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# INDEFINITE BOUNDARY VALUE PROBLEMS ON GRAPHS

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*Abstract.* We consider the spectral structure of indefinite second order boundary-value problems on graphs. A variational formulation for such boundary-value problems on graphs is given and we obtain both full and half-range completeness results. This leads to a max-min principle and as a consequence we can formulate an analogue of Dirichlet-Neumann bracketing and this in turn gives rise to asymptotic approximations for the eigenvalues.

## 1. Introduction

Let $G$ be an oriented graph with finitely many edges, say $K$, each of unit length, having the path-length metric. Suppose that $n$ of the edges have positive weight, $1$, and $K-n$ of the edges have negative weight, $-1$. We consider the second-order differential equation

$$ly := -\frac{d^2y}{dx^2} + q(x)y = \lambda By, \tag{1.1}$$

on $G$, where $q$ is real valued and essentially bounded on $G$ and $By(x) = b(x)y(x)$ with

$$b(x) := \begin{cases} 1, & \text{for } x \text{ on edges with positive weight.} \\ -1, & \text{for } x \text{ on edges with negative weight.} \end{cases}$$

At the vertices or nodes of $G$ we impose formally self-adjoint boundary conditions, see [6] for more details regarding the self-adjointness of boundary conditions.

A variational formulation for a class of indefinite self-adjoint boundary-value problems on graphs is given, see [4] and [9] for background on Sturm-Liouville problems with indefinite weight, and [5] concerning variational principles in Krein spaces. We then study the nature of the spectrum of this variational problem and obtain both full and half-range completeness results. A max-min principle for indefinite Sturm-Liouville boundary-value problems on directed graphs is then proved which enables us to develop an analogue of Dirichlet-Neumann bracketing for the eigenvalues of the boundary-value problem and consequently to obtain eigenvalue asymptotics.

In parallel to the variational aspects of boundary-value problems on graphs studied here and on trees in [21], the work of Pokornyi and Pryadiev, and Pokornyi, Pryadiev and Al-Obeid, in [17] and [18], should be noted for the extension of Sturmian oscillation

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theory to second order operators on graphs. The idea of approximating the behaviour of eigenfunctions and eigenvalues for a boundary-value problem on a graph by the behaviour of associated problems on the individual edges, used here, was studied in the definite case in [2], [11] and [22].

An extensive survey of the physical systems giving rise to boundary-value problems on graphs can be found in [15] and the bibliography thereof. Second order boundary-value problems on finite graphs arise naturally in quantum mechanics and circuit theory, [3, 12]. Multi-point boundary-value problems and periodic boundary-value problems can be considered as particular cases of boundary-value problems on graphs, [7].

In Section 2, the boundary-value problem, which forms the topic of this paper, is stated and allowable boundary conditions discussed. An operator formulation is given along with definitions of the various function spaces used. A variational reformulation of the boundary-value problem together with the definition of co-normal (elliptic) boundary conditions is given in Section 3. Here we also show that a function is a variational eigenfunction if and only if it is a classical eigenfunction. In Section 4, we study the spectrum of the variational problem. The main result of this section is that an eigenfunction is in the positive cone, with respect to the $B$ (indefinite inner product), if and only if the corresponding eigenvalue is positive and similarly for the negative cone. Following the approach used by Beals in [4] we prove both full and half-range completeness in Section 5, see Theorem 5.3 and Theorem 5.5. In Section 6, a max-min characterization of the eigenvalues of the boundary value problem is given which is then used in Section 7 to obtain a variant of Dirichlet-Neumann bracketing of the eigenvalues. Hence eigenvalue asymptotics are found. Dirichlet-Neumann bracketing for elliptic partial differential equations can be found in [8].

## 2. Preliminaries

Denote the edges of the graph G by $e_i$ for $i = 1,\ldots,K$. As $e_i$ has length $1$, $e_i$ can be considered as the interval $[0,1]$, where $0$ is identified with the initial point of $e_i$ and $1$ with the terminal point.

We recall, from [11], the following classes of function spaces:

$$\mathscr{L}^2(G) := \bigoplus_{i=1}^{K} \mathscr{L}^2(0,1),$$

$$\mathscr{H}^m(G) := \bigoplus_{i=1}^{K} \mathscr{H}^m(0,1), \quad m = 0,1,2,\ldots,$$

$$\mathscr{H}_o^m(G) := \bigoplus_{i=1}^{K} \mathscr{H}_o^m(0,1), \quad m = 0,1,2,\ldots,$$

$$\mathscr{C}^\omega(G) := \bigoplus_{i=1}^{K} \mathscr{C}^\omega(0,1), \quad \omega = \infty,0,1,2,\ldots,$$

$$\mathscr{C}_o^\omega(G) := \bigoplus_{i=1}^{K} \mathscr{C}_o^\omega(0,1), \quad \omega = \infty, 0, 1, 2, \dots .$$

The inner product on $\mathscr{H}^m(G)$ and $\mathscr{H}_0^m(G)$, denoted $(\cdot,\cdot)_m$, is defined by

$$(f,g)_m := \sum_{i=1}^{K} \sum_{j=0}^{m} \int_0^1 f|_{e_i}^{(j)} \, \overline{g}|_{e_i}^{(j)} \, dt =: \sum_{j=0}^{m} \int_G f^{(j)} \, \overline{g}^{(j)} \, dt. \tag{2.1}$$

Note that $\mathscr{L}^2(G) = \mathscr{H}^0(G) = \mathscr{H}_0^0(G)$. For brevity we will write $(\cdot,\cdot) = (\cdot,\cdot)_0$, $\|f\|_m^2 = (f,f)_m$ and $\|f\| = \|f\|_0$.

The differential equation (1.1) on the graph $G$ can be considered as the system of equations

$$-\frac{d^2 y_i}{dx^2} + q_i(x) y_i = \lambda b_i(x) y_i, \quad x \in [0,1], \, i = 1, \dots, K, \tag{2.2}$$

where $q_i$, $b_i$ and $y_i$ denote $q|_{e_i}$, $b|_{e_i}$ and $y|_{e_i}$.

As in [11], the boundary conditions at the node $v$ are specified in terms of the values of $y$ and $y'$ at $v$ on each of the incident edges. In particular, if the edges which start at $v$ are $e_i, i \in \Lambda_s(v)$, and the edges which end at $v$ are $e_i, i \in \Lambda_e(v)$, then the boundary conditions at $v$ can be expressed as

$$\sum_{j \in \Lambda_s(v)} \left[ \alpha_{ij} y_j + \beta_{ij} {y'}_j \right](0) + \sum_{j \in \Lambda_e(v)} \left[ \gamma_{ij} y_j + \delta_{ij} {y'}_j \right](1) = 0, \quad i = 1, \dots, N(v), \tag{2.3}$$

where $N(v)$ is the number of linearly independent boundary conditions at node $v$. For formally self-adjoint boundary conditions $N(v) = \sharp(\Lambda_s(v)) + \sharp(\Lambda_e(v))$ and $\sum_v N(v) = 2K$, see [6, 16] for more details.

The boundary conditions (2.3) considered over all nodes $v$, after possible relabeling, may be written as

$$\sum_{j=1}^{K} [\alpha_{ij} y_j(0) + \gamma_{ij} y_j(1)] = 0, \quad i = 1, \dots, J, \tag{2.4}$$

$$\sum_{j=1}^{K} [\alpha_{ij} y_j(0) + \beta_{ij} y_j'(0) + \gamma_{ij} y_j(1) + \delta_{ij} y_j'(1)] = 0, \quad i = J+1, \dots, 2K, \tag{2.5}$$

for some $J$, where all possible Dirichlet-like terms are in (2.4), i.e. if (2.5) is written in matrix form then Gauss-Jordan reduction will not allow any pure Dirichlet conditions linearly independent of (2.4) to be extracted.

The boundary-value problem (2.2)–(2.3) on $G$ can be formulated as an operator eigenvalue problem in $\mathscr{L}^2(G)$, [1, 6, 20], for the closed densely defined operator $BL$, where

$$Lf := -f'' + qf \tag{2.6}$$

with domain

$$\mathscr{D}(L) = \{ f \mid f, f' \in AC, Lf \in \mathscr{L}^2(G), \; f \text{ obeying (2.3)} \}. \tag{2.7}$$

The formal self-adjointness of (2.3) relative to $L$ ensures that $L$ is a closed densely defined self-adjoint operator in $\mathscr{L}^2(G)$, see [13, 16, 23], and that $BL$ is self-adjoint in $H_K$ where $H_K$ is $\mathscr{L}^2(G)$ with indefinite inner product $[f,g]=(Bf,g)$.

From [11] we have that the operator $L$ is lower semibounded in $\mathscr{L}^2(G)$.

## 3. Variational Formulation

In this section, we give a variational formulation for the boundary-value problem (2.2)–(2.3) or equivalently for the eigenvalue problem associated with the operator $BL$.

DEFINITION 3.1. (a) Let $\mathscr{D}(F)=\{y\in\mathscr{H}^1(G)\mid y \text{ obeys } (2.4)\}$, where

$$\int_{\partial G} y\,d\sigma := \sum_{i=1}^{K}[y_i(1)-y_i(0)] = \int_G y'\,dt.$$

(b) We say that the boundary conditions on a graph are co-normal or elliptic with respect to $l$ if there exists $f$ defined on $\partial G$, such that $x\in\mathscr{D}(F)$ has

$$\int_{\partial G}(fx+x')\overline{y}\,d\sigma = 0, \quad \text{for all} \quad y\in\mathscr{D}(F)$$

if and only if $x$ obeys (2.5).

(c) If the boundary conditions are co-normal and $f$ is as in (b) and $\mathscr{D}(F)$ is as in (a), then we define the sesquilinear form $F(x,y)$ for $x,y\in\mathscr{D}(F)$ by

$$F(x,y) := \int_{\partial G} fx\overline{y}\,d\sigma + \int_G (x'\overline{y}'+xq\overline{y})\,dt. \tag{3.1}$$

We note that 'Kirchhoff', Dirichlet, Neumann and periodic boundary conditions are all co-normal, but this class does not include all self-adjoint boundary-value problems on graphs.

The following lemma shows that a function is a variational eigenfunction if and only if it is a classical eigenfunction.

LEMMA 3.2. *Suppose that* (2.4)–(2.5) *are co-normal boundary conditions with respect to* $l$ *of* (1.1)*. Then* $u\in\mathscr{D}(F)$ *satisfies* $F(u,v)=\lambda(Bu,v)$ *for all* $v\in\mathscr{D}(F)$ *if and only if* $u\in\mathscr{H}^2(G)$ *and* $u$ *obeys* (1.1), (2.4)–(2.5).

*Proof.* Assume that $u\in\mathscr{H}^2(G)$ and $u$ obeys (1.1), (2.4)–(2.5). Then for each $v\in\mathscr{D}(F)$

$$\begin{aligned} F(u,v) &= \int_{\partial G} fu\overline{v}\,d\sigma + \int_G (u'\overline{v}'+qu\overline{v})\,dt \\ &= \int_{\partial G} fu\overline{v}\,d\sigma + \int_G ((u'\overline{v})'-u''\overline{v}+qu\overline{v})\,dt \\ &= \int_{\partial G} fu\overline{v}\,d\sigma + \int_G (u'\overline{v})'\,dt + \lambda(Bu,v) \end{aligned}$$

$$= \int_{\partial G} (fu + u')\overline{v}\, d\sigma + \lambda(Bu, v).$$

The assumption that (2.4)–(2.5) are co-normal boundary conditions with respect to $l$ gives that $u \in \mathscr{D}(F)$ and

$$\int_{\partial G} (fu + u')\overline{v}\, d\sigma = 0, \quad \text{for all } v \in \mathscr{D}(F),$$

completing the proof this in case.

Now assume $u \in \mathscr{D}(F)$ satisfies $F(u,v) = \lambda(Bu,v)$ for all $v \in \mathscr{D}(F)$. As $\mathscr{C}_o^\infty(G) \subset \mathscr{D}(F)$, it follows that

$$F(u,v) = \lambda(Bu,v), \quad \text{for all } v \in \mathscr{C}_0^\infty(G).$$

Hence $F(u,\cdot)$ can be extended to a continuous linear functional on $\mathscr{L}^2(G)$. In particular, since $q \in \mathscr{L}^\infty(G)$, this gives that

$$\partial u' \in \mathscr{L}^2(G) \subset \mathscr{L}^1_{\text{loc}}(G)$$

where $\partial$ denotes the distributional derivative. Then, by [20, Theorem 1.6, page 44], $u' \in AC$ and $u'' \in \mathscr{L}^1_{\text{loc}}(G)$ allowing integration by parts. Thus

$$lu = -u'' + qu \in \mathscr{L}^1_{\text{loc}}(G)$$

and consequently $lu = \lambda Bu \in \mathscr{L}^2(G)$. Now $q \in \mathscr{L}^\infty(G)$ and $\mathscr{D}(F) \subset \mathscr{L}^2(G)$, giving $u, u'' \in \mathscr{L}^2(G)$ and hence $u \in \mathscr{H}^2(G)$.

The definition of $\mathscr{D}(F)$ ensures that (2.4) holds. Integration by parts gives

$$\int_{\partial G} (fu + u')\overline{y}\, d\sigma = 0, \quad \text{for all } y \in \mathscr{D}(F),$$

which, from the definition of $f$ and the constraints on the class of boundary conditions, is equivalent to $u$ obeying (2.5). □

## 4. Nature of the spectrum

The operator $L$ is self-adjoint in $\mathscr{L}^2(G)$ with spectrum consisting of pure point spectrum and accumulating only at $+\infty$. In addition, we assume that $L$ is positive definite, thus the spectrum of $L$ may be denoted $0 < \rho_1 \leqslant \rho_2 \leqslant \ldots$ where $\lim_{n\to\infty} \rho_n = \infty$. Since $L$ is positive definite and the spectrum consists only of point spectrum, $L^{-1}$ exists and is a compact operator see, [10, p.24], moreover

$$L^{-1}y(t) = \int_G g(t,\tau)y(\tau)\, d\tau, \tag{4.1}$$

where $g(t,\tau)$ is the Green's function of $L$. Thus $L^{-1}B$ is a compact operator. Consider the eigenvalue problem

$$\mu y = L^{-1}By, \quad y \in \mathscr{L}^2(G),$$

where $\mu = \frac{1}{\lambda}$. Since $L^{-1}B$ is compact it has only point spectrum except possibly at $\mu = 0$ and the only possible accumulation point is $\mu = 0$. In addition, $\mu = 0$ is not an eigenvalue of $L^{-1}B$ since 0 is not an eigenvalue of $L^{-1}$. Thus $L^{-1}B$ has countably infinitely many eigenvalues, all non-zero, but accumulating at 0. From (4.1) it follows that

$$L^{-1}By(t) = \int_G g(t,\tau)By(\tau)\,d\tau = \int_G \tilde{g}(t,\tau)y(\tau)\,d\tau,$$

where $\tilde{g}(t,\tau) = g(t,\tau)b(\tau)$. Hence $BL$ has discrete spectrum only, with possible accumulation point at $\infty$ in the complex plane. The spectrum is also countably infinite and, as 0 is not an eigenvalue of $L$, 0 is also not an eigenvalue of $BL$.

LEMMA 4.1. *The space $\mathscr{D}(F)$ is a Hilbert space with inner product $F$. The norm generated by $F$ on $\mathscr{D}(F)$ is equivalent to the $H^1(G)$ norm, making $\mathscr{D}(F)$ a closed subspace of $H^1(G)$.*

*Proof.* By (3.1), [11, Preliminaries] and the trace theorem, see [1, p. 38] we have that there exist constants $K, c > 1$ such that

$$\frac{1}{c}||x||^2_{H^1(G)} \leqslant F(x,x) + K||x||^2 \leqslant c||x||^2_{H^1(G)}. \tag{4.2}$$

Thus the sesquilinear form $F(x,y) + K(x,y)$ is an inner product on $\mathscr{D}(F)$. From (4.2) we get directly that

$$\frac{1}{c}(F(x,x) + K||x||^2) \leqslant ||x||^2_{H^1(G)} \leqslant c(F(x,x) + K||x||^2),$$

making $F(x,y) + K(x,y)$ and $(x,y)_{H^1(G)}$ equivalent inner products on $\mathscr{D}(F)$.

We now show that $F(x,y)$ is an inner product on $\mathscr{D}(F)$ and is equivalent to the inner product $F(x,y) + K(x,y)$ on $\mathscr{D}(F)$. As $\rho_1$ is the least eigenvalue of $L$ on $\mathscr{L}^2(G)$,

$$(Ly,y) \geqslant \rho_1(y,y) = \rho_1||y||^2,$$

for all $y \in \mathscr{D}(L) \subset \mathscr{D}(F)$. Since $F(y,y) = (Ly,y)$, for all $y \in \mathscr{D}(L)$, we get

$$F(y,y) \geqslant \rho_1||y||^2,$$

for $y \in \mathscr{D}(L)$.

Now, $\mathscr{D}(L)$ is dense in $\mathscr{D}(F)$ for $\mathscr{D}(F)$ with norm $|||x|||^2 := F(x,x) + K(x,x)$. Thus, by continuity,

$$||y||^2_F := F(y,y) \geqslant \rho_1||y||^2,$$

for all $y \in \mathscr{D}(F)$, showing that $||\cdot||_F$ is a norm on $\mathscr{D}(F)$ and that $F(x,y)$ is an inner product on $\mathscr{D}(F)$. In addition

$$\left(1 + \frac{K}{\rho_1}\right)||y||^2_F = F(y,y) + \frac{K}{\rho_1}F(y,y) \geqslant F(y,y) + K(y,y) \geqslant F(y,y) = ||y||^2_F,$$

where $K$ is as given above. Thus $F(x,y)+K(x,y)$ and $F(x,y)$ are equivalent inner products on $\mathscr{D}(F)$ and since $F(x,y)+K(x,y)$ and $(x,y)_{H^1(G)}$ are equivalent inner products on $\mathscr{D}(F)$ we have that $F(x,y)$ and $(x,y)_{H^1(G)}$ are equivalent inner products on $\mathscr{D}(F)$.

We now show that, with the $F$ inner product, $\mathscr{D}(F)$ is a Hilbert space. For this, we need only show that $\mathscr{D}(F)$ is closed in $H^1(G)$. The map $\hat{T}: H^1(G) \to \mathbb{C}^J$ given by

$$\hat{T}: y \to \left( \sum_{j=1}^{K} [\alpha_{ij} y_j(0) + \gamma_{ij} y_j(1)] \right)_{i=1,\dots,J},$$

is continuous by the trace theorem, see [1], and thus the kernel of $\hat{T}$, $\mathrm{Ker}(\hat{T}) = \mathscr{D}(F)$ is closed. □

THEOREM 4.2. *The spectrum of* (1.1)*,* (2.4)*–*(2.5) *is real and all eigenvalues are semi-simple.*

*Proof.* As $\mathscr{D}(L)$ is dense in $\mathscr{D}(F)$, $L$ is a densely defined operator in $\mathscr{D}(F)$. Now $F(x,y) := (Lx,y)$ for all $x \in \mathscr{D}(L)$ and $y \in \mathscr{D}(F)$.

Let $\tilde{L} := L^{-1}B$, then $\tilde{L}: \mathscr{L}^2(G) \to \mathscr{D}(L)$ and is thus a map from $\mathscr{D}(F)$ to $\mathscr{D}(L)$.

Since $B$ and $L$ are self adjoint in $\mathscr{L}^2(G)$ we get

$$\begin{aligned} F(\tilde{L}x,y) &= F(L^{-1}Bx,y) \\ &= (Bx,y) \\ &= (x,By) \\ &= \overline{(By,x)} \\ &= \overline{F(\tilde{L}y,x)} \\ &= F(x,\tilde{L}y). \end{aligned}$$

for $x,y \in \mathscr{D}(F)$.

So $\tilde{L}$ is self adjoint in $\mathscr{D}(F)$ (with respect to $F$). Thus, in $\mathscr{D}(F)$, $\tilde{L}$ has only real spectrum and all eigenvalues are semi-simple. Therefore, by Lemma 3.2, the pencil $Lx = \lambda Bx$ has only real spectrum and all eigenvalues are semi-simple. □

Let

$$[f,g] := \sum_{i=1}^{n} \int_0^1 f|_{e_i}\, \bar{g}|_{e_i}\, dt - \sum_{i=n+1}^{K} \int_0^1 f|_{e_i}\, \bar{g}|_{e_i}\, dt = (Bf,g), \tag{4.3}$$

then $\mathscr{L}^2(G)$, with the indefinite inner product given by (4.3), is a Krein space which we denote by $H_K$.

We now define the positive, $C^+$, and negative, $C^-$, cones of $H_K$ by

$$C^+ := \{y \in H_K \mid [y,y] > 0\},$$

$$C^- := \{y \in H_K \mid [y,y] < 0\}.$$

THEOREM 4.3. *For $L$ positive definite in $\mathscr{L}^2(G)$ and $y$ an eigenfunction of* (1.1), (2.4)–(2.5) *corresponding to the eigenvalue $\lambda$ we have $y \in C^+$ if and only if $\lambda > 0$, and $y \in C^-$ if and only if $\lambda < 0$.*

*Proof.* Let $y$ be an eigenfunction corresponding to $\lambda$. Using the fact that any element, $y$, of $H_K$ may be written in the form $y = \{f,g\}$ or $y = f \oplus g$, where $f = (y|_{e_1},\dots,y|_{e_n})$ has $n$ components and $g = (y|_{e_{n+1}},\dots,y|_{e_K})$ has $K-n$ components, we get that

$$C^+ = \{\{f,g\} \mid ||f||^2_{\mathscr{L}^2(G^+)} > ||g||^2_{\mathscr{L}^2(G^-)}\},$$

and

$$C^- = \{\{f,g\} \mid ||f||^2_{\mathscr{L}^2(G^+)} < ||g||^2_{\mathscr{L}^2(G^-)}\}.$$

Here $G^+$ denotes the subgraph of $G$ where the weights are positive and $G^-$ denotes the subgraph of $G$ where the weights are negative.

Since $L > 0$ and $y = \{f,g\}$,

$$0 < (Ly,y) = (\lambda By,y) = \lambda[y,y] = \lambda(||f||^2_{\mathscr{L}^2(G^+)} - ||g||^2_{\mathscr{L}^2(G^-)}).$$

Hence, $y \in C^+$ if and only if $\lambda > 0$, and $y \in C^-$ if and only if $\lambda < 0$. □

## 5. Full and half-range completeness

In this section we prove both half and full range completeness of the eigenfunctions of (1.1), (2.4)–(2.5). In the case presented here the proof is simpler than that of Beals [4], but it is assumed that the problem is left definite, i.e. $L$ is a positive operator.

Recall that, by Lemma 4.1, $\mathscr{D}(F)$ is a Hilbert space. Define

$$\tilde{F}[u](v) := F(u,v)$$

then $\tilde{F} : \mathscr{D}(F) \longrightarrow \mathscr{D}(F)'$, where $\mathscr{D}(F)'$ is the conjugate dual of $\mathscr{D}(F)$, i.e. the space of continuous conjugate-linear maps from $\mathscr{D}(F)$ to $\mathbb{C}$.

LEMMA 5.1. *$\tilde{F}$ is an isomorphism from $\mathscr{D}(F)$ to $\mathscr{D}(F)'$.*

*Proof.* If $F(u_1,v) = F(u_2,v)$, for all $v \in \mathscr{D}(F)$, then $u_1 = u_2$ since $F$ is an inner product on $\mathscr{D}(F)$, see Lemma 4.1. Thus $\tilde{F}$ is one to one.

Now, for $\hat{v} \in \mathscr{D}(F)'$ we have that $\hat{v}(x) = F(v,x)$ for some $v \in \mathscr{D}(F)$ by the Theorem of Riesz, [19]. So $\hat{v}(x) = \tilde{F}[v](x)$ giving that $\tilde{F}[v] = \hat{v}$. Hence $\tilde{F}$ is onto.

Also $\tilde{F}$ and $\tilde{F}^{-1}$ are everywhere defined maps on a Hilbert space and are thus continuous as a consequence of the principle of uniform boundedness (Banach Steinhaus theorem), [19].

So $\tilde{F}$ is an isomorphism from $\mathscr{D}(F)$ to $\mathscr{D}(F)'$. □

Define $T[u](v) := (Bu,v)$ for $u,v \in \mathscr{D}(F)$. Then $T : \mathscr{D}(F) \longrightarrow \mathscr{D}(F)'$ is compact since $\mathscr{D}(F)$ is compactly embedded in $\mathscr{L}^2(G)$ and $Bu \in \mathscr{L}^2(G)$ with the mapping $Bu \mapsto (Bu,\cdot)$ from $\mathscr{L}^2(G)$ to $\mathscr{L}^2(G)'$ continuous. Thus $S := \tilde{F}^{-1}T$ is a compact map with $S : \mathscr{D}(F) \longrightarrow \mathscr{D}(F)$.

LEMMA 5.2. *The compact operator $S$ on $\mathscr{D}(F)$ is self-adjoint with respect to the inner product $F$.*

*Proof.* For $u,v \in \mathscr{D}(F)$

$$F(Su,v) = \tilde{F}[Su](v) = T[u](v) = (Bu,v) = (u,Bv).$$

Similarly

$$\overline{(Bv,u)} = \overline{F(Sv,u)} = F(u,Sv). \quad \square$$

As $S$ is a compact self-adjoint operator on $\mathscr{D}(F)$ and as $0$ is not an eigenvalue of $S$, the eigenfunctions, $(u_n)$, of $S$, with eigenvalues $(\lambda_n^{-1})$, can be chosen so that $(u_n)$ is an orthonormal basis for $\mathscr{D}(F)$.

NOTE. The equation $Su_n = \lambda_n^{-1}u_n$ is equivalent to the equation $Lu_n = \lambda_n Bu_n$, in the sense that if

$$\lambda_n Su_n = u_n,$$

then, by the definition of $S$,

$$\lambda_n(\tilde{F}^{-1}T)u_n = u_n.$$

Applying $\tilde{F}$ to the above gives

$$\lambda_n Tu_n = \tilde{F}u_n.$$

Thus

$$\lambda_n T[v](u_n) = \tilde{F}[v](u_n),$$

for all $v \in \mathscr{D}(F)$. From the definition of $T$, this gives

$$\lambda_n(Bv,u_n) = \tilde{F}[v](u_n).$$

Hence

$$\lambda_n(Bv,u_n) = F(v,u_n)$$

for all $v \in \mathscr{D}(F)$. Using Lemma 3.2 we we obtain that

$$\lambda_n(Bv,u_n) = (v,Lu_n).$$

Therefore

$$(v,\lambda_n Bu_n - Lu_n) = 0,$$

for all $v \in \mathscr{D}(F)$, and by the density of $\mathscr{D}(F)$ in $\mathscr{L}^2(G)$, this yields

$$Lu_n = \lambda_n Bu_n.$$

It is easy to show that if $Lu_n = \lambda_n Bu_n$, then $Su_n = \lambda_n^{-1}u_n$.

In summary, we have the following theorem:

THEOREM 5.3. (Full range completeness) *The eigenfunctions $(y_n)$ of* (1.1)*,* (2.4)*–*(2.5) *form a Riesz basis for $\mathscr{L}^2(G)$ and can be chosen to form an orthonormal basis for $\mathscr{D}(F)$ (with respect to the $F$ inner product). In addition $(y_n)$ is orthogonal with respect to $[\cdot,\cdot]$.*

*Proof.* Since $S$ is a compact self-adjoint operator on the Hilbert space $\mathscr{D}(F)$, the eigenvectors can be chosen to form an orthonormal basis in $\mathscr{D}(F)$. As shown in the note above the variational eigenfunctions coincide with those of $L^{-1}B$ (with eigenvalues mapped by $\lambda \mapsto \frac{1}{\lambda}$ and where $0$ is not in the point spectrum). Thus the eigenfunctions of $L^{-1}B$ can be chosen to form an orthonormal basis for $\mathscr{D}(F)$ and as $\mathscr{D}(F)$ is dense in $\mathscr{L}^2(G)$ they form a Riesz basis for $\mathscr{L}^2(G)$.

Finally, if $(y_n)$ is an orthonormal basis of $\mathscr{D}(F)$ of eigenfunctions then

$$\delta_{n,m} = F(y_n, y_m) = (\lambda_n B y_n, y_m) = \lambda_n (B y_n, y_m) = \lambda_n [y_n, y_m].$$

Hence $(y_n)$ is orthogonal with respect to $[\cdot,\cdot]$. □

Let $P_\pm$ be the positive and negative spectral projections of $S$. Note that $\mathrm{Ker}(S) = \{0\}$. The projections, $P_\pm$, are then defined by the property

$$P_\pm u_n = \begin{cases} u_n, & \pm\lambda_n > 0 \\ 0, & \pm\lambda_n < 0 \end{cases},$$

hence

$$|S| = S(P_+ - P_-) = (P_+ - P_-)S.$$

On $\mathscr{D}(F)$ we introduce the inner product $(u,v)_S = F(|S|u, v)$ with related norm $||u||_S = (u,u)_S^{\frac{1}{2}}$.

We must now show that this norm is equivalent to the $\mathscr{L}^2(G)$ norm, $||u|| = (u,u)^{\frac{1}{2}}$.

The operator $B$ is a self-adjoint operator in $\mathscr{L}^2(G)$ and $B$ has spectral projections $Q_\pm$, where

$$Q_\pm u(x) = \begin{cases} u(x), & b(x) = \pm 1 \\ 0, & b(x) = \mp 1 \end{cases}.$$

Thus $|B| = I = B(Q_+ + Q_-) = (Q_+ + Q_-)B$ is just the identity map, and $|T|$ is the map from $\mathscr{D}(F)$ to $\mathscr{D}(F)'$ induced by $|B|$, i.e. $|T|[u](v) = (u,v)$. But $T[u](v) := (Bu, v)$ for all $u, v \in \mathscr{D}(F)$, and thus can be extended to $u, v \in \mathscr{L}^2(G)$, i.e.

$$T : \mathscr{L}^2(G) \to \mathscr{L}^2(G)' \hookrightarrow \mathscr{D}(F)'.$$

In this sense $TQ_\pm : \mathscr{L}^2(G) \to \mathscr{D}(F)'$ is compact.

Also $T(Q_+ + Q_-)[u](v) = (B(Q_+ + Q_-)u, v) = (u,v) = |T|[u](v)$ for all $u, v \in \mathscr{L}^2(G)$ and thus for $u, v \in \mathscr{D}(F)$. We now observe that $Q'_\pm T : \mathscr{D}(F) \to \mathscr{D}(F)'$, using the extension of $T$ to $\mathscr{L}^2(G)$, is well defined as $Q'_\pm T[u](v) = T[u](Q_\pm v) = (Bu, Q_\pm v) = (Q_\pm Bu, v) = (BQ_\pm u, v)$ making $TQ_\pm = Q'_\pm T$. Hence

$$|T| = T(Q_+ - Q_-) = (Q'_+ - Q'_-)T.$$

THEOREM 5.4. *The norms* $||\cdot||_S$ *and* $||\cdot||$ *are equivalent on* $\mathscr{D}(F)$.

*Proof.* Considered as an operator in the subspace $P_+(\mathscr{D}(F))$, $S$ is a positive operator. Let $y \in \mathscr{D}(L)$. Since $L$ is a positive operator and $\mathscr{D}(F)$ is compactly embedded in $\mathscr{L}^2(G)$ we have that there exists some constant $C > 0$ such that

$$(Ly,y) = F(y,y) \geqslant C(y,y), \tag{5.1}$$

for all $y \in \mathscr{D}(L)$. Also

$$||Q_+y||^2 \leqslant ||y||^2. \tag{5.2}$$

Combining (5.1) and (5.2) we obtain that

$$C||Q_+y||^2 \leqslant C(y,y) \leqslant (Ly,y), \tag{5.3}$$

for $y \in \mathscr{D}(L)$. Let $(y_n)$ be an orthonormal basis of eigenfunctions of $S$ in $\mathscr{D}(F)$ where $y_n$ has eigenvalue $\lambda_n$ with $0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots$ and $0 > \lambda_{-1} \geqslant \lambda_{-2} \geqslant \dots$. Now

$$P_+(\mathscr{D}(F)) = \overline{< y_1, y_2, \dots >},$$

and $Ly_n = \lambda_n B y_n$ for all $n = 1, 2, \dots$.

Let $y \in P_+(\mathscr{D}(L))$ then $y = \sum_{n=1}^{\infty} \alpha_n y_n$ where $\alpha_n \in \mathbb{C}, n \in \mathbb{N}$. From (5.3) we have that

$$||Q_+y||^2 \leqslant \frac{1}{C}(Ly,y).$$

Using the orthogonality of $(y_n)$ we get

$$\frac{1}{C}(Ly,y) = \sum_{n=1}^{\infty} |\alpha_n|^2 \frac{\lambda_n}{C}(By_n, y_n),$$

thus

$$||Q_+y||^2 \leqslant \frac{\lambda_1}{C} \sum_{n=1}^{\infty} |\alpha_n|^2 (By_n, y_n).$$

But

$$\sum_{n=1}^{\infty} |\alpha_n|^2 (By_n, y_n) = (By, y),$$

hence

$$\begin{aligned} ||Q_+y||^2 &\leqslant \frac{\lambda_1}{C}(By,y) \\ &= \frac{\lambda_1}{C} T[y](y) \\ &= \frac{\lambda_1}{C} \tilde{F}[Sy](y) \\ &= \frac{\lambda_1}{C} F(Sy,y) \\ &= \frac{\lambda_1}{C} F(|S|y,y). \end{aligned}$$

So

$$||Q_+y||^2 \leqslant \frac{\lambda_1}{C}||y||_S^2$$

and setting $\sqrt{\frac{\lambda_1}{C}} := k > 0$ gives

$$||Q_+y|| \leqslant k||y||_S. \tag{5.4}$$

Similarly

$$||Q_-y||^2 \leqslant \frac{\lambda_1}{C}||y||_S^2$$

i.e.

$$||Q_-y|| \leqslant k||y||_S. \tag{5.5}$$

Since $\mathscr{D}(L)$ is dense in $\mathscr{D}(F)$, (5.4) and (5.5) hold on all $P_+(\mathscr{D}(F))$, so as $||y||^2 = ||Q_+y||^2 + ||Q_-y||^2$ we have $||y|| \leqslant \sqrt{2}k||y||_S$ for all $y \in P_+(\mathscr{D}(F))$.

Working on $P_-(\mathscr{D}(F))$ yields a similar estimate but with $\lambda_1$ replaced by $-\lambda_{-1}$. Thus there exists a constant $C_1 > 0$ so that for all $y \in \mathscr{D}(F)$,

$$||y|| \leqslant C_1||y||_S. \tag{5.6}$$

To obtain (5.7), the reverse of (5.6), we observe that

$$||y||_S^2 = F(|S|y,y) = F((SP_+ - SP_-)y,y).$$

But $SP_\pm = P_\pm S$ so

$$\begin{aligned}
||y||_S^2 &= F(Sy, P_+y - P_-y) \\
&= \tilde{F}[Sy](P_+y - P_-y) \\
&= T[y](P_+y - P_-y) \\
&= |T|[Q_+y - Q_-y](P_+y - P_-y).
\end{aligned}$$

Using Hölder's inequality we obtain that

$$|T|[Q_+y - Q_-y](P_+y - P_-y) \leqslant ||Q_+y - Q_-y||\,||P_+y - P_-y||.$$

Thus

$$||y||_S^2 \leqslant ||Q_+y - Q_-y||\,||P_+y - P_-y|| = ||y||\,||P_+y - P_-y||.$$

By (5.6)

$$||y||_S^2 \leqslant C_1||y||\,||P_+y - P_-y||_S.$$

Now

$$\begin{aligned}
||P_+y - P_-y||_S &= F(|S|(P_+ - P_-)y, (P_+ - P_-)y) \\
&= F(Sy, (P_+ - P_-)y) \\
&= F((P_+ - P_-)Sy, y) \\
&= F(|S|y, y),
\end{aligned}$$

giving

$$||y||_S^2 \leqslant C_1 ||y|| \, ||y||_S,$$

therefore

$$||y||_S \leqslant C_1 ||y||. \tag{5.7}$$

Combining (5.6) and (5.7) gives

$$\frac{1}{C_1}||y||_S \leqslant ||y|| \leqslant C_1 ||y||_S$$

and thus the two norms are equivalent in $\mathscr{D}(F)$. □

Let $H_S$ be the completion of $\mathscr{D}(F)$ with respect to $||\cdot||_S$.

THEOREM 5.5. (Half-range completeness) *For $Q_+$ and $Q_-$ as previously defined $\{Q_+y_n, \lambda_n > 0\}$ is a Riesz basis for $\mathscr{L}^2(G^+)$ and $\{Q_-y_n, \lambda_n < 0\}$ is a Riesz basis $\mathscr{L}^2(G^-)$.*

*Proof.* To prove the half-range completeness we show that $\{Q_+y_n, \lambda_n > 0\}$ and $\{Q_-y_n, \lambda_n$
$< 0\}$ are Riesz bases for $Q_+P_+(H_S)$ and $Q_-P_-(H_S)$ respectively via showing that $V := Q_+P_+ + Q_-P_-$ is an isomorphism from $H_S$ to $\mathscr{L}^2(G)$, see [4].

Let $u, v \in \mathscr{D}(F)$, then

$$(Q_\pm u, P_\pm v)_S = (Q_\pm u, P_\pm v) \tag{5.8}$$

and

$$(Q_\pm u, P_\mp v)_S = -(Q_\pm u, P_\mp v). \tag{5.9}$$

To see this, as $S$ is self-adjoint with respect to $F$ so is $|S|$, we have, for example,

$$\begin{aligned}(Q_+u, P_-v)_S &= F(|S|Q_+u, P_-v)\\ &= F(Q_+u, |S|P_-v)\\ &= F(Q_+u, S(P_+ - P_-)P_-v)\\ &= F(SQ_+u, -P_-v)\\ &= -F(SQ_+u, P_-v)\\ &= -(Q_+u, P_-v),\end{aligned}$$

because $F(SQ_+u, P_-v) = (BQ_+u, P_-v)$ and $Q_+u(x) = 0$ when $b(x) = -1$ and $Q_+u(x) = u(x)$ when $b(x) = 1$.

Now, as $P_\pm$ are self-adjoint with respect to $[\cdot,\cdot]$,

$$\begin{aligned}||u||_S^2 &= F(|S|u, u)\\ &= F((P_+ - P_-)Su, u)\\ &= F(Su, (P_+ - P_-)u)\\ &= (Bu, (P_+ - P_-)u)\end{aligned}$$

$$= ((Q_+ - Q_-)u, (P_+ - P_-)u)$$
$$= (Q_+u, P_+u) + (Q_-u, P_-u) - (Q_+u, P_-u) - (Q_-u, P_+u).$$

For $u \in \mathscr{D}(F)$,

$$\begin{aligned}
||Vu||^2 &= (Q_+P_+u, Q_+P_+u) + (Q_-P_-u, Q_-P_-u) + (Q_-P_-u, Q_+P_+u) + (Q_+P_+u, Q_-P_-u) \\
&= (Q_+P_+u, Q_+P_+u) + (Q_-P_-u, Q_-P_-u) \\
&= (Q_+(I-P_-)u, (I-Q_-)P_+u) + (Q_-(I-P_+)u, (I-Q_+)P_-u) \\
&= (Q_+u, P_+u) - (Q_+P_-u, P_+u) + (Q_-u, P_-u) - (Q_-P_+u, P_-u) \\
&= ||u||_S^2 + (Q_+u, P_-u) + (Q_-u, P_+u) - (Q_+P_-u, P_+u) - (Q_-P_+u, P_-u).
\end{aligned}$$

Setting $W := Q_+P_- + Q_-P_+$, since $Q_+ - Q_- = B$ and $P_\pm$ are self-adjoint and orthogonal with respect to $[\cdot,\cdot]$, we obtain

$$\begin{aligned}
||Vu||^2 &= ||u||_S^2 + (Q_-P_+u, Q_-P_+u) + (Q_+P_-u, Q_+P_-u) \\
&= ||u||_S^2 + ||Wu||^2.
\end{aligned}$$

As $||\cdot||$ and $||\cdot||_S$ are equivalent norms on $\mathscr{D}(F)$, the above equality holds for $u \in H_S$ and shows that the bounded operator $V$ has closed range and kernel $(0)$.

Equations (5.8) and (5.9) show that, as mappings from $H_S$ to $\mathscr{L}^2(G)$, $V$ and $W$ have adjoints $V^* = P_+Q_+ + P_-Q_-$ and $W^* = -P_+Q_- - P_-Q_+$. But $V^*$ and $W^*$ obey, by the same reasoning as above,

$$||V^*u||_S^2 = ||W^*u||_S^2 + ||u||^2. \tag{5.10}$$

Thus $V^*$ is one to one and therefore $V$ is an isomorphism. Hence we have proved the theorem. $\square$

## 6. Max-Min Property

In this section we give a maximum-minimum characterization for the eigenvalues of indefinite boundary-value problems on graphs. We refer the reader to [8, page 406] and [24] where analogous results for partial differential operators were considered.

In the following theorem $\{v_1, \ldots, v_n\}^\perp$ will denote the orthogonal complement with respect to $[\cdot,\cdot] = (B\cdot,\cdot)$ of $\{v_1, \ldots, v_n\}$. In addition, as is customary, it will be assumed that the eigenvalues, $\lambda_n > 0$, $n \in \mathbb{N}$, of (1.1), (2.4)–(2.5), are listed in increasing order and repeated according to multiplicity, and that the eigenfunctions, $y_n$, are chosen so as to form a complete orthonormal family in $\mathscr{L}^2(G) \cap C^+$. More precisely, as in Theorem 5.3, $(y_n)$, $n \in \mathbb{Z} \setminus \{0\}$ can be chosen so as to form an orthonormal basis for $\mathscr{D}(F)$ and thus for $\mathscr{L}^2(G)$ with respect to $B$. In particular $(y_n)_{n\in\mathbb{N}}$ is then an orthonormal basis for $\mathscr{L}^2(G) \cap C^+$ with respect to $B$ (i.e. $[\cdot,\cdot]$). The case of $\mathscr{L}^2(G) \cap C^-$ is similar, so for the remainder of the paper we will restrict ourselves to $\mathscr{L}^2(G) \cap C^+$.

THEOREM 6.1. *Suppose* $(L\varphi,\varphi)>0$ *for all* $\varphi\in\mathscr{D}(L)\setminus\{0\}$, *and for* $v_j\in\mathscr{L}^2(G)\cap C^+$, $j=1,2,\ldots,$ *let*

$$d_{n+1}(v_1,\ldots,v_n)=\inf\left\{\frac{F(\varphi,\varphi)}{(B\varphi,\varphi)}\;\middle|\;\varphi\in\{v_1,\ldots,v_n\}^{\perp}\cap D(F)\setminus\{0\},(B\varphi,\varphi)>0\right\}. \tag{6.1}$$

*Then*

$$\lambda_{n+1}=\sup\{d_{n+1}(v_1,\ldots,v_n)\mid v_1,\ldots,v_n\in\mathscr{L}^2(G)\cap C^+\}, \tag{6.2}$$

*for* $n=0,1,\ldots,$ *and this maximum-minimum is attained if and only if* $\varphi=y_{n+1}$ *and* $v_i=y_i$, $i=1,\ldots,n$, *where* $y_j$ *is an eigenfunction of* $L$ *with eigenvalue* $\lambda_j$, *and* $(y_j)$ *is a* $B$*-orthogonal family.*

*Proof.* Let $v_1,\ldots,v_n\in\mathscr{L}^2(G)\cap C^+$. As $\operatorname{span}\{y_1,\ldots,y_{n+1}\}$ is $n+1$ dimensional and $\operatorname{span}\{v_1,\ldots,v_n\}$ is at most $n$ dimensional there exists $\varphi$ in $\operatorname{span}\{y_1,\ldots,y_{n+1}\}\setminus\{0\}$ having

$$(B\varphi,v_i)=0,\quad\text{for all}\quad i=1,\ldots,n.$$

In particular, this ensures that $\varphi\in\mathscr{D}(F)$ as each $y_i$ is in $\mathscr{D}(F)$.

Denote $\varphi=\sum_{k=1}^{n+1}c_ky_k$, then

$$\begin{aligned}
F(\varphi,\varphi)&=\sum_{i,k=1}^{n+1}c_i\bar{c}_kF(y_i,y_k)\\
&=\sum_{i=1}^{n+1}|c_i|^2F(y_i,y_i)\\
&=\sum_{i=1}^{n+1}|c_i|^2(Ly_i,y_i)\\
&=\sum_{i=1}^{n+1}|c_i|^2(\lambda_iBy_i,y_i)\\
&=\sum_{i=1}^{n+1}|c_i|^2\lambda_i(By_i,y_i)\\
&\leqslant\lambda_{n+1}\sum_{i=1}^{n+1}|c_i|^2(By_i,y_i)\\
&=\lambda_{n+1}(B\varphi,\varphi),
\end{aligned}$$

thus showing that

$$d_{n+1}(v_1,\ldots,v_n)\leqslant\lambda_{n+1}\quad\text{for all } v_1,\ldots,v_n\in\mathscr{L}^2(G)\cap C^+.$$

Hence

$$\sup\{d_{n+1}(v_1,\ldots,v_n)\,|\,v_1,\ldots,v_n\in\mathscr{L}^2(G)\cap C^+\}\leqslant\lambda_{n+1}.$$

Now suppose $\lambda_{n+1} > d_{n+1}(y_1,\dots,y_n)$. Then there exists $u \in \mathscr{D}(F)\setminus\{0\}$, $u \in \{y_1,\dots,y_n\}^{\perp}$, such that $B(u,u)=1$ and

$$F(u,u) < d_{n+1}(y_1,\dots,y_n) + \frac{1}{2}\left(\lambda_{n+1} - d_{n+1}(y_1,\dots,y_n)\right). \tag{6.3}$$

By Theorem 5.3 we can write $u = \sum_{j\notin\{1,\dots,n\}} \alpha_j y_j$. Therefore

$$\begin{aligned} F(u,u) &= \sum_{i,j\notin\{1,\dots,n\}} \alpha_i \bar{\alpha}_j F(y_i,y_j) \\ &= \sum_{i`\notin\{1,\dots,n\}} |\alpha_i|^2 F(y_i,y_i) \\ &= \sum_{i\notin\{1,\dots,n\}} |\alpha_i|^2 (Ly_i,y_i) \\ &= \sum_{i\notin\{1,\dots,n\}} |\alpha_i|^2 (\lambda_i By_i,y_i). \end{aligned}$$

Now as $\lambda_i(By_i,y_i) = F(y_i,y_i) > 0$ for all $i$, we have

$$\begin{aligned} F(u,u) &= \sum_{i>n} |\alpha_i|^2 \lambda_i (By_i,y_i) + \sum_{i\leqslant -1} |\alpha_i|^2 \lambda_i (By_i,y_i) \\ &\geqslant \sum_{i>n} |\alpha_i|^2 \lambda_i (By_i,y_i) \\ &\geqslant \lambda_{n+1} \sum_{i>n} |\alpha_i|^2 (By_i,y_i) \\ &= \lambda_{n+1}\left(B\sum_{i>n}\alpha_i y_i, \sum_{j>n}\alpha_j y_j\right) \\ &= \lambda_{n+1}(BP_+u,P_+u). \end{aligned}$$

Combining the above with (6.3) and noting that $(Bu,u)=1$, gives

$$\lambda_{n+1} - \frac{1}{2}\left(\lambda_{n+1} - d_{n+1}(y_1,\dots,y_n)\right) > \lambda_{n+1}(BP_+u,P_+u).$$

Thus

$$(Bu,u) - \frac{\lambda_{n+1} - d_{n+1}(y_1,\dots,y_n)}{2\lambda_{n+1}} = 1 - \frac{\lambda_{n+1} - d_{n+1}(y_1,\dots,y_n)}{2\lambda_{n+1}} > (BP_+u,P_+u).$$

Using the self-adjointness of the projections $P_\pm$ with respect to $[\cdot,\cdot]$ now gives

$$(BP_-u,P_-u) > \frac{\lambda_{n+1} - d_{n+1}(y_1,\dots,y_n)}{2\lambda_{n+1}} > 0.$$

But $P_-u \in C^-$, so we have a contradiction and therefore $\lambda_{n+1} \leqslant d_{n+1}(y_1,\dots,y_n)$.

We have shown that $\lambda_{n+1} = d_{n+1}(y_1,\dots,y_n)$, (6.2) holds and $d_{n+1}$ attains its supremum for $(y_1,\dots,y_n)$. Also a direct computation gives $F(y_{n+1},y_{n+1}) = \lambda_{n+1}(By_{n+1},y_{n+1})$.

It remains to be shown that if $u \in \mathscr{D}(F)$ is such that the maximum or minimum is attained for $u,y_1,\dots,y_n$ then $u$ is an eigenfunction with eigenvalue $\lambda = d_{n+1}(y_1,\dots,y_n)$.

Let $u \in \mathscr{D}(F)$ with $(Bu,u) = 1$ and

$$J(\varphi,\varepsilon) = \frac{F(u+\varepsilon\varphi,u+\varepsilon\varphi)}{(B(u+\varepsilon\varphi),u+\varepsilon\varphi)} \quad \text{for all } \varphi \in \mathscr{D}(F), \varepsilon \in \mathbb{R},\ |\varepsilon| \text{ small.}$$

Differentiation with respect to $\varepsilon$ of $J(\varphi,\varepsilon)$ gives

$$0 = \frac{\partial}{\partial\varepsilon}J(\varphi,\varepsilon)|_{\varepsilon=0} = 2\Re[F(\varphi,u) - d_{n+1}(y_1,\dots,y_n)(B\varphi,u)],$$

for all $\varphi \in \mathscr{D}(F)$ and $(Bu,u) = 1$. Since everything in the above expression is real we obtain that

$$F(\varphi,u) = d_{n+1}(y_1,\dots,y_n)(B\varphi,u), \tag{6.4}$$

for all $\varphi \in \mathscr{D}(F)$ and $(Bu,u) = 1$.

Now $F(u,u) > 0$ therefore $d_{n+1}(y_1,\dots,y_n)(Bu,u) > 0$ which, since $(Bu,u) = 1$, gives $d_{n+1}(y_1,\dots,y_n) > 0$. From (6.4), for $\varphi \in \mathscr{C}_0^\infty(G)$, we get that

$$(L\varphi,u) - d_{n+1}(y_1,\dots,y_n)(B\varphi,u) = 0,$$

giving

$$(\varphi,(l - d_{n+1}(y_1,\dots,y_n)B)u) = 0.$$

Hence, by the proof of Lemma 3.2, $u \in H^2(G) \cap \mathscr{D}(F)$ and obeys (1.1) and (2.4). We must still show that $u$ obeys the boundary condition (2.5).

From the proof of Lemma 3.2 we see that, for $\varphi \in \mathscr{D}(F)$,

$$F(u,\varphi) = \int_{\partial G}(fu+u')\bar{\varphi}d\sigma + d_{n+1}(y_1,\dots,y_n)(Bu,\varphi).$$

This together with (6.4) gives that

$$0 = \int_{\partial G}(fu+u')\bar{\varphi}d\sigma \tag{6.5}$$

for all $\varphi \in \mathscr{D}(F)$.

As, (6.5) holds for all $\varphi \in \mathscr{D}(F)$, $u$ obeys (2.5), giving that $u$ is an eigenfunction of (1.1), (2.4)–(2.5) with eigenvalue $\lambda = d_{n+1}(y_1,\dots,y_n)$.

Thus $\lambda = \lambda_{n+1}$ and $u = y_{n+1}$. As this holds for the case of $d_1$, the result has been proved by induction. □

## 7. Eigenvalue Bracketing and Asymptotics

If the boundary conditions (2.4)–(2.5) are replaced by the Dirichlet condition $y=0$ at each node of $G$, i.e.

$$y_i(1)=0 \quad \text{and} \quad y_i(0)=0, \quad i=1,\dots,K, \tag{7.1}$$

then the graph $G$ becomes disconnected with each edge $e_i$ becoming a component sub-graph, $G_i$, with Dirichlet boundary conditions at its two nodes (ends). The boundary value problem on each sub-graph $G_i$ is equivalent to a Sturm-Liouville boundary value problem on $[0,1]$ with Dirichlet boundary conditions. Depending on whether the edge has positive or negative weight the resulting boundary value problem is

$$-y_i''+q_iy_i=\mu y_i, \quad i=1,\dots,n, \tag{7.2}$$

or

$$-y_i''+q_iy_i=-\mu y_i, \quad i=n+1,\dots,K, \tag{7.3}$$

with boundary conditions (7.1).

Let $\lambda_1^D \leqslant \lambda_2^D \leqslant \dots$ be the eigenvalues (repeated according to multiplicity) of the system (7.1) with (7.2) and (7.3) for which the eigenvectors are in $\mathscr{L}^2(G)\cap C^+$. Let $\Lambda_1^D < \Lambda_2^D < \dots$ be the eigenvalues of the system (7.1) with (7.2) and (7.3) not repeated by multiplicity. Denote by $\nu_j^D$ the dimension of the maximal positive (with respect to $[\cdot,\cdot]$) subspace of the eigenspace $E_j^D$ to $\Lambda_j^D$.

Observe that if $\mu$ is an eigenvalue of the system (7.1) with (7.2) and (7.3), with multiplicity $\nu$ and eigenspace $E$, then there are precisely $\nu$ indices $i_1,\dots,i_\nu$ such that $\mu$ is an eigenvalue of

$$-y_i''+q_iy_i=b_i\mu y_i, \tag{7.4}$$

with boundary conditions (7.1). In particular, if

$$Y_j^i := \begin{cases} 0, & j\neq i, \\ y_i, & j=i, \end{cases}$$

where $j\in\{1,\dots,K\}$, then $Y^{i_1},\dots,Y^{i_\nu}$ are eigenfunctions to (7.1) with (7.2) and (7.3) and form a basis for $E$, which is orthogonal with respect to both $(\cdot,\cdot)$ and $[\cdot,\cdot]$. Hence, by [14, Corollary 10.1.4], the maximal $B$-positive subspace of $E$ has dimension

$$\nu^+ = \#(\{i_1,\dots,i_\nu\}\cap\{1,\dots,n\}).$$

I.e. $\nu^+$ is the multiplicity of $\mu$ as an eigenvalue of (7.1) with (7.2).

Hence $\lambda_j^D$ is the $j$th eigenvalue of (7.1) with (7.2), i.e. of (1.1) with (7.1) considered only on $G^+$.

Similarly if we consider the equation (2.2) with the non-Dirichlet conditions

$$y_i'(1)=f(1)y_i(1) \quad \text{and} \quad y_i'(0)=f(0)y_i(0), \quad i=1,\dots,K, \tag{7.5}$$

where $f$ is given in (3.1), then, as in the Dirichlet case, above, $G$ decomposes into a union of disconnected graphs $G_1,\dots,G_K$. Again, depending on whether the edge has positive or negative weight, we have the equation

$$-y_i'' + q_i y_i = \mu y_i, \quad i = 1,\dots,n, \tag{7.6}$$

or

$$-y_i'' + q_i y_i = -\mu y_i, \quad i = n+1,\dots,K, \tag{7.7}$$

with boundary conditions (7.5).

Let $\lambda_1^N \leqslant \lambda_2^N \leqslant \dots$ be the eigenvalues (repeated according to multiplicity) of the system (7.5) with (7.6) and (7.7) for which the eigenvectors are in $\mathscr{L}^2(G)\cap C^+$. By the same reasoning as above, $\lambda_j^N$ is the $j$th eigenvalue of (7.5) with (7.6), i.e. of (1.1) with (7.5) considered only on $G^+$.

Thus, from Theorem 6.1 and [11] we have that, in $\mathscr{L}^2(G)\cap C^+$, the eigenvalues of (2.2), (2.4)–(2.5) are ordered by

$$\lambda_n^N \leqslant \lambda_n \leqslant \lambda_n^D, \quad n = 1,2,\dots. \tag{7.8}$$

The asymptotics for $\lambda_n^N$ and $\lambda_n^D$ are well known, in particular, using the results in [11] for (1.1) on $G^+$, with (7.1) and (7.5) we obtain the following theorem:

THEOREM 7.1. *Let $G$ be a compact graph with finitely many nodes. If the boundary value problem* (2.2)*,* (2.4)*–*(2.5) *has co-normal (elliptic) boundary conditions, then the eigenvalues in $\mathscr{L}^2(G)\cap C^+$ obey the asymptotic development*

$$\sqrt{\lambda_n} = \frac{n\pi}{length(G^+)} + O(1), \quad as\ n\to\infty.$$

By formally replacing $\lambda$ by $-\lambda$ in (1.1) a similar result is obtained for $\mathscr{L}^2(G)\cap C^-$.

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