} : c(x) = 0\}$ of $\mathscr{A}$. Let a,b in $\mathscr{A}$ be such that $a(x) = \alpha^*, b(x) = \beta$. Then $ab \in I$. By a result of Akemann and Pedersen [2] (see also [9, Lemma 4.14]), we can find a', b' in $\mathscr{A}$ such that $a'(x) = \alpha^*$, $b'(x) = \beta$ and $a'b' = 0$. Now $\theta(a'^*)^*\theta(b') = 0$ implies $H_y(\alpha)^*H_y(\beta) = 0$. So each H_y preserves the orthogonality of ranges.

So far we know that $H_y \colon A_{\varphi(y)} \to B_y$ is a bijective linear map and preserves range orthogonality in both sense for each y in Y. Note that $A_{\varphi(y)}$ and B_y are elementary C*-algebras and hence $A_{\varphi(y)}$ and B_y consists of compact operators on Hilbert spaces $\mathscr{H}_{\varphi(y)}$ and $\mathscr{K}_y$, respectively. It is showed in [22, Theorem 3] that H_y is bounded and assumes the following form $H_y(a) = U_y a S_y$, where $U_y \colon \mathscr{H}_{\varphi(y)} \to \mathscr{K}_y$ is a unitary operator and $S_y \colon \mathscr{K}_y \to \mathscr{H}_{\varphi(y)}$ is an invertible bounded operator.

Next, we show that $\sup \|H_y\| < +\infty$. For else, there is a sequence $\{y_n\}$ in Y such that $\lim \|H_{y_n}\| = +\infty$. By Lemma 8, we can assume all $\varphi(y_n) = x$ in X. Let $e \in A_x$ and $f \in \mathscr{A}$ such that $f(x) = e$. Then

$$\|H_{y_n}(e)\| = \|\theta(f)(y_n)\| \leqslant \|\theta(f)\|, \quad n = 1, 2, \dots.$$

It follows from the uniform boundedness principle that $\sup \|H_{y_n}\| < +\infty$, a contradiction. The inequality

$$\begin{aligned} \|\theta\| &= \sup\{\|\theta(f)\| : f \in \mathscr{A} \text{ with } \|f\| = 1\} \\ &= \sup\{\|H_y(f(\varphi(y)))\| : f \in \mathscr{A} \text{ with } \|f\| = 1, y \in Y\} \\ &\leqslant \sup\{\|H_y\| : y \in Y\} \end{aligned}$$

implies that θ is bounded with $\|\theta\| = \sup\{\|H_y\| : y \in Y\}$.

Finally, we make use of a result of Lee [17, Lemma 2] which asserts that the multiplier algebras $M(A)$ and $M(B)$ can be represented as families of bounded operator fields in $(X, \{M(A_x)\})$ and $(Y, \{M(B_y)\})$, respectively. By restricting the double dual map of θ to $M(A)$, we see that the invertible multiplier $\theta^{**}(1)(y) = U_y S_y$. It is plain that the *-algebra isomorphism $\Psi = \theta\theta^{**}(1)^{-1}$ is given by sending a continuous operator field $\{f(x)\}$ to $\{U_y f(\varphi(y)) U_y^*\}$. $\square$

As a consequence of Theorem 3, the linear and orthogonality structures suffice to determine a C*-algebras with continuous traces.

THEOREM 9. *Two C*-algebras $\mathscr{A}, \mathscr{B}$ with continuous traces are *-algebra isomorphic (respectively, anti-*-algebra isomorphic) if and only if they have the same* *linear and orthogonality (respectively, reversed orthogonality) structures, i.e. there is a bijective linear map* $\theta : \mathscr{A} \to \mathscr{B}$ *satisfying Case 1 or Case 2 (respectively, Case 3 or Case 4) in Theorem* 3.

We remark that the only place in our arguments we need to utilize the existence of a continuous trace for the C*-algebras (i.e., Fell's condition) is in the verifying of the Claim in the proof of Lemma 5. It seems to us that one might be able to get rid of this technical assumption, and extend Theorems 3 and 9 to the wider class of CCR C*-algebras with Hausdorff spectrum. We hope to finish this task in future.

In the case of doubly orthogonality preservers, we do not need to assume the Fell's condition. We thank the referee for suggesting us to include the following result.

THEOREM 10. *Two CCR C*-algebras* $\mathscr{A},\mathscr{B}$ *with Hausdorff spectrum are isomorphic as JB*-algebras if and only if they have the same linear and double orthogonality structures. More precisely, let* $\theta \colon \mathscr{A} \to \mathscr{B}$ *be a bijective linear map such that*

$$a^*b = ab^* = 0 \text{ in } \mathscr{A} \quad \text{if and only if} \quad \theta(a)^*\theta(b) = \theta(a)\theta(b)^* = 0 \text{ in } \mathscr{B}.$$

Then θ *is automatically bounded. Indeed,*

$$\theta(\cdot) = \theta^{**}(1)\Psi_1(\cdot) = \Psi_2(\cdot)\theta^{**}(1),$$

where $\theta^{**}(1)$ *is an invertible central multiple of a unitary in the multiplier algebra of* $\mathscr{B}$*, and* Ψ_1*,* Ψ_2 *are Jordan *-algebra isomorphisms from* $\mathscr{A}$ *onto* $\mathscr{B}$*.*

Proof. As we have already pointed out that the proof of Theorem 3, except for the part in verifying the Claim in Lemma 5, does not assume Fell's condition. We can use similar arguments for the current case. In other words, it suffices to establish that S_y is a singleton without assuming Fell's condition in this situation. We will adapt the arguments in [19, Lemma 2.6] to achieve this goal.

Suppose that S_y contains two distinct points x_1,x_2. Let U_1,U_2 be disjoint open neighborhoods of x_1,x_2, respectively. Let f_1,f_2 be in $\mathscr{A}_{U_1},\mathscr{A}_{U_2}$, respectively. So, $f_1^*f_2 = f_1f_2^* = 0$. Thus,

$$\theta(f_1)^*\theta(f_2) = 0, \quad \forall f_1 \in \mathscr{A}_{U_1}, \forall f_2 \in \mathscr{A}_{U_2}. \tag{3}$$

Let E_1 be the orthogonal complement to the sum of the range spaces of all members in $\theta(A_{U_1})(y)$. It follows from (3) and the non-triviality of $\theta(\mathscr{A}_{U_2})(y)$ that E_1 is a nonzero subspace of the underlying Hilbert space E_y on which the elementary C*-algebra B_y is acting. Let V be any open set in X with compact closure contained in U_1. For any h in $\mathscr{A}_V$, let g be a continuous scalar function on X_∞ such that $g = 1$ on the support of h and g vanishes outside V. Then for any f in $\mathscr{A}$, we have $fg \in \mathscr{A}_{U_1}$, and thus $\theta(fg)(y)^* = 0$ on E_1. On the other hand, we have $h^*(f(1-g)) = h(f(1-g))^* = 0$. This forces $\theta(h)(y)\theta(f)^*(y) = \theta(h)(y)\theta(fg)^*(y) = 0$ on E_1. Note that $\theta(A)^* = \theta(\mathscr{A}) = \mathscr{B}$, and $B_yE_1 = E_y$ since $E_1 \neq 0$. Consequently, $\theta(h)(y) = 0$ for all h in $\mathscr{A}_{U_1}$. This gives a contradiction to the non-triviality of $\theta(A_{U_1})$ at y. Hence, S_y is a singleton.

Argue as in other parts of the proof of Theorem 3, we will have a field of bijective linear fiber maps $H_y : A_{\varphi(y)} \to B_y$ such that both H_y and its inverse H_y^{-1} preserve double orthogonality. By [22, Theorem 3], H_y is bounded, and there exist a nonzero scalar λ_y, a unitary operator $U_y\colon E_y \to E_y$ and Jordan *-isomorphisms $\Psi_{1y}, \Psi_{2y} : A_{\varphi(y)} \to B_y$ such that

$$H_y(\cdot) = \lambda_y U_y \Psi_{1y}(\cdot) = \lambda_y \Psi_{2y}(\cdot) U_y, \quad \forall y \in Y.$$

By a similar argument as in the proof of Theorem 3 again, we will see that θ is bounded, and $\theta^{**}(1)$ is determined by the field of scalar multiples of unitary operators $y \mapsto \lambda_y U_y$. By [17], we see that the map $y \mapsto \lambda_y$ determines an invertible central element, and the map $y \mapsto U_y$ determines a unitary, in the multiplier algebra of $\mathscr{B}$. It is now plain that $\Psi_1(\cdot) := \theta^{**}(1)^{-1}\theta(\cdot)$ and $\Psi_2 := \theta(\cdot)\theta^{**}(1)^{-1}$ are both Jordan *-isomorphisms from $\mathscr{A}$ onto $\mathscr{B}$. $\square$

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