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JORDAN $*$–HOMOMORPHISMS ON C^*–ALGEBRAS

M. ESHAGHI GORDJI, N. GHOBADIPOUR AND CHOONKIL PARK

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Abstract. In this paper, we investigate Jordan $*$-homomorphisms on C^*-algebras associated with the following functional inequality $\left\|f\left(\frac{b-a}{3}\right)+f\left(\frac{a-3c}{3}\right)+f\left(\frac{3a+3c-b}{3}\right)\right\| \leqslant \|f(a)\|$. We moreover prove the superstability and the generalized Hyers-Ulam stability of Jordan $*$-homomorphisms on C^*-algebras associated with the following functional equation

$$f\left(\frac{b-a}{3}\right)+f\left(\frac{a-3c}{3}\right)+f\left(\frac{3a+3c-b}{3}\right)=f(a).$$

1. Introduction

The stability of functional equations was first introduced by Ulam [27] in 1940. More precisely, he proposed the following problem: Given a group G_1, a metric group (G_2,d) and a positive number ε, does there exist a $\delta>0$ such that if a function $f: G_1 \longrightarrow G_2$ satisfies the inequality $d(f(xy),f(x)f(y))<\delta$ for all $x,y \in G1$, then there exists a homomorphism $T: G_1 \to G_2$ such that $d(f(x),T(x))<\varepsilon$ for all $x \in G_1$? As mentioned above, when this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable. In 1941, Hyers [6] gave a partial solution of Ulam's problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. In 1978, Th. M. Rassias [22] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. This phenomenon of stability that was introduced by Th. M. Rassias [22] is called *generalized Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability*.

THEOREM 1.1. *Let $f: E \longrightarrow E'$ be a mapping from a norm vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y)-f(x)-f(y)\| \leqslant \varepsilon(\|x\|^p+\|y\|^p) \tag{1.1}$$

for all $x,y \in E$, where ε and p are constants with $\varepsilon>0$ and $p<1$. Then there exists a unique additive mapping $T: E \longrightarrow E'$ such that

$$\|f(x)-T(x)\| \leqslant \frac{2\varepsilon}{2-2^p}\|x\|^p \tag{1.2}$$

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for all $x \in E$. If $p < 0$ then inequality (1.1) holds for all $x, y \neq 0$, and (1.2) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E' is continuous for each fixed $x \in E$, then T is $\mathbb{R}$-linear.

During the last decades several stability problems of functional equations have been investigated by many mathematicians. A large list of references concerning the stability of functional equations can be found in [1, 2, 3, 11, 16, 25, 26, 28].

DEFINITION 1.2. Let A, B be two C^*-algebras. A $\mathbb{C}$-linear mapping $f : A \to B$ is called a Jordan $*$-homomorphism if

$$\begin{cases} f(a^2) = f(a)^2, \\ f(a^*) = f(a)^* \end{cases}$$

for all $a \in A$.

C. Park [19] introduced and investigated Jordan $*$-derivations on C^*- algebras associated with the following functional inequality

$$\|f(a) + f(b) + kf(c)\| \leqslant \left\| kf\left(\frac{a+b}{k} + c\right) \right\|$$

for some integer k greater than 1 and proved the generalized Hyers-Ulam stability of Jordan $*$-derivations on C^*-algebras associated with the following functional equation

$$f\left(\frac{a+b}{k} + c\right) = \frac{f(a) + f(b)}{k} + f(c)$$

for some integer k greater than 1 (see also [20, 14, 15, 17, 21]).

In this paper, we investigate Jordan $*$-homomorphisms on C^*-algebras associated with the following functional inequality

$$\left\| f\left(\frac{b-a}{3}\right) + f\left(\frac{a-3c}{3}\right) + f\left(\frac{3a+3c-b}{3}\right) \right\| \leqslant \|f(a)\|.$$

We moreover prove the generalized Hyers-Ulam stability of Jordan $*$-homomorphisms on C^*-algebras associated with the following functional equation

$$f\left(\frac{b-a}{3}\right) + f\left(\frac{a-3c}{3}\right) + f\left(\frac{3a+3c-b}{3}\right) = f(a).$$

2. Jordan $*$-homomorphisms

In this section, we investigate Jordan $*$-homomorphisms on C^*- algebras. Throughout this section, assume that A, B are two C^*-algebras.

LEMMA 2.1. *Let $f : A \to B$ be a mapping such that*

$$\left\| f\left(\frac{b-a}{3}\right) + f\left(\frac{a-3c}{3}\right) + f\left(\frac{3a+3c-b}{3}\right) \right\|_B \leqslant \|f(a)\|_B \tag{2.1}$$

for all $a,b,c \in A$. Then f is additive.

Proof. Letting $a = b = c = 0$ in (2.1), we get

$$\|3f(0)\|_B \leqslant \|f(0)\|_B.$$

So $f(0) = 0$. Letting $a = b = 0$ in (2.1), we get

$$\|f(-c) + f(c)\|_B \leqslant \|f(0)\|_B = 0$$

for all $c \in A$. Hence $f(-c) = -f(c)$ for all $c \in A$. Letting $a = 0$ and $b = 6c$ in (2.1), we get

$$\|f(2c) - 2f(c)\|_B \leqslant \|f(0)\|_B = 0$$

for all $c \in A$. Hence

$$f(2c) = 2f(c)$$

for all $c \in A$. Letting $a = 0$ and $b = 9c$ in (2.1), we get

$$\|f(3c) - f(c) - 2f(c)\|_B \leqslant \|f(0)\|_B = 0$$

for all $c \in A$. Hence

$$f(3c) = 3f(c)$$

for all $c \in A$. Letting $a = 0$ in (2.1), we get

$$\left\| f\left(\frac{b}{3}\right) + f(-c) + f\left(c - \frac{b}{3}\right) \right\|_B \leqslant \|f(0)\|_B = 0$$

for all $a,b,c \in A$. So

$$f\left(\frac{b}{3}\right) + f(-c) + f\left(c - \frac{b}{3}\right) = 0 \tag{2.2}$$

for all $a,b,c \in A$. Let $t_1 = c - \frac{b}{3}$ and $t_2 = \frac{b}{3}$ in (2.2). Then

$$f(t_2) - f(t_1 + t_2) + f(t_1) = 0$$

for all $t_1, t_2 \in A$. This means that f is additive. $\square$

Now we prove the superstability problem for Jordan $*$-homomorphisms as follows.

THEOREM 2.2. *Let* $p<1$ *and* θ *be nonnegative real numbers, and let* $f:A\to B$ *be a mapping such that*

$$\left\|f\left(\frac{b-a}{3}\right)+f\left(\frac{a-3\mu c}{3}\right)+\mu f\left(\frac{3a+3c-b}{3}\right)\right\|_B \leqslant \|f(a)\|_B, \tag{2.2}$$

$$\|f(a^2)-f(a)^2\|_B \leqslant \theta\|a\|^{2p}, \tag{2.3}$$

$$\|f(a^*)-f(a)^*\|_B \leqslant \theta\|a^*\|^p \tag{2.4}$$

for all $\mu\in\mathbb{T}^1:=\{\lambda\in\mathbb{C}\ ;|\lambda|=1\}$ *and all* $a,b,c\in A$*. Then the mapping* $f:A\to B$ *is a Jordan* $*$*-homomorphism.*

Proof. Let $\mu=1$ in (2.2). By Lemma 2.1, the mapping $f:A\to B$ is additive. Letting $a=b=0$ in (2.2), we get

$$\|f(-\mu c)+\mu f(c)\|_B \leqslant \|f(0)\|_B = 0$$

for all $c\in A$ and all $\mu\in\mathbb{T}^1$. So

$$-f(\mu c)+\mu f(c)=f(-\mu c)+\mu f(c)=0$$

for all $c\in A$ and all $\mu\in\mathbb{T}^1$. Hence $f(\mu c)=\mu f(c)$ for all $c\in A$ and all $\mu\in\mathbb{T}^1$. By Theorem 2.1 of [18], the mapping $f:A\to B$ is $\mathbb{C}$-linear. It follows from (2.3) that

$$\begin{aligned}\|f(a^2)-f(a)^2\|_B &= \left\|\frac{1}{n^2}f(n^2a^2)-\left(\frac{1}{n}f(na)\right)^2\right\|_B\\ &=\frac{1}{n^2}\|f(n^2a^2)-f(na)^2\|_B\\ &\leqslant \frac{\theta}{n^2}n^{2p}\|a\|^{2p}\end{aligned}$$

for all $a\in A$. Thus, since $p<1$, by letting n tend to ∞ in last inequality, we obtain $f(a^2)=f(a)^2$ for all $a\in A$. On the other hand, it follows from (2.4) that

$$\begin{aligned}\|f(a^*)-f(a)^*\|_B &= \left\|\frac{1}{n}f(na^*)-\left(\frac{1}{n}f(na)\right)^*\right\|_B\\ &=\frac{1}{n}\|f(na^*)-f(na)^*\|_B\\ &\leqslant \frac{\theta}{n}n^p\|a^*\|^p\end{aligned}$$

for all $a\in A$. Thus, since $p<1$, by letting n tend to ∞ in last inequality, we obtain $f(a^*)=f(a)^*$ for all $a\in A$. Hence the mapping $f:A\to B$ is a Jordan $*$-homomorphism. □

THEOREM 2.3. *Let* $p>1$ *and* θ *be a nonnegative real number, and let* $f:A\to B$ *be a mapping satisfying* (2.2), (2.3) *and* (2.4)*. Then the mapping* $f:A\to B$ *is a Jordan* $*$*-homomorphism.*

Proof. The proof is similar to the proof of Theorem 2.2. $\square$

We prove the generalized Hyers-Ulam stability of Jordan $*$-homomorphisms on C^*-algebras.

THEOREM 2.4. *Suppose that $f : A \to B$ is an odd mapping for which there exists a function $\varphi : A \times A \times A \to \mathbb{R}^+$ such that*

$$\sum_{i=0}^{\infty} 3^i \varphi\left(\frac{a}{3^i}, \frac{b}{3^i}, \frac{c}{3^i}\right) < \infty, \tag{2.5}$$

$$\lim_{n\to\infty} 3^{2n} \varphi\left(\frac{a}{3^n}, \frac{b}{3^n}, \frac{c}{3^n}\right) = 0, \tag{2.6}$$

$$\|f(a^*) - f(a)^*\|_B \leqslant \varphi(a,a,a), \tag{2.7}$$

$$\left\| f\left(\frac{\mu b - a}{3}\right) + f\left(\frac{a-3c}{3}\right) + \mu f\left(\frac{3a-b}{3} + c\right) - f(a) + f(c^2) - f(c)^2 \right\|_B \leqslant \varphi(a,b,c) \tag{2.8}$$

for all $a,b,c \in A$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique Jordan $*$-homomorphism $h : A \to B$ such that

$$\|h(a) - f(a)\|_B \leqslant \sum_{i=0}^{\infty} 3^i \varphi\left(\frac{a}{3^i}, \frac{2a}{3^i}, 0\right) \tag{2.9}$$

for all $a \in A$.

Proof. Letting $\mu = 1$, $b = 2a$ and $c = 0$ in (2.8), we get

$$\left\| 3f\left(\frac{a}{3}\right) - f(a) \right\|_B \leqslant \varphi(a, 2a, 0)$$

for all $a \in A$. Using the induction method, we have

$$\left\| 3^n f\left(\frac{a}{3^n}\right) - f(a) \right\| \leqslant \sum_{i=0}^{n-1} 3^i \varphi\left(\frac{a}{3^i}, \frac{2a}{3^i}, 0\right) \tag{2.10}$$

for all $a \in A$. In order to show the functions $h_n(a) = 3^n f\left(\frac{a}{3^n}\right)$ form a convergent sequence, we use the Cauchy convergence criterion. Indeed, replace a by $\frac{a}{3^m}$ and multiply by 3^m in (2.10), where m is an arbitrary positive integer. We find that

$$\left\| 3^{m+n} f\left(\frac{a}{3^{m+n}}\right) - 3^m f\left(\frac{a}{3^m}\right) \right\| \leqslant \sum_{i=m}^{m+n-1} 3^i \varphi\left(\frac{a}{3^i}, \frac{2a}{3^i}, 0\right) \tag{2.11}$$

for all positive integers. Hence by the Cauchy criterion the limit $h(a) = \lim_{n\to\infty} h_n(a)$ exists for each $a \in A$. By taking the limit as $n \to \infty$ in (2.10) we see that

$$\|h(a) - f(a)\| \leqslant \sum_{i=0}^{\infty} 3^i \varphi\left(\frac{a}{3^i}, \frac{2a}{3^i}, 0\right)$$

and (2.9) holds for all $a\in A$. Let $\mu=1$ and $c=0$ in (2.8), we get

$$\left\|f\left(\frac{b-a}{3}\right)+f\left(\frac{a}{3}\right)+f\left(\frac{3a-b}{3}\right)-f(a)\right\|_B\leqslant\varphi(a,b,0) \tag{2.12}$$

for all $a,b,c\in A$. Multiplying both sides (2.12) by 3^n and Replacing a,b by $\frac{a}{3^n},\frac{b}{3^n}$, respectively, we get

$$\left\|3^nf\left(\frac{b-a}{3^{n+1}}\right)+3^nf\left(\frac{a}{3^{n+1}}\right)+3^nf\left(\frac{3a-b}{3^{n+1}}\right)-3^nf\left(\frac{a}{3^n}\right)\right\|_B\leqslant 3^n\varphi\left(\frac{a}{3^n},\frac{b}{3^n},0\right) \tag{2.13}$$

for all $a,b,c\in A$. Taking the limit as $n\to\infty$, we obtain

$$h\left(\frac{b-a}{3}\right)+h\left(\frac{a}{3}\right)+h\left(\frac{3a-b}{3}\right)-h(a)=0 \tag{2.14}$$

for all $a,b,c\in A$. Putting $b=2a$ in (2.14), we get $3h\left(\frac{a}{3}\right)=h(a)$ for all $a\in A$. Replacing a by $2a$ in (2.14), we get

$$h(b-2a)+h(6a-b)=2h(2a) \tag{2.15}$$

for all $a,b\in A$. Letting $b=2a$ in (2.15), we get $h(4a)=2h(2a)$ for all $a\in A$. So $h(2a)=2h(a)$ for all $a\in A$. Letting $3a-b=s$ and $b-a=t$ in (2.14), we get

$$h\left(\frac{t}{3}\right)+h\left(\frac{s+t}{6}\right)+h\left(\frac{t}{3}\right)=h\left(\frac{s+t}{2}\right)$$

for all $s,t\in A$. Hence $h(s)+h(t)=h(s+t)$ for all $s,t\in A$. So, h is additive. Letting $a=c=0$ in (2.12) and using the above method, we have $h(\mu b)=\mu h(b)$ for all $b\in A$ and all $\mu\in\mathbb{T}$. Hence by the Theorem 2.1 of [18], the mapping $f:A\to B$ is $\mathbb{C}$-linear.

Now, let $h':A\to B$ be another $\mathbb{C}$-linear mapping satisfying (2.9). Then we have

$$\begin{aligned}\|h(a)-h'(a)\|_B&=3^n\left\|h\left(\frac{a}{3^n}\right)-h'\left(\frac{a}{3^n}\right)\right\|_B\\&\leqslant 3^n\left[\left\|h\left(\frac{a}{3^n}\right)-f\left(\frac{a}{3^n}\right)\right\|_B+\left\|h'\left(\frac{a}{3^n}\right)-f\left(\frac{a}{3^n}\right)\right\|_B\right]\\&\leqslant 2\sum_{i=n}^{\infty}3^i\varphi\left(\frac{a}{3^i},\frac{2a}{3^i},0\right)=0\end{aligned}$$

for all $a\in A$. Letting $\mu=1$ and $a=b=0$ in (2.8), we get $\|f(c^2)-f(c)^2\|_B\leqslant\varphi(0,0,c)$ for all $c\in A$. So

$$\|h(c^2)-h(c)^2\|_B=\lim_{n\to\infty}3^{2n}\left\|f\left(\frac{c^2}{3^{2n}}\right)-f\left(\frac{c}{3^n}\right)^2\right\|_B\leqslant\lim_{n\to\infty}3^{2n}\varphi\left(0,0,\frac{c}{3^n}\right)=0$$

for all $c\in A$. Hence $h(c^2)=h(c)^2$ for all $c\in A$. On the other hand we have

$$\|h(c^*)-h(c)^*\|_B=\lim_{n\to\infty}3^n\left\|f\left(\frac{c^*}{3^n}\right)-f\left(\frac{c}{3^n}\right)^*\right\|_B\leqslant\lim_{n\to\infty}3^n\varphi\left(\frac{c}{3^n},\frac{c}{3^n},\frac{c}{3^n}\right)=0$$

for all $c\in A$. Hence $h(c^*)=h(c)^*$ for all $c\in A$. Hence $h:A\to B$ is a unique Jordan $*$-homomorphism. $\square$

COROLLARY 2.5. *Suppose that* $f: A \to B$ *is a mapping with* $f(0) = 0$ *for which there exists constant* $\theta \geqslant 0$ *and* $p_1, p_2, p_3 > 1$ *such that*

$$\left\| f\left(\frac{\mu b - a}{3}\right) + f\left(\frac{a - 3c}{3}\right) + \mu f\left(\frac{3a - b}{3} + c\right) - f(a) + f(c^2) - f(c)^2 \right\|_B \leqslant \theta(\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3}),$$

$$\|f(a^*) - f(a)^*\|_B \leqslant \theta(\|a\|^{p_1} + \|a\|^{p_2} + \|a\|^{p_3})$$

for all $a, b, c \in A$ *and all* $\mu \in \mathbb{T}$. *Then there exists a unique Jordan* $*$*-homomorphism* $h: A \to B$ *such that*

$$\|f(a) - h(a)\|_B \leqslant \frac{\theta\|a\|^{p_1}}{1 - 3^{(1-p_1)}} + \frac{\theta 2^{p_2}\|a\|^{p_2}}{1 - 3^{(1-p_2)}}$$

for all $a \in A$.

Proof. Letting $\varphi(a,b,c) := \theta(\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3})$ in Theorem 2.4, we obtain the result. □

The following corollary is the Isac-Rassias stability.

COROLLARY 2.6. *Let* $\psi: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ *be a function with* $\psi(0) = 0$ *such that*

$$\lim_{t \to 0} \frac{\psi(t)}{t} = 0,$$

$$\psi(st) \leqslant \psi(s)\psi(t) \qquad s, t \in \mathbb{R}^+,$$

$$3\psi\left(\frac{1}{3}\right) < 1.$$

Suppose that $f: A \to B$ *is a mapping with* $f(0) = 0$ *satisfying* (2.7) *and* (2.8) *such that*

$$\left\| f\left(\frac{\mu b - a}{3}\right) + f\left(\frac{a - 3c}{3}\right) + \mu f\left(\frac{3a - b}{3} + c\right) - f(a) + f(c^2) - f(c)^2 \right\|_B \leqslant \theta[\psi(\|a\|) + \psi(\|b\|) + \psi(\|c\|)]$$

for all $a, b, c \in A$ *where* $\theta > 0$ *is a constant. Then there exists a unique Jordan* $*$*-homomorphism* $h: A \to B$ *such that*

$$\|h(a) - f(a)\|_B \leqslant \frac{\theta(1 + \psi(2))\psi(\|a\|)}{1 - 3\psi(\frac{1}{3})}$$

for all $a \in A$.

Proof. Letting $\varphi(a,b,c) := \theta[\psi(\|a\|) + \psi(\|b\|) + \psi(\|c\|)]$ in Theorem 2.4, we obtain the result. □

THEOREM 2.7. *Suppose that* $f: A \to B$ *is a mapping with* $f(0) = 0$ *for which there exists a function* $\varphi: A \times A \times A \to B$ *satisfying* (2.7), (2.8) *and* (2.8) *such that*

$$\sum_{i=1}^{\infty} 3^{-i}\varphi(3^i a, 3^i b, 3^i c) < \infty, \tag{2.16}$$

$$\lim_{n\to\infty} 3^{-2n}\varphi(3^i a, 3^i b, 3^i c) = 0 \tag{2.17}$$

for all $a,b,c \in A$. *Then there exists a unique Jordan* $*$*-homomorphism* $h: A \to B$ *such that*

$$\|h(a) - f(a)\|_B \leqslant \sum_{i=1}^{\infty} 3^{-i}\varphi(3^i a, 3^i 2a, 0) \tag{2.18}$$

for all $a \in A$.

Proof. Letting $\mu = 1$, $b = 2a$ and $c = 0$ in (2.8), we get

$$\left\| 3f\left(\frac{a}{3}\right) - f(a) \right\|_B \leqslant \varphi(a, 2a, 0) \tag{2.19}$$

for all $a \in A$. Replacing a by $3a$ in (2.19), we get

$$\|3^{-1}f(3a) - f(a)\|_B \leqslant 3^{-1}\varphi(3a, 2(3a), 0)$$

for all $a \in A$. On can apply the induction method to prove that

$$\|3^{-n}f(3^n a) - f(a)\|_B \leqslant \sum_{i=1}^{n} 3^{-i}\varphi(3^i a, 2(3^i a), 0) \tag{2.20}$$

for all $a \in A$. In order to show the functions $h_n(a) = 3^{-n}f(3^n a)$ form a convergent sequence, we use the Cauchy convergence criterion. Indeed, replace a by $3^m a$ and multiply by 3^{-m} in (2.20), where m is an arbitrary positive integer. We find that

$$\|3^{-(m+n)}f(3^{m+n}a) - 3^{-m}f(3^m a)\| \leqslant \sum_{i=m+1}^{m+n} 3^{-i}\varphi(3^i a, 2(3^i a), 0) \tag{2.21}$$

for all positive integers. Hence by the Cauchy criterion the limit $h(a) = \lim_{n\to\infty} h_n(a)$ exists for each $a \in A$. By taking the limit as $n \to \infty$ in (2.20) we see that

$$\|h(a) - f(a)\| \leqslant \sum_{i=1}^{\infty} 3^{-i}\varphi(3^i a, 2(3^i a), 0)$$

and (2.18) holds for all $a \in A$.

The rest of the proof is similar to the proof of Theorem 2.4. $\square$

COROLLARY 2.8. *Suppose that $f: A \to B$ is a mapping with $f(0) = 0$ for which there exists constant $\theta \geqslant 0$ and $p_1, p_2, p_3 < 1$ such that*

$$\left\| f\left(\frac{\mu b - a}{3}\right) + f\left(\frac{a - 3c}{3}\right) + \mu f\left(\frac{3a - b}{3} + c\right) - f(a) + f(c^2) - f(c)^2 \right\|_B \leqslant \theta(\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3}),$$

$$\|f(a^*) - f(a)^*\|_B \leqslant \theta(\|a\|^{p_1} + \|a\|^{p_2} + \|a\|^{p_3})$$

for all $a, b, c \in A$ and all $\mu \in \mathbb{T}$. Then there exists a unique Jordan $*$-homomorphism $h: A \to B$ such that

$$\|f(a) - h(a)\|_B \leqslant \frac{\theta\|a\|^{p_1}}{3^{(1-p_1)} - 1} + \frac{\theta 2^{p_2}\|a\|^{p_2}}{3^{(1-p_2)} - 1}$$

for all $a \in A$.

Proof. Letting $\varphi(a,b,c) := \theta(\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3})$ in Theorem 2.7, we obtain the result. $\square$

The following corollary is the Isac-Rassias stability.

COROLLARY 2.9. *Let $\psi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ be a function with $\psi(0) = 0$ such that*

$$\lim_{t \to 0} \frac{\psi(t)}{t} = 0,$$

$$\psi(st) \leqslant \psi(s)\psi(t) \qquad s, t \in \mathbb{R}^+,$$

$$3^{-1}\psi(3) < 1.$$

Suppose that $f: A \to B$ is a mapping with $f(0) = 0$ satisfying (2.7) and (2.8) such that

$$\left\| f\left(\frac{\mu b - a}{3}\right) + f\left(\frac{a - 3c}{3}\right) + \mu f\left(\frac{3a - b}{3} + c\right) - f(a) + f(c^2) - f(c)^2 \right\|_B \leqslant \theta[\psi(\|a\|) + \psi(\|b\|) + \psi(\|c\|)]$$

for all $a, b, c \in A$ where $\theta > 0$ is a constant. Then there exists a unique Jordan $*$-homomorphism $h: A \to B$ such that

$$\|h(a) - f(a)\|_B \leqslant \frac{\theta(1 + \psi(2))\psi(\|a\|)}{1 - 3^{-1}\psi(3)}$$

for all $a \in A$.

Proof. Letting $\varphi(a,b,c) := \theta[\psi(\|a\|) + \psi(\|b\|) + \psi(\|c\|)]$ in Theorem 2.7, we obtain the result. $\square$

One can get easily the stability of Hyers-Ulam by the following corollary.

COROLLARY 2.10. *Suppose that $f: A \to B$ is a mapping with $f(0)=0$ for which there exists constant $\theta \geqslant 0$ such that*

$$\left\| f\left(\frac{\mu b-a}{3}\right)+f\left(\frac{a-3c}{3}\right)+\mu f\left(\frac{3a-b}{3}+c\right)-f(a)+f(c^2)-f(c)^2\right\|_B \leqslant \theta,$$

$$\|f(a^*)-f(a)^*\|_B \leqslant \theta$$

for all $a,b,c \in A$ and all $\mu \in \mathbb{T}$. Then there exists a unique Jordan $*$-homomorphism $h: A \to B$ such that

$$\|f(a)-h(a)\|_B \leqslant \theta$$

for all $a \in A$.

Proof. Letting $p_1 = p_2 = p_3 = 0$ in Corollary 2.8, we obtain the result. □

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M. Eshaghi Gordji
Department of Mathematics, Semnan University
P. O. Box 35195-363, Semnan
Iran
Center of Excellence in Nonlinear Analysis and Applications (CENAA)
Semnan University
Iran
e-mail: madjid.eshaghi@gmail.com

N. Ghobadipour
Department of Mathematics, Semnan University
P. O. Box 35195-363, Semnan
Iran
e-mail: ghobadipour.n@gmail.com

Choonkil Park
Department of Mathematics, Hanyang University
Seoul 133-791
South Korea
e-mail: baak@hanyang.ac.kr