JORDAN $\ast$-HOMOMORPHISMS ON $C^\ast$-ALGEBRAS

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Abstract. In this paper, we investigate Jordan $\ast$-homomorphisms on $C^\ast$-algebras associated with the following functional inequality

$$\|f\left(\frac{b-a}{3}\right) + f\left(\frac{a-3c}{3}\right) + f\left(\frac{3a+3c-b}{3}\right)\| \leq \|f(a)\|.$$

We moreover prove the superstability and the generalized Hyers-Ulam stability of Jordan $\ast$-homomorphisms on $C^\ast$-algebras associated with the following functional equation

$$f\left(\frac{b-a}{3}\right) + f\left(\frac{a-3c}{3}\right) + f\left(\frac{3a+3c-b}{3}\right) = f(a).$$

1. Introduction

The stability of functional equations was first introduced by Ulam [27] in 1940. More precisely, he proposed the following problem: Given a group $G_1$, a metric group $(G_2,d)$ and a positive number $\varepsilon$, does there exist a $\delta > 0$ such that if a function $f : G_1 \rightarrow G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x,y \in G_1$, then there exists a homomorphism $T : G_1 \rightarrow G_2$ such that $d(f(x), T(x)) < \varepsilon$ for all $x \in G_1$? As mentioned above, when this problem has a solution, we say that the homomorphisms from $G_1$ to $G_2$ are stable. In 1941, Hyers [6] gave a partial solution of Ulam’s problem for the case of approximate additive mappings under the assumption that $G_1$ and $G_2$ are Banach spaces. In 1978, Th. M. Rassias [22] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. This phenomenon of stability that was introduced by Th. M. Rassias [22] is called generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability.

THEOREM 1.1. Let $f : E \rightarrow E'$ be a mapping from a norm vector space $E$ into a Banach space $E'$ subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon (\|x\|^p + \|y\|^p) \quad (1.1)$$

for all $x, y \in E$, where $\varepsilon$ and $p$ are constants with $\varepsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \rightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon}{2-p} \|x\|^p \quad (1.2)$$


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for all \( x \in E \). If \( p < 0 \) then inequality (1.1) holds for all \( x, y \neq 0 \), and (1.2) for \( x \neq 0 \). Also, if the function \( t \mapsto f(tx) \) from \( \mathbb{R} \) into \( E' \) is continuous for each fixed \( x \in E \), then \( T \) is \( \mathbb{R} \)-linear.

During the last decades several stability problems of functional equations have been investigated by many mathematicians. A large list of references concerning the stability of functional equations can be found in [1, 2, 3, 11, 16, 25, 26, 28].

**Definition 1.2.** Let \( A, B \) be two \( C^* \)-algebras. A \( \mathbb{C} \)-linear mapping \( f : A \to B \) is called a Jordan \(*\)-homo- morphism if

\[
\begin{align*}
& f(a^2) = f(a)^2, \\
& f(a^*) = f(a)^* 
\end{align*}
\]

for all \( a \in A \).

C. Park [19] introduced and investigated Jordan \(*\)-derivations on \( C^* \)-algebras associated with the following functional inequality

\[
\|f(a) + f(b) + kf(c)\| \leq \|k f\left(\frac{a+b}{k} + c\right)\|
\]

for some integer \( k \) greater than 1 and proved the generalized Hyers-Ulam stability of Jordan \(*\)-derivations on \( C^* \)-algebras associated with the following functional equation

\[
f\left(\frac{a+b}{k} + c\right) = f(a) + f(b) + f(c)
\]

for some integer \( k \) greater than 1 (see also [20, 14, 15, 17, 21]).

In this paper, we investigate Jordan \(*\)-homomorphisms on \( C^* \)-algebras associated with the following functional inequality

\[
\|f\left(\frac{b-a}{3}\right) + f\left(\frac{a-3c}{3}\right) + f\left(\frac{3a+3c-b}{3}\right)\| \leq \|f(a)\|.
\]

We moreover prove the generalized Hyers-Ulam stability of Jordan \(*\)-homomorphisms on \( C^* \)-algebras associated with the following functional equation

\[
f\left(\frac{b-a}{3}\right) + f\left(\frac{a-3c}{3}\right) + f\left(\frac{3a+3c-b}{3}\right) = f(a).
\]

### 2. Jordan \(*\)-homomorphisms

In this section, we investigate Jordan \(*\)-homomorphisms on \( C^* \)-algebras. Throughout this section, assume that \( A, B \) are two \( C^* \)-algebras.
LEMMA 2.1. Let $f : A \to B$ be a mapping such that

$$
\left\| f\left( \frac{b-a}{3} \right) + f\left( \frac{a-3c}{3} \right) + f\left( \frac{3a+3c-b}{3} \right) \right\|_B \leq \|f(a)\|_B \quad (2.1)
$$

for all $a, b, c \in A$. Then $f$ is additive.

Proof. Letting $a = b = c = 0$ in (2.1), we get

$$
\|3f(0)\|_B \leq \|f(0)\|_B.
$$

So $f(0) = 0$. Letting $a = b = 0$ in (2.1), we get

$$
\|f(-c) + f(c)\|_B \leq \|f(0)\|_B = 0
$$

for all $c \in A$. Hence $f(-c) = -f(c)$ for all $c \in A$. Letting $a = 0$ and $b = 6c$ in (2.1), we get

$$
\|f(2c) - 2f(c)\|_B \leq \|f(0)\|_B = 0
$$

for all $c \in A$. Hence

$$
f(2c) = 2f(c)
$$

for all $c \in A$. Letting $a = 0$ and $b = 9c$ in (2.1), we get

$$
\|f(3c) - f(c) - 2f(c)\|_B \leq \|f(0)\|_B = 0
$$

for all $c \in A$. Hence

$$
f(3c) = 3f(c)
$$

for all $c \in A$. Letting $a = 0$ in (2.1), we get

$$
\left\| f\left( \frac{b}{3} \right) + f\left( -c \right) + f\left( c - \frac{b}{3} \right) \right\|_B \leq \|f(0)\|_B = 0
$$

for all $a, b, c \in A$. So

$$
f\left( \frac{b}{3} \right) + f\left( -c \right) + f\left( c - \frac{b}{3} \right) = 0 \quad (2.2)
$$

for all $a, b, c \in A$. Let $t_1 = c - \frac{b}{3}$ and $t_2 = \frac{b}{3}$ in (2.2). Then

$$
f(t_2) - f(t_1 + t_2) + f(t_1) = 0
$$

for all $t_1, t_2 \in A$. This means that $f$ is additive. \hfill \Box

Now we prove the superstability problem for Jordan $\ast$-homomorphisms as follows.
THEOREM 2.2. Let \( p < 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : A \rightarrow B \) be a mapping such that

\[
\left\| f\left( \frac{b-a}{3} \right) + f\left( \frac{a-3\mu c}{3} \right) + \mu f\left( \frac{3a+3c-b}{3} \right) \right\|_B \leq \|f(a)\|_B, \tag{2.2}
\]

\[
\|f(a^2) - f(a)^2\|_B \leq \theta \|a\|^{2p}, \tag{2.3}
\]

\[
\|f(a^*) - f(a)^*\|_B \leq \theta \|a^*\|^{p} \tag{2.4}
\]

for all \( \mu \in T^1 := \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \) and all \( a, b, c \in A \). Then the mapping \( f : A \rightarrow B \) is a Jordan \( * \)-homomorphism.

**Proof.** Let \( \mu = 1 \) in (2.2). By Lemma 2.1, the mapping \( f : A \rightarrow B \) is additive. Letting \( a = b = 0 \) in (2.2), we get

\[
\|f(-\mu c) + \mu f(c)\|_B \leq \|f(0)\|_B = 0
\]

for all \( c \in A \) and all \( \mu \in T^1 \). So

\[
-f(\mu c) + \mu f(c) = f(-\mu c) + \mu f(c) = 0
\]

for all \( c \in A \) and all \( \mu \in T^1 \). Hence \( f(\mu c) = \mu f(c) \) for all \( c \in A \) and all \( \mu \in T^1 \). By Theorem 2.1 of [18], the mapping \( f : A \rightarrow B \) is \( \mathbb{C} \)-linear. It follows from (2.3) that

\[
\|f(a^2) - f(a)^2\|_B = \left\| \frac{1}{n^2} f(n^2 a^2) - \left( \frac{1}{n} f(na) \right)^2 \right\|_B
\]

\[
= \frac{1}{n^2} \|f(n^2 a^2) - f(na)^2\|_B
\]

\[
\leq \frac{\theta}{n^2} \|a\|^{2p}
\]

for all \( a \in A \). Thus, since \( p < 1 \), by letting \( n \) tend to \( \infty \) in last inequality, we obtain \( f(a^2) = f(a)^2 \) for all \( a \in A \). On the other hand, it follows from (2.4) that

\[
\|f(a^*) - f(a)^*\|_B = \left\| \frac{1}{n} f(na^*) - \left( \frac{1}{n} f(na) \right)^* \right\|_B
\]

\[
= \frac{1}{n} \|f(na^*) - f(na)^*\|_B
\]

\[
\leq \frac{\theta}{n^p} \|a^*\|^p
\]

for all \( a \in A \). Thus, since \( p < 1 \), by letting \( n \) tend to \( \infty \) in last inequality, we obtain \( f(a^*) = f(a)^* \) for all \( a \in A \). Hence the mapping \( f : A \rightarrow B \) is a Jordan \( * \)-homomorphism. \( \square \)

THEOREM 2.3. Let \( p > 1 \) and \( \theta \) be a nonnegative real number, and let \( f : A \rightarrow B \) be a mapping satisfying (2.2), (2.3) and (2.4). Then the mapping \( f : A \rightarrow B \) is a Jordan \( * \)-homomorphism.
Proof. The proof is similar to the proof of Theorem 2.2. □

We prove the generalized Hyers-Ulam stability of Jordan ∗-homomorphisms on $C^*$-algebras.

**Theorem 2.4.** Suppose that $f : A \to B$ is an odd mapping for which there exists a function $\varphi : A \times A \times A \to \mathbb{R}^+$ such that

$$\sum_{i=0}^{\infty} 3^i \varphi \left( \frac{a}{3^i}, \frac{b}{3^i}, \frac{c}{3^i} \right) < \infty, \quad (2.5)$$

$$\lim_{n \to \infty} 3^{2n} \varphi \left( \frac{a}{3^n}, \frac{b}{3^n}, \frac{c}{3^n} \right) = 0, \quad (2.6)$$

$$\|f(a^*) - f(a)^*\|_B \leq \varphi(a, a, a), \quad (2.7)$$

$$\left\| f \left( \frac{\mu b - a}{3} \right) + f \left( \frac{a - 3c}{3} \right) + \mu f \left( \frac{3a - b}{3} + c \right) - f(a) + f(c^2) - f(c)^2 \right\|_B \leq \varphi(a, b, c) \quad (2.8)$$

for all $a, b, c \in A$ and all $\mu \in \mathbb{T}$. Then there exists a unique Jordan ∗-homomorphism $h : A \to B$ such that

$$\|h(a) - f(a)\|_B \leq \sum_{i=0}^{\infty} 3^i \varphi \left( \frac{a}{3^i}, \frac{2a}{3^i}, 0 \right) \quad (2.9)$$

for all $a \in A$.

Proof. Letting $\mu = 1$, $b = 2a$ and $c = 0$ in (2.8), we get

$$\left\| 3f \left( \frac{a}{3} \right) - f(a) \right\|_B \leq \varphi(a, 2a, 0)$$

for all $a \in A$. Using the induction method, we have

$$\left\| 3^n f \left( \frac{a}{3^n} \right) - f(a) \right\| \leq \sum_{i=0}^{n-1} 3^i \varphi \left( \frac{a}{3^i}, \frac{2a}{3^i}, 0 \right) \quad (2.10)$$

for all $a \in A$. In order to show the functions $h_n(a) = 3^n f \left( \frac{a}{3^n} \right)$ form a convergent sequence, we use the Cauchy convergence criterion. Indeed, replace $a$ by $\frac{a}{3^n}$ and multiply by $3^m$ in (2.10), where $m$ is an arbitrary positive integer. We find that

$$\left\| 3^{m+n} f \left( \frac{a}{3^{m+n}} \right) - 3^m f \left( \frac{a}{3^m} \right) \right\| \leq \sum_{i=m}^{m+n-1} 3^i \varphi \left( \frac{a}{3^i}, \frac{2a}{3^i}, 0 \right) \quad (2.11)$$

for all positive integers. Hence by the Cauchy criterion the limit $h(a) = \lim_{n \to \infty} h_n(a)$ exists for each $a \in A$. By taking the limit as $n \to \infty$ in (2.10) we see that

$$\|h(a) - f(a)\| \leq \sum_{i=0}^{\infty} 3^i \varphi \left( \frac{a}{3^i}, \frac{2a}{3^i}, 0 \right)$$
and (2.9) holds for all \(a \in A\). Let \(\mu = 1\) and \(c = 0\) in (2.8), we get
\[
\left\| f\left(\frac{b-a}{3}\right) + f\left(\frac{a}{3}\right) + f\left(\frac{3a-b}{3}\right) - f(a) \right\|_B \leq \varphi(a,b,0) \tag{2.12}
\]
for all \(a, b, c \in A\). Multiplying both sides (2.12) by \(3^n\) and Replacing \(a, b\) by \(\frac{a}{3^n}, \frac{b}{3^n}\), respectively, we get
\[
\left\| 3^n f\left(\frac{b-a}{3^{n+1}}\right) + 3^n f\left(\frac{a}{3^{n+1}}\right) + 3^n f\left(\frac{3a-b}{3^{n+1}}\right) - 3^n f\left(\frac{a}{3^n}\right) \right\|_B \leq 3^n \varphi\left(\frac{a}{3^n}, \frac{b}{3^n}, 0\right) \tag{2.13}
\]
for all \(a, b, c \in A\). Taking the limit as \(n \to \infty\), we obtain
\[
h\left(\frac{b-a}{3}\right) + h\left(\frac{a}{3}\right) + h\left(\frac{3a-b}{3}\right) - h(a) = 0 \tag{2.14}
\]
for all \(a, b, c \in A\). Putting \(b = 2a\) in (2.14), we get \(3h\left(\frac{a}{3}\right) = h(a)\) for all \(a \in A\). Replacing \(a\) by \(2a\) in (2.14), we get
\[
h(b-2a) + h(6a-b) = 2h(2a) \tag{2.15}
\]
for all \(a, b \in A\). Letting \(b = 2a\) in (2.15), we get \(h(4a) = 2h(2a)\) for all \(a \in A\). So \(h(2a) = 2h(a)\) for all \(a \in A\). Letting \(3a - b = s\) and \(b - a = t\) in (2.14), we get
\[
h\left(\frac{t}{3}\right) + h\left(\frac{s+t}{6}\right) + h\left(\frac{t}{3}\right) = h\left(\frac{s+t}{2}\right)
\]
for all \(s, t \in A\). Hence \(h(s) + h(t) = h(s+t)\) for all \(s, t \in A\). So, \(h\) is additive. Letting \(a = c = 0\) in (2.12) and using the above method, we have \(h(\mu b) = \mu h(b)\) for all \(b \in A\) and all \(\mu \in \mathbb{T}\). Hence by the Theorem 2.1 of [18], the mapping \(f : A \to B\) is \(\mathbb{C}\)-linear.

Now, let \(h' : A \to B\) be another \(\mathbb{C}\)-linear mapping satisfying (2.9). Then we have
\[
\left\| h(a) - h'(a) \right\|_B = 3^n \left\| h\left(\frac{a}{3^n}\right) - h'\left(\frac{a}{3^n}\right) \right\|_B \\
\leq 3^n \left[ \left\| h\left(\frac{a}{3^n}\right) - f\left(\frac{a}{3^n}\right) \right\|_B + \left\| h'\left(\frac{a}{3^n}\right) - f\left(\frac{a}{3^n}\right) \right\|_B \right] \\
\leq 2 \sum_{i=n}^{\infty} 3^i \varphi\left(\frac{a}{3^i}, \frac{2a}{3^i}, 0\right) = 0
\]
for all \(a \in A\). Letting \(\mu = 1\) and \(a = b = 0\) in (2.8), we get \(\left\| f(c^2) - f(c)^2 \right\|_B \leq \varphi(0,0,c)\) for all \(c \in A\). So
\[
\left\| h(c^2) - h(c)^2 \right\|_B = \lim_{n \to \infty} 3^{2n} \left\| f\left(\frac{c^2}{3^{2n}}\right) - f\left(\frac{c}{3^{2n}}\right)^2 \right\|_B \\
\leq \lim_{n \to \infty} 3^{2n} \varphi\left(0,0,\frac{c}{3^n}\right) = 0
\]
for all \(c \in A\). Hence \(h(c^2) = h(c)^2\) for all \(c \in A\). On the other hand we have
\[
\left\| h(c^*) - h(c)^* \right\|_B = \lim_{n \to \infty} 3^n \left\| f\left(\frac{c^*}{3^n}\right) - f\left(\frac{c}{3^n}\right)^* \right\|_B \\
\leq \lim_{n \to \infty} 3^n \varphi\left(\frac{c}{3^n}, \frac{c}{3^n}, \frac{c}{3^n}\right) = 0
\]
for all \(c \in A\). Hence \(h(c^*) = h(c)^*\) for all \(c \in A\). Hence \(h : A \to B\) is a unique Jordan \(*\)-homomorphism. \(\square\)
Corollary 2.5. Suppose that $f : A \to B$ is a mapping with $f(0) = 0$ for which there exists constant $\theta \geq 0$ and $p_1, p_2, p_3 > 1$ such that
\[ \left\| f \left( \frac{\mu b - a}{3} \right) + f \left( \frac{a - 3c}{3} \right) + \mu f \left( \frac{3a - b}{3} + c \right) - f(a) + f(c^2) - f(c)^2 \right\|_B \leq \theta(\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3}), \]
\[ \|f(a^*) - f(a)^*\|_B \leq \theta(\|a\|^{p_1} + \|a\|^{p_2} + \|a\|^{p_3}) \]
for all $a, b, c \in A$ and all $\mu \in \mathbb{T}$. Then there exists a unique Jordan $\ast$-homomorphism $h : A \to B$ such that
\[ \|f(a) - h(a)\|_B \leq \frac{\theta \|a\|^{p_1}}{1 - 3(1 - p_1)} + \frac{\theta 2^{p_2} \|a\|^{p_2}}{1 - 3(1 - p_2)} \]
for all $a \in A$.

Proof. Letting $\varphi(a, b, c) := \theta(\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3})$ in Theorem 2.4, we obtain the result. □

The following corollary is the Isac-Rassias stability.

Corollary 2.6. Let $\psi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ be a function with $\psi(0) = 0$ such that
\[ \lim_{t \to 0} \frac{\psi(t)}{t} = 0, \]
\[ \psi(st) \leq \psi(s) \psi(t) \quad s, t \in \mathbb{R}^+, \]
\[ 3\psi \left( \frac{1}{3} \right) < 1. \]

Suppose that $f : A \to B$ is a mapping with $f(0) = 0$ satisfying (2.7) and (2.8) such that
\[ \left\| f \left( \frac{\mu b - a}{3} \right) + f \left( \frac{a - 3c}{3} \right) + \mu f \left( \frac{3a - b}{3} + c \right) - f(a) + f(c^2) - f(c)^2 \right\|_B \leq \theta[\psi(\|a\|) + \psi(\|b\|) + \psi(\|c\|)]] \]
for all $a, b, c \in A$ where $\theta > 0$ is a constant. Then there exists a unique Jordan $\ast$-homomorphism $h : A \to B$ such that
\[ \|h(a) - f(a)\|_B \leq \frac{\theta(1 + \psi(2)) \psi(\|a\|)}{1 - 3\psi\left( \frac{1}{3} \right)} \]
for all $a \in A$.

Proof. Letting $\varphi(a, b, c) := \theta[\psi(\|a\|) + \psi(\|b\|) + \psi(\|c\|)]$ in Theorem 2.4, we obtain the result. □
THEOREM 2.7. Suppose that $f : A \to B$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : A \times A \times A \to B$ satisfying (2.7), (2.8) and (2.8) such that

$$\sum_{i=1}^{\infty} 3^{-i} \varphi(3^i a, 3^i b, 3^i c) < \infty,$$  \hfill (2.16)

$$\lim_{n \to \infty} 3^{-2n} \varphi(3^i a, 3^i b, 3^i c) = 0$$ \hfill (2.17)

for all $a, b, c \in A$. Then there exists a unique Jordan $\ast$-homomorphism $h : A \to B$ such that

$$\|h(a) - f(a)\|_B \leq \sum_{i=1}^{\infty} 3^{-i} \varphi(3^i a, 3^i 2 a, 0)$$  \hfill (2.18)

for all $a \in A$.

Proof. Letting $\mu = 1$, $b = 2a$ and $c = 0$ in (2.8), we get

$$\left\|3 f\left(\frac{a}{3}\right) - f(a)\right\|_B \leq \varphi(a, 2a, 0)$$  \hfill (2.19)

for all $a \in A$. Replacing $a$ by $3a$ in (2.19), we get

$$\|3^{-1} f(3a) - f(a)\|_B \leq 3^{-1} \varphi(3a, 2(3a), 0)$$

for all $a \in A$. On can apply the induction method to prove that

$$\|3^{-n} f(3^n a) - f(a)\|_B \leq \sum_{i=1}^{n} 3^{-i} \varphi(3^i a, 2(3^i a), 0)$$  \hfill (2.20)

for all $a \in A$. In order to show the functions $h_n(a) = 3^{-n} f(3^n a)$ form a convergent sequence, we use the Cauchy convergence criterion. Indeed, replace $a$ by $3^m a$ and multiply by $3^{-m}$ in (2.20), where $m$ is an arbitrary positive integer. We find that

$$\|3^{-(m+n)} f(3^{m+n} a) - 3^{-m} f(3^m a)\| \leq \sum_{i=m+1}^{m+n} 3^{-i} \varphi(3^i a, 2(3^i a), 0)$$  \hfill (2.21)

for all positive integers. Hence by the Cauchy criterion the limit $h(a) = \lim_{n \to \infty} h_n(a)$ exists for each $a \in A$. By taking the limit as $n \to \infty$ in (2.20) we see that

$$\|h(a) - f(a)\| \leq \sum_{i=1}^{\infty} 3^{-i} \varphi(3^i a, 2(3^i a), 0)$$

and (2.18) holds for all $a \in A$.

The rest of the proof is similar to the proof of Theorem 2.4. \hfill □
Suppose that $f : A \to B$ is a mapping with $f(0) = 0$ for which there exists constant $\theta \geq 0$ and $p_1, p_2, p_3 < 1$ such that
\[
\left\| f \left( \frac{\mu b - a}{3} \right) + f \left( \frac{a - 3c}{3} \right) + \mu f \left( \frac{3a - b}{3} + c \right) - f(a) + f(c^2) - f(c)^2 \right\|_B 
\leq \theta (\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3}),
\]
for all $a, b, c \in A$ and all $\mu \in \mathbb{T}$. Then there exists a unique Jordan $\ast$-homomorphism $h : A \to B$ such that
\[
\left\| f(a) - h(a) \right\|_B \leq \frac{\theta \|a\|^{p_1}}{3(1-p_1) - 1} + \frac{\theta 2^{p_2} \|a\|^{p_2}}{3(1-p_2) - 1}
\]
for all $a \in A$.

Proof. Letting $\varphi(a, b, c) := \theta (\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3})$ in Theorem 2.7, we obtain the result. □

The following corollary is the Isac-Rassias stability.

Corollary 2.9. Let $\psi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ be a function with $\psi(0) = 0$ such that
\[
\lim_{t \to 0} \frac{\psi(t)}{t} = 0,
\]
\[
\psi(st) \leq \psi(s) \psi(t) \quad s, t \in \mathbb{R}^+,
\]
\[
3^{-1} \psi(3) < 1.
\]
Suppose that $f : A \to B$ is a mapping with $f(0) = 0$ satisfying (2.7) and (2.8) such that
\[
\left\| f \left( \frac{\mu b - a}{3} \right) + f \left( \frac{a - 3c}{3} \right) + \mu f \left( \frac{3a - b}{3} + c \right) - f(a) + f(c^2) - f(c)^2 \right\|_B 
\leq \theta \left[ \psi(\|a\|) + \psi(\|b\|) + \psi(\|c\|) \right]
\]
for all $a, b, c \in A$ where $\theta > 0$ is a constant. Then there exists a unique Jordan $\ast$-homomorphism $h : A \to B$ such that
\[
\left\| h(a) - f(a) \right\|_B \leq \frac{\theta (1 + \psi(2)) \psi(\|a\|)}{1 - 3^{-1} \psi(3)}
\]
for all $a \in A$.

Proof. Letting $\varphi(a, b, c) := \theta \left[ \psi(\|a\|) + \psi(\|b\|) + \psi(\|c\|) \right]$ in Theorem 2.7, we obtain the result. □

One can get easily the stability of Hyers-Ulam by the following corollary.
COROLLARY 2.10. Suppose that \( f : A \rightarrow B \) is a mapping with \( f(0) = 0 \) for which there exists constant \( \theta \geq 0 \) such that

\[
\| f \left( \frac{\mu b - a}{3} \right) + f \left( \frac{a - 3c}{3} + \frac{3a - b}{3} + c \right) - f(a) + f(c^2) - f(c)^2 \|_B \leq \theta,
\]

\[
\| f(a^*) - f(a)^* \|_B \leq \theta
\]

for all \( a, b, c \in A \) and all \( \mu \in \mathbb{T} \). Then there exists a unique Jordan \(*\)-homomorphism \( h : A \rightarrow B \) such that

\[
\| f(a) - h(a) \|_B \leq \theta
\]

for all \( a \in A \).

Proof. Letting \( p_1 = p_2 = p_3 = 0 \) in Corollary 2.8, we obtain the result. \( \square \)

REFERENCES


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