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ON THE PERTURBATION OF SINGULAR ANALYTIC MATRIX FUNCTIONS: A GENERALIZATION OF LANGER AND NAJMAN'S RESULTS

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Abstract. Given a singular $n\times n$ matrix function $A(\lambda)$, analytic in a neighborhood of an eigenvalue $\lambda_0 \in \mathbb{C}$, and perturbations, $B(\lambda,\varepsilon)$, such that $B(\lambda,0)\equiv 0$ and analytic in λ and ε near $(\lambda_0,0)$, we provide sufficient conditions on these perturbations for the existence of eigenvalue expansions of the perturbed matrix $A(\lambda)+B(\lambda,\varepsilon)$ near λ_0. We also describe the first order term of these expansions. This extends to the singular case some results by Langer and Najman.

1. Introduction

An $n\times n$ matrix function $A(\lambda)$ of the complex scalar variable λ is said to be *regular* if $\det A(\lambda)$ is not identically zero as a function of λ (from now on, this condition will be denoted by $\det A(\lambda)\not\equiv 0$), and it is *singular* otherwise. The *normal rank* of $A(\lambda)$, denoted by $\operatorname{nrank} A(\lambda)$, is the dimension of the largest non-identically zero minor of $A(\lambda)$, and an *eigenvalue* of $A(\lambda)$ is a number $\lambda_0\in\mathbb{C}$ such that

$$\operatorname{rank} A(\lambda_0) < \operatorname{nrank} A(\lambda).$$

This notion extends the well-known definition of eigenvalues of singular matrix pencils, introduced by Sun in [7], to general singular matrix functions. It also generalizes the notion of eigenvalues of regular matrix functions to the singular case. Note that if $A(\lambda)$ is regular then $\operatorname{nrank} A(\lambda)=n$ and the eigenvalues of $A(\lambda)$ are the solutions of the equation

$$\det A(\lambda)=0,$$

but, if $A(\lambda)$ is singular then this equation becomes an identity, and it is satisfied not only by the eigenvalues but also by all complex values.

In the present paper we are concerned with the local behavior of a given eigenvalue λ_0 of an $n\times n$ singular matrix function $A(\lambda)$, which is analytic in a neighborhood of λ_0, when this matrix is perturbed by another $n\times n$ matrix $B(\lambda,\varepsilon)$ which is assumed to be analytic in a neighborhood of $(\lambda_0,0)$. In other words, we are interested in knowing whether there are eigenvalues of $A(\lambda)+B(\lambda,\varepsilon)$ approaching λ_0 as ε approaches 0 and, in this case, in describing the first order term of the asymptotic expansions. Notice that, in order to recover the original matrix $A(\lambda)$ at $\varepsilon=0$ we need to impose $B(\lambda,0)\equiv$

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0 (this condition will be assumed throughout the paper). It is well known that if $A(\lambda)$ is regular then λ_0 changes continuously with ε and, moreover, the eigenvalues of the perturbed matrix $A(\lambda)+B(\lambda,\varepsilon)$ approaching λ_0 as ε tends to zero can be expanded as a (fractional) power series in ε. In a remarkable series of papers [5, 6], Langer and Najman obtained the first order term of these expansions under certain generic conditions.

The behavior of eigenvalues of regular matrix functions described above is in stark contrast with the one in the singular case. If $A(\lambda)$ is singular, there exist arbitrarily small perturbations placing the eigenvalues anywhere in the complex plane (see [3, Section 1] for an example in the case of matrix pencils). Nonetheless, in [3] (respectively in [2]) it is shown that in the case of matrix pencils (resp. matrix polynomials of higher degree) the eigenvalues change continuously, as power series expansions in ε, for most perturbations (that is, for all perturbations except those in a subset of Lebesgue measure zero). Moreover, in [2, 3] sufficient conditions are given on the set of perturbations for the continuity of the eigenvalues, and a considerable algebraic effort is performed to relate these sufficient conditions with relevant subspaces associated with λ_0 and $A(\lambda)$. The procedure followed in these works consists of "regularizing" the problem (see Section 1 in [2]) and then applying the techniques by Langer and Najman in [6]. In the present paper, the same approach is extended to deal, for the first time, with singular matrix functions analytic in a neighborhood of an eigenvalue λ_0. An important difference with the preceding papers is the following: in [3] and [2] the set of perturbations for which there are eigenvalue expansions and also the first order terms of these expansions are described using, respectively, *reducing subspaces* [9] of the unperturbed matrix pencil and *singular spaces at* λ_0 (introduced in [2]) of the unperturbed matrix polynomial. In the case of analytic matrix functions we are not able to give an analogous description. We just make use of some particular bases of the left and right null spaces of the matrix $A(\lambda_0)$ associated with the local Smith form at λ_0 (see the paragraph following the proof of Theorem 1).

Only in the particular case of semisimple eigenvalues we are able to provide a description of the first order coefficients in terms of arbitrary bases of the left and right null spaces of $A(\lambda_0)$ (see Theorem 4). This has been previously done in [4] for the regular case, and the leading terms of the eigenvalue expansions we obtain when specialized to a regular matrix function $A(\lambda)$ coincide with the ones obtained in [4].

2. The local Smith form

In this section we recall one of the main tools that will be used in the paper: the local Smith form. We also introduce some definitions related with this canonical form and some notation.

Given an eigenvalue λ_0 of the $n\times n$ matrix $A(\lambda)$, which is assumed to be analytic in a neighborhood of λ_0, there exist positive integers $0<m_1\leqslant\ldots\leqslant m_g$ and two $n\times n$ matrix functions $W(\lambda)$ and $V(\lambda)$, which are analytic and invertible in a neighborhood of λ_0, such that

$$W(\lambda)A(\lambda)V(\lambda)=\Delta(\lambda), \tag{1}$$

where

$$\Delta(\lambda) = \begin{bmatrix} (\lambda-\lambda_0)^{m_1} & & & & \\ & \ddots & & & \\ & & (\lambda-\lambda_0)^{m_g} & & \\ & & & I & \\ & & & & 0_{d\times d} \end{bmatrix}, \qquad d = n - \mathrm{nrank} A(\lambda). \tag{2}$$

The matrix $\Delta(\lambda)$ in (2) is known as the *local Smith form of* $A(\lambda)$ *at* λ_0 [1, p. 10]. The numbers $m_1,\ldots,m_g$ are known as the *partial multiplicities* at λ_0, and the number of partial multiplicities, g, is the *geometric multiplicity of* λ_0. The sum of the partial multiplicities $a := m_1 + \cdots + m_g$ is the *algebraic multiplicity of* λ_0. An eigenvalue λ_0 of $A(\lambda)$ is said to be *semisimple* if $m_1 = \ldots = m_g = 1$. Notice that if λ_0 is semisimple then the local Smith form of $A(\lambda)$ at λ_0 simplifies to

$$\Delta(\lambda) = \begin{bmatrix} (\lambda-\lambda_0)I_g & & \\ & I & \\ & & 0_{d\times d} \end{bmatrix}, \qquad d = n - \mathrm{nrank} A(\lambda). \tag{3}$$

Finally, given a complex value μ, we will denote by $\mathscr{N}(A(\mu))$ (resp. $\mathscr{N}_T(A(\mu))$) the right (resp. left) null space of the matrix $A(\mu) \in \mathbb{C}^{n\times n}$. The subscript $\cdot_T$ stands for the fact that its elements are row vectors. From now on, we will follow the convention of using row vectors for left null spaces and column vectors for right null spaces.

3. Eigenvalue expansions

Let us partition the matrices $W(\lambda)$ and $V(\lambda)$ transforming $A(\lambda)$ into the local Smith form at λ_0 (1) in the form

$$W(\lambda) = \begin{bmatrix} W_1(\lambda) \\ \star \\ W_2(\lambda) \end{bmatrix} \quad \text{and} \quad V(\lambda) = \begin{bmatrix} V_1(\lambda) \star V_2(\lambda) \end{bmatrix},$$

with

$$W_1(\lambda) = (W(\lambda))(1:g,:), \quad W_2(\lambda) = (W(\lambda))(n-d+1:n,:) \tag{4}$$

and

$$V_1(\lambda) = (V(\lambda))(:,1:g), \quad V_2(\lambda) = (V(\lambda))(:,n-d+1:n), \tag{5}$$

where we use MATLAB's notation for submatrices. The blocks denoted with $\star$ will not be of interest in our arguments.

The following result provides sufficient conditions for the existence of eigenvalue expansions near λ_0.

THEOREM 1. *Let $A(\lambda)$ be an $n\times n$ matrix function which is analytic in a neighborhood of an eigenvalue $\lambda_0 \in \mathbb{C}$, and whose Smith local form at λ_0 is given by* (2)*. Let $W_2(\lambda)$ and $V_2(\lambda)$ be defined as in* (4) *and* (5)*. Let $B(\lambda,\varepsilon)$ be another $n\times n$ matrix* *function, analytic in some neighborhood of* $(\lambda_0,0)$*, with* $B(\lambda,0)\equiv 0$*, and such that* $\det\left(W_2(\lambda)\frac{\partial B}{\partial\varepsilon}(\lambda,0)V_2(\lambda)\right)\not\equiv 0$. *Then*

1. *There exists a constant* $b>0$ *such that the matrix function* $A(\lambda)+B(\lambda,\varepsilon)$ *is regular whenever* $0<|\varepsilon|<b$.

2. *For* $0<|\varepsilon|<b$ *the eigenvalues of* $A(\lambda)+B(\lambda,\varepsilon)$ *approaching* λ_0 *as* ε *tends to zero are the solutions of the equation in* λ

$$p_\varepsilon(\lambda)=0,$$

where $p_\varepsilon(\lambda)$ *is analytic near* λ_0 *and the coefficients of the power expansion in* λ *of* $p_\varepsilon(\lambda)$ *are functions in* ε *which are analytic near* $\varepsilon=0$. *In addition, when* $\varepsilon=0$,

$$p_0(\lambda)=(\lambda-\lambda_0)^a\det\left(W_2(\lambda)\frac{\partial B}{\partial\varepsilon}(\lambda,0)V_2(\lambda)\right), \tag{6}$$

where $a=m_1+\cdots+m_g$ *is the algebraic multiplicity of* λ_0 *as an eigenvalue of* $A(\lambda)$.

3. *There are at least* a *eigenvalues* $\{\lambda_1(\varepsilon),\ldots,\lambda_a(\varepsilon)\}$ *of* $A(\lambda)+B(\lambda,\varepsilon)$ *such that*

$$\lim_{\varepsilon\to 0}\lambda_j(\varepsilon)=\lambda_0,\quad j=1,\ldots,a,$$

and these can be expanded as a (fractional) power series in ε.

Proof. The proof follows the arguments in the first part of the proof of Theorem 2 in [3]. It is based on the local Smith form (2). First, we consider the transformation to the local Smith form at λ_0,

$$W(\lambda)(A(\lambda)+B(\lambda,\varepsilon))V(\lambda)=\Delta(\lambda)+W(\lambda)B(\lambda,\varepsilon)V(\lambda)\equiv\widehat{\Delta}(\lambda)+G(\lambda,\varepsilon), \tag{7}$$

where

$$\widehat{\Delta}(\lambda)=\begin{bmatrix}D(\lambda)&&\\&0&\\&&0_{d\times d}\end{bmatrix}\quad\text{and}\quad G(\lambda,\varepsilon)=\begin{bmatrix}G_{11}(\lambda,\varepsilon)&G_{12}(\lambda,\varepsilon)&G_{13}(\lambda,\varepsilon)\\G_{21}(\lambda,\varepsilon)&I+G_{22}(\lambda,\varepsilon)&G_{23}(\lambda,\varepsilon)\\G_{31}(\lambda,\varepsilon)&G_{32}(\lambda,\varepsilon)&G_{33}(\lambda,\varepsilon)\end{bmatrix}$$

(with $D(\lambda)=\operatorname{diag}((\lambda-\lambda_0)^{m_1},\ldots,(\lambda-\lambda_0)^{m_g})$ the "relevant" part of the local Smith form) are partitioned conformally, and $[G_{ij}(\lambda,\varepsilon)]_{i,j=1}^3=W(\lambda)B(\lambda,\varepsilon)V(\lambda)$. Moreover, by hypothesis we have $B(\lambda,\varepsilon)=\varepsilon\widetilde{B}(\lambda,\varepsilon)$, with $\widetilde{B}(\lambda,\varepsilon)$ analytic in a neighborhood of $(\lambda_0,0)$ so, we can write $G_{ij}(\lambda,\varepsilon)=\varepsilon\widetilde{G}_{ij}(\lambda,\varepsilon)$, for $i,j=1,2,3$. In particular, $\widetilde{G}_{33}(\lambda,\varepsilon)=W_2(\lambda)\left(\frac{\partial B}{\partial\varepsilon}(\lambda,\varepsilon)+O(\varepsilon)\right)V_2(\lambda)$. Therefore,

$$f(\lambda,\varepsilon)=\det(A(\lambda)+B(\lambda,\varepsilon))=\delta(\lambda)\varepsilon^d\widetilde{f}(\lambda,\varepsilon),$$

where

$$\widetilde{f}(\lambda,\varepsilon)=\det(\widehat{\Delta}(\lambda)+\widehat{G}(\lambda,\varepsilon))$$

and

$$\widehat{G}(\lambda,\varepsilon)=\begin{bmatrix}G_{11}(\lambda,\varepsilon)&G_{12}(\lambda,\varepsilon)&\widetilde{G}_{13}(\lambda,\varepsilon)\\G_{21}(\lambda,\varepsilon)&I+G_{22}(\lambda,\varepsilon)&\widetilde{G}_{23}(\lambda,\varepsilon)\\G_{31}(\lambda,\varepsilon)&G_{32}(\lambda,\varepsilon)&\widetilde{G}_{33}(\lambda,\varepsilon)\end{bmatrix}.$$

In addition, the function $\delta(\lambda)$ is given by $\delta(\lambda) = p(\lambda)q(\lambda)$ where, $\det(W(\lambda)) = 1/p(\lambda)$ and $\det(V(\lambda)) = 1/q(\lambda)$. So $\delta(\lambda)$ is analytic near λ_0 with $\delta(\lambda_0) \neq 0$ and does not depend on the perturbation $B(\lambda,\varepsilon)$. These facts imply that, for $\varepsilon \neq 0$, the matrix $A(\lambda)+B(\lambda,\varepsilon)$ is regular if and only if $\widetilde{f}(\lambda,\varepsilon) \not\equiv 0$. Since

$$\widetilde{f}(\lambda,0) = \det(D(\lambda))\det(\widetilde{G}_{33}(\lambda,0)) = (\lambda-\lambda_0)^a \det\left(W_2(\lambda)\frac{\partial B}{\partial \varepsilon}(\lambda,0)V_2(\lambda)\right),$$

and, by hypothesis, $\det\left(W_2(\lambda)\frac{\partial B}{\partial \varepsilon}(\lambda,0)V_2(\lambda)\right) \not\equiv 0$, we conclude, by continuity, that $A(\lambda)+B(\lambda,\varepsilon)$ is regular in a punctured disk $0<|\varepsilon|<b$. This proves the first claim.

To prove the second one, notice that, when $A(\lambda)+B(\lambda,\varepsilon)$ is regular, the eigenvalues of $A(\lambda)+B(\lambda,\varepsilon)$ approaching λ_0 as ε tends to zero are those zeros, $\lambda(\varepsilon)$, of $\widetilde{f}(\lambda,\varepsilon)$ whose limit is λ_0. Obviously, $\widetilde{f}(\lambda,\varepsilon)$ is a quotient of functions which are analytic in λ near λ_0. Moreover, the coefficients of the numerator are functions in ε, analytic at $\varepsilon = 0$, and the denominator is precisely $\delta(\lambda)$. So, the second claim follows by taking $p_\varepsilon(\lambda) := \widetilde{f}(\lambda,\varepsilon)$.

Finally, for the third claim we first observe, as in [4, p. 607], that the Weierstrass preparation theorem allows us to see the equation $\widetilde{f}(\lambda,\varepsilon) = 0$ locally as a polynomial equation in λ, $\sum_{i=0}^k b_i(\varepsilon)(\lambda-\lambda_0)^i = 0$, for some $k \geqslant a$, where $b_i(\varepsilon)$, for $i = 0,\ldots,k$, are analytic near $\varepsilon = 0$. Now the claim is an immediate consequence of the second one and some well-know results of the Analytic Function Theory (see, for instance, [8, Th. 5.1]). □

Our aim in the following is to describe the first order term of the eigenvalue expansions mentioned in the third claim of Theorem 1. To this end, set

$$W(\lambda) = \begin{bmatrix} w_1(\lambda) \\ \vdots \\ w_n(\lambda) \end{bmatrix} \quad \text{and} \quad V(\lambda) = \begin{bmatrix} v_1(\lambda) & \ldots & v_n(\lambda) \end{bmatrix}.$$

Notice that, since $W(\lambda_0)$ and $V(\lambda_0)$ are both nonsingular, the set of vectors

$$\left\{w_1(\lambda_0),\ldots,w_g(\lambda_0),w_{n-d+1}(\lambda_0),\ldots,w_n(\lambda_0)\right\}$$

and

$$\left\{v_1(\lambda_0),\ldots,v_g(\lambda_0),v_{n-d+1}(\lambda_0),\ldots,v_n(\lambda_0)\right\}$$

are bases of, respectively, $\mathscr{N}_T(A(\lambda_0))$ and $\mathscr{N}(A(\lambda_0))$.

Now, we rename the partial multiplicities at λ_0, appearing in the local Smith form of $A(\lambda)$ at λ_0 (2), as

$$\{\underbrace{n_1,\ldots,n_1}_{r_1},\ldots,\underbrace{n_q,\ldots,n_q}_{r_q}\} \equiv \{m_1,\ldots,m_g\}, \tag{8}$$

where we assume that,

$$0 < n_1 < n_2 < \cdots < n_q. \tag{9}$$

Note that the algebraic and geometric multiplicities of λ_0 are given, respectively, by

$$a = \sum_{i=1}^{q} r_i n_i \quad \text{and} \quad g = \sum_{i=1}^{q} r_i.$$

Let us define the sequence

$$f_j = \sum_{i=j}^{q} r_i, \quad j = 1, \ldots, q, \quad \text{and} \quad f_{q+1} = 0,$$

so $f_1 = g$. We consider also the following submatrices of the matrices $W(\lambda_0)$ and $V(\lambda_0)$

$$W_{1j} = (W(\lambda_0))(g - f_j + 1 : g, :) \text{ and } V_{1j} = (V(\lambda_0))(:, g - f_j + 1 : g), \text{ for } j = 1, \ldots, q,$$
$$W_2 = W_2(\lambda_0) \text{ and } V_2 = V_2(\lambda_0),$$

with $W_2(\lambda)$ and $V_2(\lambda)$ as in (4) and (5). Note that $W_{11} = W_1(\lambda_0)$ and $V_{11} = V_1(\lambda_0)$, according to the notation introduced in (4) and (5). We will denote this matrices by, respectively, W_1 and V_1. Notice also that, as mentioned above, the rows of $[W_1^T \; W_2^T]^T$ (resp. the columns of $[V_1 \; V_2]$) form a very specific basis of $\mathscr{N}_T(A(\lambda_0))$ (resp. of $\mathscr{N}(A(\lambda_0))$). Now, we can build up the matrices

$$\Phi_j = \begin{bmatrix} W_{1j} \\ W_2 \end{bmatrix} \frac{\partial B}{\partial \varepsilon}(\lambda_0, 0) \begin{bmatrix} V_{1j} & V_2 \end{bmatrix}, \quad j = 1, \ldots, q, \quad \text{and} \quad \Phi_{q+1} = W_2 \frac{\partial B}{\partial \varepsilon}(\lambda_0, 0) V_2. \tag{10}$$

Notice that

$$\Phi_1 = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \frac{\partial B}{\partial \varepsilon}(\lambda_0, 0) \begin{bmatrix} V_1 & V_2 \end{bmatrix},$$

and that Φ_j is the $(f_j + d) \times (f_j + d)$ lower right principal submatrix of Φ. Finally, we define

$$E_j = \operatorname{diag}(I_{r_j}, 0_{(f_{j+1}+d)\times(f_{j+1}+d)}), \quad j = 1, \ldots, q. \tag{11}$$

The pencils $\Phi_j + \zeta E_j$, $j = 1, \ldots, q$ are relevant in obtaining the leading coefficient of the perturbation expansions. The main property of these pencils relating to this is given in the first claim of the following lemma.

LEMMA 1. [3, Lemma 6] *Let* Φ_j, Φ_{q+1} *and* E_j, $j = 1, \ldots, q$, *be the matrices defined, respectively, in* (10) *and* (11). *If the matrix* Φ_{j+1} *is nonsingular then*

1. *The pencil* $\Phi_j + \zeta E_j$ *is regular and has exactly* r_j *finite eigenvalues.*

2. *The finite eigenvalues of* $\Phi_j + \zeta E_j$ *are minus the eigenvalues of the Schur complement of* Φ_{j+1} *in* Φ_j.

3. *If, in addition,* Φ_j *is nonsingular then the* r_j *finite eigenvalues of* $\Phi_j + \zeta E_j$ *are all different from zero.*

The r_j eigenvalues of the pencil $\Phi_j + \zeta E_j$ (under the assumption $\det\Phi_{j+1} \neq 0$ in the statement Lemma 1) will determine the leading coefficients of r_j eigenvalue expansions near λ_0 with leading exponent $1/n_j$. Moreover, the third claim of Lemma 1 provides a sufficient condition for these being the only expansions with leading coefficient $1/n_j$. This is part of the main result of this section, which is the generalization to analytic matrices of [2, Theorem 5], that is valid only for matrix polynomials.

THEOREM 2. *Let $A(\lambda)$ be an $n\times n$ matrix function (singular or not) which is analytic in a neighborhood of $\lambda_0 \in \mathbb{C}$. Assume that λ_0 is an eigenvalue of $A(\lambda)$ such that the partial multiplicities at λ_0 satisfy* (8) *and* (9). *Let $B(\lambda,\varepsilon)$ be another $n\times n$ matrix which is analytic near $(\lambda_0,0)$ and with $B(\lambda,0)\equiv 0$, and let Φ_j, $j=1,\dots,q+1$, and E_j, $j=1,\dots,q$, be the matrices defined in* (10) *and* (11). *If* $\det\Phi_{j+1}\neq 0$ *for some $j\in\{1,2,...,q\}$, let $\xi_1,\dots,\xi_{r_j}$ be the r_j finite eigenvalues of the pencil $\Phi_j+\zeta E_j$, and $(\xi_t)_s^{1/n_j}$, $s=1,\dots,n_j$, be the n_j determinations of the n_jth root. Then, in a neighborhood of $\varepsilon=0$, the matrix $A(\lambda)+B(\lambda,\varepsilon)$ has r_jn_j eigenvalues satisfying*

$$\lambda_j^{rs}(\varepsilon) = \lambda_0 + (\xi_t)_s^{1/n_j}\,\varepsilon^{1/n_j} + o(\varepsilon^{1/n_j}),\ \ t=1,2,...,r_j\ ,\ \ s=1,2,...,n_j\,, \tag{12}$$

where ε^{1/n_j} is the principal determination of the n_jth root of ε. Moreover, the matrix $A(\lambda)+B(\lambda,\varepsilon)$ is regular in the same neighborhood for $\varepsilon\neq 0$. If, in addition, $\det\Phi_j\neq 0$, *then all ξ_t in* (12) *are nonzero, and* (12) *are all the expansions near λ_0 with leading exponent $1/n_j$.*

Proof. The proof of this theorem is a continuation of the proof of Theorem 1. Recall that the function $\widetilde{f}(\lambda,\varepsilon)$ in that proof may be seen as a function in ε which is analytic in a neighborhood of $\varepsilon=0$ and whose coefficients are functions in λ which are analytic near λ_0. Let us study more carefully this function $\widetilde{f}(\lambda,\varepsilon)$.

In the first place, note that according to the identity (7) and the definitions (10), we have

$$\Phi_1 = \begin{bmatrix} \widetilde{G}_{11}(\lambda_0,0) & \widetilde{G}_{13}(\lambda_0,0) \\ \widetilde{G}_{31}(\lambda_0,0) & \widetilde{G}_{33}(\lambda_0,0)\end{bmatrix}, \quad \text{and} \quad \Phi_{q+1} = \widetilde{G}_{33}(\lambda_0,0). \tag{13}$$

We now make use of the Lemma in [6, p. 799], on determinants of the type $\det(D+G)$ with D diagonal, to expand $\widetilde{f}(\lambda,\varepsilon)$ as

$$\widetilde{f}(\lambda,\varepsilon) = \det\widehat{G}(\lambda,\varepsilon) + \sum(\lambda-\lambda_0)^{m_{\nu_1}}\cdots(\lambda-\lambda_0)^{m_{\nu_r}}\det\widehat{G}(\lambda,\varepsilon)(\{\nu_1,\dots,\nu_r\}'), \tag{14}$$

where for any matrix C, $C(\{\nu_1,\dots,\nu_r\}')$ denotes the matrix obtained by removing from C the rows and columns with indices $\nu_1,\dots,\nu_r$. The sum runs over all $r\in\{1,\dots,g\}$ and all $\nu_1,\dots,\nu_r$ such that $1\leqslant\nu_1<\ldots<\nu_r\leqslant g$. Finally, note that

$$\det\widehat{G}(\lambda,\varepsilon) = \varepsilon^g(\det\Phi_1 + Q_0(\lambda,\varepsilon)), \tag{15}$$

for $Q_0(\lambda,\varepsilon)$ analytic in λ with $Q_0(\lambda_0,0)=0$, and

$$\det\widehat{G}(\lambda,\varepsilon)(\{\nu_1,\dots,\nu_r\}') = \varepsilon^{g-r}(\det\Phi_1(\{\nu_1,\dots,\nu_r\}') + Q_{\nu_1\dots\nu_r}(\lambda,\varepsilon)), \tag{16}$$

with $Q_{v_1\dots v_r}$ analytic in λ and $Q_{v_1\dots v_r}(\lambda_0,0)=0$. From now on, it suffices to repeat the arguments in [6, pp. 799-800]. The only remark to be made is that equations (14-15-16) show that $\widetilde{f}(\lambda,\varepsilon)\not\equiv 0$, since $\det\Phi_{j+1}=\det\Phi_1(\{1,\dots,\sum_{i=1}^{j} r_i\}')\neq 0$ is the coefficient of $\varepsilon^{f_{j+1}}(\lambda-\lambda_0)^{r_1n_1+\dots+r_jn_j}$ in the two variable Taylor expansion of $\widetilde{f}(\lambda,\varepsilon)$ □

It is worth noticing that that although the algebraic multiplicity of λ_0 in $A(\lambda)$ is $r_1n_1+\ldots+r_qn_q$, the condition $\det\Phi_{j+1}\neq 0$ in Theorem 2 only guarantees the existence of r_jn_j expansions with the leading exponents and coefficients in (12). We want also to point out that condition $\det\Phi_{q+1}\neq 0$ in Theorem 2 implies $\det(W_2(\lambda)\frac{\partial B}{\partial\varepsilon}(\lambda,0)$ $V_2(\lambda))\not\equiv 0$ of Theorem 1. This follows easily from the definition of Φ_{q+1} in (10). In particular, $\det\Phi_{q+1}\neq 0$ not only guarantees the existence of r_qn_q expansions with first order term as in (12) (with $j=q$), but also the existence of at least a expansions near λ_0, as stated in Theorem 1.

3.1. Expansions for semisimple eigenvalues

In Section 3 we have obtained sufficient conditions for the existence of expansions near an arbitrary eigenvalue of $A(\lambda)$, and also formulas for the first order term of these expansions. In this section we specialize on a given semisimple eigenvalue λ_0 of the singular square matrix function $A(\lambda)$. We will see that, in this case, the corresponding formulas for the first order terms can be greatly simplified.

We will make use of the following Lemma. Here and hereafter $A'(\lambda)$ denotes the derivative of $A(\lambda)$ with respect the variable λ.

LEMMA 2. *Let $W(\lambda)$ and $V(\lambda)$ be the matrices leading $A(\lambda)$ to its local Smith form at λ_0, and assume, in addition, that the eigenvalue λ_0 of $A(\lambda)$ is semisimple. Then*

$$\begin{bmatrix} w_1(\lambda_0) \\ \vdots \\ w_g(\lambda_0) \\ w_{n-d+1}(\lambda_0) \\ \vdots \\ w_n(\lambda_0) \end{bmatrix} A'(\lambda_0)\left[\, v_1(\lambda_0)\ \dots\ v_g(\lambda_0)\ v_{n-d+1}(\lambda_0)\ \dots\ v_n(\lambda_0)\,\right] = \begin{bmatrix} I_g & \\ & 0_{d\times d}\end{bmatrix}.$$

Proof. Taking derivatives in the identity $W(\lambda)A(\lambda)V(\lambda)=\Delta(\lambda)$, where $\Delta(\lambda)$ is given by (3), we achieve

$$W'(\lambda)A(\lambda)V(\lambda)+W(\lambda)A'(\lambda)V(\lambda)+W(\lambda)A(\lambda)V'(\lambda)=\Delta'(\lambda). \qquad (17)$$

Since $w_i(\lambda_0)A(\lambda_0)=0$ and $A(\lambda_0)v_i(\lambda_0)=0$, for $i=1,\dots,g,n-d+1,\dots,n$ and, from (3)

$$\Delta'(\lambda_0)=\begin{bmatrix} I_g & \\ & 0_{d\times d}\end{bmatrix},$$

the result follows from evaluating at λ_0 the equation (17). □

Now, since λ_0 is semisimple, the Φ_j matrices in (10) reduce to

$$\Phi = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \frac{\partial B}{\partial \varepsilon}(\lambda_0, 0) \begin{bmatrix} V_1 & V_2 \end{bmatrix}. \tag{18}$$

Associated with Φ we introduce the $(g+d) \times (g+d)$ matrix pencil

$$\mathscr{P}(\zeta) = \Phi + \zeta \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} A'(\lambda_0) \begin{bmatrix} V_1 & V_2 \end{bmatrix}, \tag{19}$$

and note that, by virtue of Lemma 2,

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} A'(\lambda_0) \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} I_g & \\ & 0_{d\times d} \end{bmatrix}.$$

Lemma 3 states some relevant properties of the pencil $\mathscr{P}(\zeta)$ needed in the main results of this subsection. This lemma is the specialization of Lemma 1 to the semisimple case.

LEMMA 3. *Let Φ be the matrix defined in* (18) *and $\mathscr{P}(\zeta)$ the pencil in* (19). *Then the following statements hold.*

1) *$\mathscr{P}(\zeta)$ is regular and has exactly g finite eigenvalues if and only if the $d \times d$ matrix $W_2 \frac{\partial B}{\partial \varepsilon}(\lambda_0, 0) V_2$ is nonsingular.*

2) *If $W_2 \frac{\partial B}{\partial \varepsilon}(\lambda_0, 0) V_2$ is nonsingular, then the g finite eigenvalues of $\mathscr{P}(\zeta)$ are all different from zero if and only if Φ is nonsingular.*

Proof. Let us express

$$\Phi = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & W_2 \frac{\partial B}{\partial \varepsilon}(\lambda_0, 0) V_2 \end{bmatrix}.$$

By Lemma 2 we have

$$\mathscr{P}(\zeta) = \Phi + \zeta \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} A'(\lambda_0) \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} C_{11} + \zeta I_g & C_{12} \\ C_{21} & W_2 \frac{\partial B}{\partial \varepsilon}(\lambda_0, 0) V_2 \end{bmatrix}.$$

Therefore,

$$\det \mathscr{P}(\zeta) = \zeta^g \det\left(W_2 \frac{\partial B}{\partial \varepsilon}(\lambda_0, 0) V_2 \right) + \zeta^{g-1} b_{g-1} + \cdots + \zeta b_1 + \det \Phi, \tag{20}$$

where the coefficients $b_{g-1}, \ldots, b_1$ in the previous polynomial are of no interest in this argument. Now both claims follow easily. $\square$

Now we are in the position to state the main result of this section, which is the specialization of Theorem 2 to semisimple eigenvalues.

THEOREM 3. *Let $A(\lambda)$ be an arbitrary $n\times n$ matrix function analytic in a neighborhood of λ_0, which is a semisimple eigenvalue of $A(\lambda)$ with geometric multiplicity g. Let $B(\lambda,\varepsilon)$ be another matrix function with the same dimension, analytic near $(\lambda_0,0)$ and with $B(\lambda,0)\equiv 0$. Let $W=[W_1^T\, W_2^T]^T := [W_1(\lambda_0)^T\, W_2(\lambda_0)^T]^T$ and $V=[V_1\, V_2] := [V_1(\lambda_0)\, V_2(\lambda_0)]$, where $W_1(\lambda), W_2(\lambda), V_1(\lambda)$ and $V_2(\lambda)$ are the matrices defined in* (4) *and* (5). *Let also Φ be the matrix defined in* (18) *and $\mathscr{P}(\zeta)$ be the pencil defined in* (19). *If $W_2\frac{\partial B}{\partial\varepsilon}(\lambda_0,0)V_2$ is nonsingular, then the perturbed matrix $A(\lambda)+B(\lambda,\varepsilon)$ is regular and has exactly g eigenvalues in a neighborhood of $\varepsilon=0$ satisfying*

$$\lambda_j(\varepsilon)=\lambda_0+\zeta_j\varepsilon+o(\varepsilon)\quad j=1,\dots,g, \tag{21}$$

where $\zeta_1,\dots,\zeta_g$ are the finite eigenvalues of the pencil $\mathscr{P}(\zeta)$. If, in addition, Φ is nonsingular, then $\zeta_1,\dots,\zeta_g$ are all nonzero and all the expansions near λ_0 have leading exponent equal to one. If $g=1$, i.e., λ_0 is simple, then W_1 has only one row vector and V_1 only one column vector, and (21) *is equal to*

$$\lambda(\varepsilon)=\lambda_0-\frac{\det(W\frac{\partial B}{\partial\varepsilon}(\lambda_0,0)V)}{(W_1A'(\lambda_0)V_1)\cdot\det(W_2\frac{\partial B}{\partial\varepsilon}(\lambda_0,0)V_2)}\,\varepsilon+O(\varepsilon^2). \tag{22}$$

Proof. If $W_2\frac{\partial B}{\partial\varepsilon}(\lambda_0,0)V_2$ is nonsingular, condition $\det\left(W_2(\lambda)\frac{\partial B}{\partial\varepsilon}(\lambda,0)V_2(\lambda)\right)\not\equiv 0$ in Theorem 1 holds, and the equation $p_0(\lambda)=0$, with $p_0(\lambda)$ as in (6), has exactly g roots equal to λ_0. Therefore, Theorem 1 guarantees that $A(\lambda)+B(\lambda,\varepsilon)$ is regular in a neighborhood of $\varepsilon=0$ and has exactly g eigenvalues whose expansions tend to λ_0 when ε tends to zero. The expressions for the first order term of these expansions now follow from Theorem 2.

The expansion (22) when $g=1$ follows from (20). The only point to justify is why $o(\varepsilon)$ is replaced by $O(\varepsilon^2)$. This follows from the fact that the equation $p_0(\lambda)=0$, with $p_0(\lambda)$ as in (6), has only one root equal to zero, so the corresponding root of $p_\varepsilon(\lambda)$ is analytic in ε. □

As mentioned in the paragraph after the proof of Theorem 1, the rows of W and the columns of V are particular bases of, respectively, $\mathscr{N}_T(A(\lambda_0))$ and $\mathscr{N}(A(\lambda_0))$. Hence, the sufficient condition stated in Theorem 3 for the expansions near λ_0 being of the form (21) depends on the basis of these vector subspaces. Nonetheless, based on Lemma 3, we may enunciate a sufficient condition which does not depend on these bases.

THEOREM 4. *Let $A(\lambda)$ be an arbitrary $n\times n$ matrix function (singular or not) which is analytic in a neighborhood of a semisimple eigenvalue λ_0 having geometric multiplicity g. Let $B(\lambda,\varepsilon)$ be another matrix function with the same dimension, analytic in a neighborhood of $(\lambda_0,0)$ and with $B(\lambda,0)\equiv 0$. Denote by W a matrix whose rows form any basis of $\mathscr{N}_T(A(\lambda_0))$ and by V a matrix whose columns form any basis of $\mathscr{N}(A(\lambda_0))$. Then, if the pencil $W\frac{\partial B}{\partial\varepsilon}(\lambda_0,0)V+\zeta WA'(\lambda_0)V$ is regular and has exactly g finite eigenvalues equal to $\zeta_1,\dots,\zeta_g$, there are exactly g eigenvalues of*

$A(\lambda)+B(\lambda,\varepsilon)$ such that

$$\lambda_j(\varepsilon)=\lambda_0+\zeta_j\,\varepsilon+o(\varepsilon),\quad j=1,\ldots,g, \tag{23}$$

as ε tends to zero. If $g=1$, i.e., λ_0 is a simple eigenvalue, then $o(\varepsilon)$ can be replaced by $O(\varepsilon^2)$ in the previous expansions.

Proof. First, notice that the eigenvalues and the regularity of the pencil $W\frac{\partial B}{\partial \varepsilon}(\lambda_0,0)V + \zeta\, WA'(\lambda_0)V$ are independent on the bases W and V of the left and right null spaces of $A(\lambda_0)$, because any change of bases transforms the pencil into a strictly equivalent one. Hence, we can choose $W=\begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$ and $V=\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ be bases as the ones in Theorem 3. With this choice $W\frac{\partial B}{\partial \varepsilon}(\lambda_0,0)V+\zeta\, WA'(\lambda_0)V$ is precisely $\mathscr{P}(\zeta)$ in (19). Lemma 3 states that $W_2\frac{\partial B}{\partial \varepsilon}(\lambda_0,0)V_2$ is nonsingular if and only if $W\frac{\partial B}{\partial \varepsilon}(\lambda_0,0)V+\zeta\, WA'(\lambda_0)V$ is regular with exactly g finite eigenvalues. Now the result follows from Theorem 3 and Lemma 3. □

If the unperturbed matrix $A(\lambda)$ is regular, matrices W_2 and V_2 in (19) do not appear and, by Lemma 3, the pencil $\mathscr{P}(\lambda)$ is always regular and has exactly g finite eigenvalues. In this case, Theorem 3 is equivalent to the first paragraph of Theorem 6 in [4].

We want to make some final comments about the genericity of the main results included in the paper. These results, stated in Theorems 1, 2 and 3, hold under certain sufficient conditions. In Theorem 1, the condition assuring the existence of eigenvalue expansions near λ_0 is $\det\left(W_2\frac{\partial B}{\partial \varepsilon}(\lambda_0,0)V_2\right)\not\equiv 0$, so only those perturbations $B(\lambda,\varepsilon)$ for which the previous determinant is identically zero are not covered by this Theorem. In Theorem 2 the sufficient condition for the expansions near λ_0 being of the form (12) is $\det\Phi_{j+1}\neq 0$. Finally, in Theorem 3, the condition reduces to $\det W_2\frac{\partial B}{\partial \varepsilon}(\lambda_0,0)V_2\neq 0$. Then, the set of perturbations not covered by Theorems 2 and 3 are those for which these determinants vanish. These, and the one of Theorem 1, translate into very specific conditions on the perturbations, and it seems that the results stated in these Theorems will be satisfied for most perturbations. In the case of matrix polynomials of degree $k\geqslant 1$ [2] (see also [3] for matrix pencils) such conditions can properly termed as *generic*, because the set of perturbations not satisfying them are contained in a proper algebraic manifold of the set of all perturbations $B(\lambda,\varepsilon)$.

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