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ON A COMMUTATIVE WJ^*–ALGEBRA OF D_1^+–CLASS AND ITS BICOMMUTANT

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Abstract. We study different properties of a commutative WJ^*-algebra in a Krein space that has a maximal non-negative subspace represented as a direct sum of its one-dimensional isotropic subspace and a uniformly positive one. In particulary we give a criteria for the equality between of a WJ^*-algebra of this class and its bicommutant.

Introduction

A well-known theorem of J. von Neumann says that the bicommutant of an arbitrary W^*-algebra (all definitions can be found below) coincides with the algebra. If we replace a W^*-algebra by a WJ^*-algebra, the corresponding result is false even for a finite-dimensional Pontryagin space with the index of indefiniteness equal one (i.e. for a finite-dimensional space Π_1). On the other hand, if we consider only commutative WJ^*-algebras, then for the Pontryagin space Π_1 (including infinite-dimensional case) an analog of J. von Neumann's Theorem is true, but this result cannot be extended even for the case of the space Π_2. Here we study different properties of a commutative WJ^*-algebra in a Krein space that has a maximal non-negative subspace represented as a direct sum of its one-dimensional isotropic subspace and a uniformly positive one. In particulary we give a criteria for the equality between of a WJ^*-algebra of this class and its bicommutant. Section 1 contains some known results and definitions with an exception of Subsection 1.3, where some ideas are shown using the simple case of a single operator. In Section 2 a complete model representation for a commutative WJ^*-algebra of D_1^+-class is given (Theorems 44, 45 and 46) and Section 3 is devoted to the bicommutant problem.

1. Definitions and previous results

1.1. Main objects

In what follows the term "Krein space" means a (complex) vector space $\mathscr{H}$ with a Hermitian sesquilinear indefinite form $[\cdot,\cdot]$ if for $\mathscr{H}$ there is at least one scalar product $(\cdot,\cdot)$ that converts $\mathscr{H}$ to a Hilbert space and

$$[x,y] = (Jx,y), \ x,y \in \mathscr{H}, \ J = J^{-1}. \tag{1.1}$$

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The operator J is called a *canonical symmetry*. By the definition of the canonical symmetry J we have $J = P_+ - P_-$, where P_+ and P_- are ortho-projections $P_+ + P_- = I$ and $\mathscr{H}_+ = P_+\mathscr{H}$, $\mathscr{H}_- = P_-\mathscr{H}$. If at least one of the eigen-subspaces of J (corresponding to the eigenvalues $+1$ and -1, respectively) is finite, then the Krein space is said to be a Pontryagin space (a space Π_κ, $\kappa = min\{dim\mathscr{H}_+, dim\mathscr{H}_-\}$). The decomposition $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$ is called *a canonical decomposition*. Here and everywhere below the symbol $\oplus$ means the orthogonal sum with respect to the scalar product $(\cdot,\cdot)$ from (1.1), that will be called the *canonical scalar product*. Let us note that there exist different canonical scalar products, canonical symmetries and canonical decompositions on the same Krein space, but if we fix one of these elements then the other two canonical elements would be uniquely defined via the corresponding formulae. Let us observe also that all canonical scalar products define the same topology on $\mathscr{H}$. In the present work we shall consider the case of *separable* $\mathscr{H}$ only and in what follows this condition will not be mentioned.

In this paper we shall use the terminology from [2]. This remark concerns the natural definitions of positive, negative, definite and neutral vectors or lineals, uniformly positive lineals, maximal non-negative subspaces, regular subspaces, J-orthogonal vectors, J-self-adjoint (J-s.a.) operators, etc. The set of all maximal non-negative subspaces of the Krein space $\mathscr{H}$ is denoted $\mathfrak{M}^+(\mathscr{H})$.

A subspace $\mathscr{L}$ is called *pseudo-regular* ([11]) if it can be presented in the form $\mathscr{L} = \widehat{\mathscr{L}} \dot{+} \mathscr{L}_1$,where $\widehat{\mathscr{L}}$ is a regular subspace and $\mathscr{L}_1 = \mathscr{L} \cap \mathscr{L}^{[\perp]}$ (i.e. $\mathscr{L}_1$ is the isotropic part of $\mathscr{L}$), $[\perp]$ is the symbol of J-orthogonality.

PROPOSITION 1. ([3]) *Let:*

- $\mathfrak{L}_+$ *be a pseudo-regular subspace belonging to* $\mathfrak{M}^+(\mathscr{H})$*;*
- $\mathfrak{L}_1$ *be the isotropic subspace of* $\mathfrak{L}_+$*;*
- $(\cdot,\cdot)'$ *be a scalar product on* $\mathfrak{L}_1$*, such that the norm* $\sqrt{(x,x)'}$ *is equivalent to the original one;*
- $\mathfrak{L}_- = \mathfrak{L}_+^{[\perp]}$*;*

and let

$$\mathfrak{L}_+ = \widehat{\mathfrak{L}}_+ \dot{+} \mathfrak{L}_1,\ \mathfrak{L}_- = \widehat{\mathfrak{L}}_- \dot{+} \mathfrak{L}_1, \tag{1.2}$$

where $\widehat{\mathfrak{L}}_+$ *and* $\widehat{\mathfrak{L}}_-$ *are uniformly definite subspaces. Then one can define on* $\mathscr{H}$ *a canonical scalar product* $(\cdot,\cdot)$ *such that:*

$$\left.\begin{array}{lll} a)\ \text{on } \mathfrak{L}_1 & : & (\cdot,\cdot) \equiv (\cdot,\cdot)' \\ b)\ \mathfrak{L}_1 \perp \widehat{\mathfrak{L}}_+\ , & & \mathfrak{L}_1 \perp \widehat{\mathfrak{L}}_- \\ c)\ \text{on } \widehat{\mathfrak{L}}_+ & : & (\cdot,\cdot) = [\cdot,\cdot] \\ d)\ \text{on } \widehat{\mathfrak{L}}_- & : & (\cdot,\cdot) = -[\cdot,\cdot] \end{array}\right\} \tag{1.3}$$

DEFINITION 2. If a canonical scalar product of a Krein space $\mathscr{H}$ has the properties (1.3), it is said to be *compatible* with Decomposition (1.2) and the choice of the scalar product $(\cdot,\cdot)'$ on $\mathfrak{L}_1$.

Define a special case of pseudo-regular subspaces: a non-negative (non-positive) subspace $\mathscr{L}$ is called *a subspace of the class* h^+ (h^-) if it is pseudo-regular and $\dim(\mathscr{L} \cap \mathscr{L}^{[\perp]}) < \infty$. In Pontryagin spaces every subspace is pseudo-regular and every semi-definite subspace belongs to class h^+ or h^-.

Here the term "operator" means "bounded linear operator". If $\mathfrak{Y}$ is an operator family then the symbol $\mathfrak{Y}'$ refers to the *commutant* of $\mathfrak{Y}$, i.e. to the algebra of all operators B such that $AB = BA$ for every $A \in \mathfrak{Y}$. The algebra $\mathfrak{Y}'' = (\mathfrak{Y}')'$ is said to be a *bicommutant* of $\mathfrak{Y}$. An algebra $\mathfrak{A}$ is called *reflexive* if $\mathfrak{A}'' = \mathfrak{A}$.

By the symbol $B^{\#}$ we denote the operator J-adjoint (J-a.) to an operator B. An operator algebra $\mathfrak{A}$ is said to be WJ^*-algebra if it is closed in the weak operator topology, J-symmetric and contains the identity I. The symbol $\operatorname{Alg}\mathfrak{Y}$ means the minimal WJ^*-algebra which contains $\mathfrak{Y}$.

DEFINITION 3. A J-symmetric operator family $\mathfrak{Y}$ belongs to the class D^+_κ if there is a subspace $\mathscr{L}_+$ in $\mathscr{H}$, such that

- $\mathscr{L}_+$ is $\mathfrak{Y}$-invariant,
- $\mathscr{L}_+ \in \mathfrak{M}^+(\mathscr{H}) \cap h^+$,
- $\dim(\mathscr{L}_+ \cap \mathscr{L}_+^{[\perp]}) = \kappa$.

1.2. Some function spaces

Assume that $\sigma(t)$ is a non-decreasing function defined on the segment $[-1;1]$, continuous in the points -1 and 1, continuous (at least) from the left in all other points of the segment and having an infinite number of growth points, where zero is one of these points. The mentioned function generates on $[-1;1]$ the Lebesgue-Stieltjes measure μ_σ and spaces (L^2_σ, L^∞_σ, etc.) of complex-valued functions. At the same time we shall consider also some spaces of vector-valued functions so from time to time we shall note after a symbol of a space a symbol of a range for the functions forming this space, for instance, $L^2_\sigma(\mathbb{C})$. Let us pass to some notation relating to direct integrals of Hilbert spaces and corresponding model descriptions of self-adjoint operators (see [20], §41; [8], Chapter 7; [9], Chapter 4.4; [21], Chapter VII). We shall use definitions close to the "coordinate notation" given in [20]. A difference between [20] and the definitions that follow is related to the fact that direct integrals here will be used not only for a resolution of Hilbert spaces but also for a resolution of Krein spaces. Let $\mathscr{E}$ be some separable Hilbert space ($\mathscr{E}$ can be finite-dimensional), let $\{d_j\}_1^\alpha$ be an orthonormalized basis of this space, let $\sigma(t)$ be be the same as above. Let $\{\rho_j(t)\}_1^\alpha$ be a system of non-negative μ_σ-measurable functions defined almost everywhere (a.e) on the segment $[-1;1]$ and such that every function of the system is the indicator of some set of non-zero measure and $\mu_\sigma\{t : \rho_j(t) = 0, \quad j = 1,2,\ldots,\alpha\} = 0$. Denote

$$d\vec{\sigma}(t) = \sum_{j=1}^{\alpha} d_j \rho_j(t) d\sigma(t). \tag{1.4}$$

In this case the sum in the right part of the formula is considered as a formal expression without any suggestion of its convergence or divergence.

Here the space $M_{\vec{\sigma}}(\mathscr{E})$ means the space of vector-valued functions $\{f(t)\}$ defined a.e. (with respect to μ_σ) on the segment $[-1;1]$ and taking values in $\mathscr{E}$ under the conditions

$$f(t)=\sum_{j=1}^{\alpha}\beta_j(t)d_j,$$

where $\beta_j(t)$ runs the set of all μ_σ-measurable a.e. finite scalar functions, such that

$$\left.\begin{array}{ll}\text{a)} & \beta_j(t)=\rho_j(t)\beta_j(t),\ j=1,2,\ldots\alpha;\\ \text{b)} & \text{a.e. } \|f(t)\|^2_{\mathscr{E}}=\sum_{j=1}^{\alpha}|\beta_j(t)|^2<\infty.\end{array}\right\} \tag{1.5}$$

The topology on $M_{\vec{\sigma}}(\mathscr{E})$ is introduced by a base for neighborhoods of zero, where any neighborhood of the base is defined by a couple of positive numbers ε and δ (the couples are different for the different neighborhoods) and contains all functions satisfying the condition $\mu_\sigma\{t\colon \|f(t)\|^2_{\mathscr{E}}\geqslant\delta\}<\varepsilon$. Next, the symbol $L^2_{\vec{\sigma}}(\mathscr{E})$ means here a Hilbert space of functions $f(t)\in M_{\vec{\sigma}}(\mathscr{E})$, such that $\int_{-1}^{1}\|f(t)\|^2_{\mathscr{E}}d\sigma(t)<\infty$.

The spaces $M_{\vec{\sigma}}(\mathscr{E})$ and $L^2_{\vec{\sigma}}(\mathscr{E})$ are said to be a *standard space of measurable functions* and a *standard Hilbert space* respectively.

Next, let spaces $L^2_{\vec{\sigma}_+}(\mathscr{E}_+)$ and $L^2_{\vec{\sigma}_-}(\mathscr{E}_-)$ be based (in the sense (1.4)) on monotonous scalar functions $\sigma_+(t)$ and $\sigma_-(t)$, such that

$$\begin{array}{l}\sigma_+(t)=\int_{-1}^{t}\rho_+(\lambda)d\sigma(\lambda),\quad \sigma_-(t)=\int_{-1}^{t}\rho_-(\lambda)d\sigma(\lambda),\quad \rho_+^2(\lambda)=\rho_+(\lambda),\\ \rho_-^2(\lambda)=\rho_-(\lambda),\quad \sigma(t)=\int_{-1}^{t}\Big(\rho_+(\lambda)+\rho_-(\lambda)-\rho_+(\lambda)\rho_-(\lambda)\Big)d\sigma(\lambda).\end{array} \tag{1.6}$$

Put

$$\left.\begin{array}{c}\mathscr{E}=\mathscr{E}_+\oplus\mathscr{E}_-,\quad \mathscr{J}\text{-}L^2_{\vec{\sigma}}(\mathscr{E}):=L^2_{\vec{\sigma}_+}(\mathscr{E}_+)\oplus L^2_{\vec{\sigma}_-}(\mathscr{E}_-)\\ \text{and}\\ [f,g]=\int_{-1}^{1}\big[f(t),g(t)\big]_{\mathscr{E}}d\sigma(t):=\\ \int_{-1}^{1}\Big(\big(f_+(t),g_+(t)\big)_{\mathscr{E}_+}-\big(f_-(t),g_-(t)\big)_{\mathscr{E}_-}\Big)d\sigma(t),\end{array}\right\} \tag{1.7}$$

where $f(t),g(t)\in L^2_{\vec{\sigma}}(\mathscr{E})$, $f_+(t),g_+(t)\in L^2_{\vec{\sigma}_+}(\mathscr{E}_+)$ and $f_-(t),g_-(t)\in L^2_{\vec{\sigma}_+}(\mathscr{E}_-)$. Thus, $\mathscr{J}\text{-}L^2_{\vec{\sigma}}(\mathscr{E})$ is a Krein space. In what follows it is called a *standard Krein space*. As a slight abuse of the previous notation put also

$$M_{\vec{\sigma}}(\mathscr{E}):=M_{\vec{\sigma}_+}(\mathscr{E}_+)\oplus M_{\vec{\sigma}_-}(\mathscr{E}_-). \tag{1.8}$$

Next, let us consider a slightly different construction. Let additionally $\sigma(t)$ be continuous in zero and $G(t)$ be a μ_σ-measurable function defined a.e. on $[-1;1]$ and such that

- a.e. $G(t)\geqslant 1$,
- $\int_{-1}^{-\tau}G(t)d\sigma(t)<\infty$, $\int_{\tau}^{1}G(t)d\sigma(t)<\infty$ for every $\tau\in(0;1]$,

- $\int_{-1}^{1} G(t)d\sigma(t) = \infty$.

Set

$$\left.\begin{aligned} \nu(\tau) &= \begin{cases} \int_{-1}^{\tau} G(t)d\sigma(t), & \text{if } \tau \in [-1;0); \\ \\ -\int_{\tau}^{1} G(t)d\sigma(t), & \text{if } \tau \in (0;1]. \end{cases} \\ \\ \eta(\tau) &= \int_{-1}^{\tau}(1/G(t))d\sigma(t) \text{ for } \tau \in [-1;1]. \end{aligned}\right\} \tag{1.9}$$

The function $\nu(t)$ is non-decreasing in both segments $[-1;0)$ and $(0;1]$ but it is unbounded in neighborhoods of zero. Define for it a corresponding function space. Let $f(t)$ and $g(t)$ be arbitrary functions continuous in $[-1;1]$ and vanishing in some neighborhoods (different in the general case for $f(t)$ and $g(t)$) of zero. Then the integral $\int_{-1}^{1} f(t)\overline{g(t)}d\nu(t)$ is well defined and generates a structure of pre-Hilbert space on the set of all such functions. The completion of the space will be denote L^2_ν (or $L^2_\nu(\mathbb{C})$). In a similar way one can define the space L^1_ν. At the same time the function $\eta(t)$ is non-decreasing on the whole interval $[-1;1]$, hence $\eta(t)$ defines on this interval the ordinary Lebesgue-Stieltjes measure μ_η that is absolutely continuous with respect to μ_σ. Thus, the space L^2_η and others are defined as usual.

Note that due to (1.9) the spaces L^∞_σ and L^2_ν, as well as the spaces L^1_σ and L^2_η, form compatible pairs or Banach pairs (for details see [7] or [15]). Thus, the spaces $L^1_\sigma + L^2_\eta$ and $L^\infty_\sigma \cap L^2_\nu$ are well defined. In particular, the standard norm on $L^1_\sigma + L^2_\eta$ is given by the formula

$$\|f\| = \inf_{f_1+f_2=f}\{\|f_1\|_{L^1_\sigma} + \|f_2\|_{L^2_\eta}\}.$$

The space $L^\infty_\sigma \cap L^2_\nu$ can be considered as adjoint to the space $L^1_\sigma + L^2_\eta$ if the duality between these space is given by the formula $\langle f(t), g(t)\rangle = \int_{-1}^{1} f(t)\overline{g(t)}d\sigma(t)$, where $f(t) \in L^1_\sigma + L^2_\eta$ and $g(t) \in L^\infty_\sigma \cap L^2_\nu$.

Let us pass to some notations relating to direct integrals of Hilbert spaces and corresponding model descriptions of self-adjoint operators (see [20], §41; [8], Chapter 7; [9], Chapter 4.4; [21], Chapter VII). Let $\mathscr{E}$ be some separable Hilbert space ($\mathscr{E}$ can be finite-dimensional), let $\sigma(t)$ be as above. Consider a mapping $t \mapsto \mathscr{E}_t$, $t \in [-1;1]$, where $\mathscr{E}_t \subset \mathscr{E}$, $\dim(\mathscr{E}_t)$ is a μ_σ-measurable (but not necessarily finite a.e.) function, and if $\dim(\mathscr{E}_{t_1}) = \dim(\mathscr{E}_{t_2})$, then $\mathscr{E}_{t_1} = \mathscr{E}_{t_2}$. Denote by $M_{\vec{\sigma}}(\mathscr{E})$ the space of the vector-valued functions $f(t)\colon\ t \mapsto \mathscr{E}_t$ μ_σ-measurable in the weak sense, defined a.e. and finite a.e. on the segment $[-1;1]$. Next, the symbol $L^2_{\vec{\sigma}}(\mathscr{E})$ means here a Hilbert space of functions $f(t) \in M_{\vec{\sigma}}(\mathscr{E})$, such that $\int_{-1}^{1} \|f(t)\|^2_{\mathscr{E}} d\sigma(t) < \infty$.

Introduce some notation related with multiplication operators by scalar function. Everywhere below we assume a scalar function $\varphi(t)$ to be defined a.e. on $[-1;1]$, μ_σ-measurable and a.e. bounded. For $f(t) \in M_{\vec{\sigma}}(\mathscr{E})$ set

$$(\Phi f)(t) = \varphi(t)f(t). \tag{1.10}$$

It is clear that $(\Phi f)(t) \in M_{\vec{\sigma}}(\mathscr{E})$, so equality (1.10) defines on $M_{\vec{\sigma}}(\mathscr{E})$ the continuous operator Φ (= the multiplication operator by the function $\varphi(t)$). If $\varphi(t)$ satisfies some additional conditions one can consider the operator Φ as acting simultaneously on different spaces. If, for instance, $\varphi(t)$ is continuous then the operator Φ is well defined on every space $M_\sigma(\mathscr{E})$ independently of $\vec{\sigma}(t)$ and $\mathscr{E}$. If $\varphi(t) \in L^\infty_\sigma(\mathbb{C})$ then $L^2_{\vec{\sigma}}(\mathscr{E})$ can also be taken as a domain of Φ. So, if it is necessary, we'll mention simultaneously the operator Φ and its domain using the notation $\{\Phi, \mathfrak{D}(\Phi)\}$, say, $\{\Phi, L^2_{\vec{\sigma}}(\mathscr{E})\}$.

REMARK 4. ([26], Remark 2.8; cf. [7], Theorem 5.2.1) Let

$$x_1(t) = \begin{cases} x(t), & \text{if } |x(t)| < G(t); \\ G(t) \cdot e^{i \arg x(t)}, & \text{if } |x(t)| \geqslant G(t); \end{cases} \quad x_2(t) = x(t) - x_1(t).$$

Then $x(t) \in L^1_\sigma + L^2_\eta$ if and only if simultaneously $x_1(t) \in L^2_\eta$ and $x_2(t) \in L^1_\sigma$.

1.3. Some remarks for a single operator

In this subsection we study an operator $A = A^\# \in D_1^+$ with real spectrum. Thus, there is a subspace $\mathscr{L}_+$ in $\mathscr{H}$, such as in Definition 3. Let $\mathscr{L}_- = \mathscr{L}_+^{[\perp]}$. Set

$$\mathscr{L}_1 = \mathscr{L}_+ \cap \mathscr{L}_-, \mathscr{L}_2 = \mathscr{L}_+ \ominus \mathscr{L}_1, \mathscr{L}_3 = \mathscr{L}_- \ominus \mathscr{L}_1, \mathscr{L}_0 = J\mathscr{L}_1. \tag{1.11}$$

As the subspaces $\mathscr{L}_2$ and $\mathscr{L}_3$ are uniformly definite we can suppose that our scalar product is compatible with (1.11), so

$$\mathscr{H} = \mathscr{L}_0 \oplus \mathscr{L}_1 \oplus \mathscr{L}_2 \oplus \mathscr{L}_3. \tag{1.12}$$

Next, the spaces $\mathscr{L}_0$ and $\mathscr{L}_1$ are one-dimensional, so the space $\mathscr{L}_1$ is an eigen-space for A. With no loss of generality we can assume that the corresponding eigen-value of A is equal zero, so with respect to the decomposition (1.12) the operator A has the following representation:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ A_{10} & 0 & A_{12} & A_{13} \\ A_{20} & 0 & A_{22} & 0 \\ A_{30} & 0 & 0 & A_{33} \end{pmatrix}, \tag{1.13}$$

where $A_{22} = A_{22}^*$, $A_{33} = A_{33}^*$. We omit other relations among the elements of the representation (1.13). Since the operators A_{22} and A_{33} are self-adjoint in the ordinary Hilbert sense, they can be described as a multiplication operator by independent variable acting in suitable spaces. With no loss of generality we can assume that $\|A_{22}\| < 1$ and $\|A_{33}\| < 1$, so we identify the operators A_{22} and A_{33} as operators of multiplication operators by independent variable acting in spaces $L^2_{\vec{\sigma}_+}(\mathscr{E}_+)$ and $L^2_{\vec{\sigma}_-}(\mathscr{E}_-)$ respectively. Here $\mathscr{E}_+$ and $\mathscr{E}_-$ are some Hilbert (maybe finite-dimensional) spaces, $\vec{\sigma}_+(t)$ and $\vec{\sigma}_-(t)$ can be discontinuous in zero. Then

$$\mathscr{H} = \mathbb{C} \oplus \mathbb{C} \oplus L^2_{\vec{\sigma}_+}(\mathscr{E}_+) \oplus L^2_{\vec{\sigma}_-}(\mathscr{E}_-), \tag{1.14}$$

and the operators from (1.13) have the following representation

$$A_{22}:\ L^2_{\vec{\sigma}_+}(\mathscr{E}_+)\ \mapsto\ L^2_{\vec{\sigma}_+}(\mathscr{E}_+),\ f(t)\mapsto tf(t),$$

$$A_{33}:\ L^2_{\vec{\sigma}_-}(\mathscr{E}_-)\ \mapsto\ L^2_{\vec{\sigma}_-}(\mathscr{E}_-),\ f(t)\mapsto tf(t),$$

$$A_{12}:\ L^2_{\vec{\sigma}_+}(\mathscr{E}_+)\ \mapsto\ \mathbb{C},\ f(t)\mapsto\int_{-1}^{1}(f(t),a_{12}(t))_{\mathscr{E}_+}d\vec{\sigma}_+(t)\,,$$

$$A_{13}:\ L^2_{\vec{\sigma}_-}(\mathscr{E}_-)\ \mapsto\ \mathbb{C},\ f(t)\mapsto\int_{-1}^{1}(f(t),a_{12}(t))_{\mathscr{E}_-}d\vec{\sigma}_-(t)\,,$$

$$A_{20}:\ \mathbb{C}\mapsto L^2_{\vec{\sigma}_+}(\mathscr{E}_+),\ 1\mapsto a_{12}(t)\,,$$

$$A_{30}:\ \mathbb{C}\mapsto L^2_{\vec{\sigma}_-}(\mathscr{E}_-),\ 1\mapsto -a_{13}(t)\,,$$

$$A_{10}:\ \mathbb{C}\mapsto\mathbb{C},1\mapsto\alpha\,,\quad \alpha=\overline{\alpha}$$

A direct calculation shows that the resolvent $R_\xi(A)$ can be defined for values $\xi\in\mathbb{C}\backslash[-1;1]$ and has the following form

$$R_\xi(A)=\begin{pmatrix}R_{00}&0&0&0\\R_{10}&R_{11}&R_{12}&R_{13}\\R_{20}&0&R_{22}&0\\R_{30}&0&0&R_{33}\end{pmatrix},\tag{1.15}$$

where

$$R_{00}:\mathbb{C}\mapsto\mathbb{C},\ 1\mapsto-\frac{1}{\xi}\,,$$

$$R_{11}:\mathbb{C}\mapsto\mathbb{C},\ 1\mapsto-\frac{1}{\xi}\,,$$

$$R_{10}:\mathbb{C}\mapsto\mathbb{C},\ 1\mapsto\frac{1}{\xi^2}\left\{-\alpha+\int_{-1}^{1}\frac{\|a_{12}(t)\|^2_{\mathscr{E}_+}}{t-\xi}d\vec{\sigma}_+(t)-\int_{-1}^{1}\frac{\|a_{13}(t)\|^2_{\mathscr{E}_-}}{t-\xi}d\vec{\sigma}_-(t)\right\},$$

$$R_{12}:\ L^2_{\vec{\sigma}_+}(\mathscr{E}_+)\ \mapsto\ \mathbb{C},\ f(t)\mapsto\frac{1}{\xi}\cdot\int_{-1}^{1}\frac{(f(t),a_{12}(t))_{\mathscr{E}_+}}{t-\xi}d\vec{\sigma}_+(t)\,,$$

$$R_{13}:\ L^2_{\vec{\sigma}_-}(\mathscr{E}_-)\ \mapsto\ \mathbb{C},\ f(t)\mapsto\frac{1}{\xi}\cdot\int_{-1}^{1}\frac{(f(t),a_{13}(t))_{\mathscr{E}_-}}{t-\xi}d\vec{\sigma}_-(t)\,,$$

$$R_{20}:\ \mathbb{C}\mapsto L^2_{\vec{\sigma}_+}(\mathscr{E}_+),\ 1\mapsto\ \frac{1}{\xi(t-\xi)}\cdot a_{12}(t)\,,$$

$$R_{22}:\ L^2_{\vec{\sigma}_+}(\mathscr{E}_+)\ \mapsto\ L^2_{\vec{\sigma}_+}(\mathscr{E}_+),\ f(t)\mapsto\frac{1}{t-\xi}\cdot f(t),$$

$$R_{30}:\ \mathbb{C}\mapsto L^2_{\vec{\sigma}_-}(\mathscr{E}_-),\ 1\mapsto-\frac{1}{\xi(t-\xi)}\cdot a_{13}(t)\,,$$

$$R_{33}:\ L^2_{\vec{\sigma}_-}(\mathscr{E}_-)\ \mapsto\ L^2_{\vec{\sigma}_-}(\mathscr{E}_-),\ f(t)\mapsto\frac{1}{t-\xi}\cdot f(t)\,.$$

Using these formulas and the standard improper contour integral one can calculate the spectral resolution E^A_λ of A for every $\lambda \in [-1;0)\cup(0;1]$. In particular, for every interval $\Delta = [a;b) \subset [-1;0)\cup(0;1]$ with $b \neq 0$ we have

$$E^A(\Delta) = E^A_b - E^A_a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ E^A_{10}(\Delta) & 0 & E^A_{12}(\Delta) & E^A_{13}(\Delta) \\ E^A_{20}(\Delta) & 0 & E^A_{22}(\Delta) & 0 \\ E^A_{30}(\Delta) & 0 & 0 & E^A_{33}(\Delta) \end{pmatrix}, \tag{1.16}$$

where

$$E^A_{10}(\Delta) : \mathbb{C} \mapsto \mathbb{C},\ 1 \mapsto \left\{ \int_a^b \frac{\|a_{12}(t)\|^2_{\mathscr{E}_+}}{t^2} d\vec{\sigma}_+(t) - \int_a^b \frac{\|a_{13}(t)\|^2_{\mathscr{E}_-}}{t^2} d\vec{\sigma}_-(t) \right\},$$

$$E^A_{12}(\Delta):\ L^2_{\vec{\sigma}_+}(\mathscr{E}_+)\ \mapsto\ \mathbb{C},\ f(t) \mapsto \int_a^b \frac{(f(t), a_{12}(t))_{\mathscr{E}_+}}{t} d\vec{\sigma}_+(t),$$

$$E^A_{13}(\Delta):\ L^2_{\vec{\sigma}_-}(\mathscr{E}_-)\ \mapsto\ \mathbb{C},\ f(t) \mapsto \int_a^b \frac{(f(t), a_{13}(t))_{\mathscr{E}_-}}{t} d\vec{\sigma}_-(t),$$

$$E^A_{20}(\Delta):\ \mathbb{C} \mapsto L^2_{\vec{\sigma}_+}(\mathscr{E}_+),\ 1 \mapsto \frac{1}{t}\cdot a_{12}(t)\cdot \chi_\Delta(t),$$

$$E^A_{22}(\Delta):\ L^2_{\vec{\sigma}_+}(\mathscr{E}_+)\ \mapsto\ L^2_{\vec{\sigma}_+}(\mathscr{E}_+),\ f(t) \mapsto f(t)\cdot \chi_\Delta(t),$$

$$E^A_{30}(\Delta):\ \mathbb{C} \mapsto L^2_{\vec{\sigma}_-}(\mathscr{E}_-),\ 1 \mapsto -\frac{1}{t}\cdot a_{13}(t)\cdot \chi_\Delta(t),$$

$$E^A_{33}(\Delta):\ L^2_{\vec{\sigma}_-}(\mathscr{E}_-)\ \mapsto\ L^2_{\vec{\sigma}_-}(\mathscr{E}_-),\ f(t) \mapsto f(t)\cdot \chi_\Delta(t).$$

It is evident that E^A_λ can be bounded or unbounded. It depends of the functions $\frac{a_{12}(t)}{t}$ and $\frac{a_{13}(t)}{t}$. If $\frac{a_{12}(t)}{t} \in L^2_{\vec{\sigma}_+}(\mathscr{E}_+)$ and $\frac{a_{13}(t)}{t} \in L^2_{\vec{\sigma}_-}(\mathscr{E}_-)$, then E^A_λ is bounded, if at least one of these conditions is not fulfilled, E^A_λ is unbounded. If E^A_λ is bounded, then A is a spectral operator (see [10] and an explanation below) and the projection $E^A(\{0\}) = E^A_{+0} - E^A_0$ is correctly defined. If under the latter condition $AE^A(\{0\}) \neq 0$, the operator A has a non-trivial nilpotent part. In the case of unbounded spectral resolution the representations of A and E^A_λ given above contain a simple idea for a model representation of these objects: one can take into account not only the Hilbert spaces $L^2_{\vec{\sigma}_+}(\mathscr{E}_+)$ and $L^2_{\vec{\sigma}_-}(\mathscr{E}_-)$ but also the functions $\frac{a_{12}(t)}{t}$ and $\frac{a_{13}(t)}{t}$ improper for these spaces. The latter idea was developed for operator families of D^+_κ-class in [28], here we use this approach to study the problem of reflexivity for commutative algebras in the specific case of D^+_1-class. As a preliminary step let us consider the structure of the space $\bigcap_{\varepsilon>0} E^A([-\varepsilon,\varepsilon])\mathscr{H}$.

LEMMA 5. *Let an operator A be under the same conditions as in the beginning of this subsection and let its spectral resolution be unbounded. Then in the terms of Representations* (1.12) *and* (1.16) *we have*

$$\bigcap_{\varepsilon>0} E^A([-\varepsilon,\varepsilon])\mathscr{H} = \mathscr{L}_1 \oplus E_{22}(\{0\})\mathscr{L}_2 \oplus E_{33}(\{0\})\mathscr{L}_3,$$

where $E_{22}(\{0\})$ and $E_{33}(\{0\})$ have the usual sense.

Proof. By (1.16) we have

$$E([-\varepsilon,\varepsilon]) = \begin{pmatrix} I_0 & 0 & 0 & 0 \\ -E^A_{10}(X_\varepsilon) & I_1 & -E^A_{12}(X_\varepsilon) & -E^A_{13}(X_\varepsilon) \\ -E^A_{20}(X_\varepsilon) & 0 & I_2 - E^A_{22}(X_\varepsilon) & 0 \\ -E^A_{30}(X_\varepsilon) & 0 & 0 & I_3 - E^A_{33}(X_\varepsilon) \end{pmatrix},$$

where $X_\varepsilon = [-1,-\varepsilon)\cup(\varepsilon,1]$. In virtue of the latter formulae it is evident that $\mathscr{L}_1 \oplus E_{22}(\{0\})\mathscr{L}_2 \oplus E_{33}(\{0\})\mathscr{L}_3 \subseteq \bigcap_{\varepsilon>0} E^A([-\varepsilon,\varepsilon])\mathscr{H}$, so we need only to prove that the vector $e_1 := 0\oplus 1\oplus 0\oplus 0$ (see (1.14)) belongs to the closure of the linear manifold $\bigcup_{\varepsilon>0} E^A(X_\varepsilon)\mathscr{H}$. Since E^A_λ is unbounded, at least $\frac{a_{12}(t)}{t} \notin L^2_{\vec{\sigma}_+}(\mathscr{E}_+)$ or $\frac{a_{13}(t)}{t} \notin L^2_{\vec{\sigma}_-}(\mathscr{E}_-)$. Let, for instance, $\frac{a_{12}(t)}{t} \notin L^2_{\vec{\sigma}_+}(\mathscr{E}_+)$. Then in terms of Representation (1.14) we have

$$\frac{\chi_{X_\varepsilon}(t)\cdot a_{12}(t)}{t} \oplus e_1 \cdot \left(\int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1}\right) \frac{\|a_{12}(t)\|^2_{\mathscr{E}_+}}{t^2} d\vec{\sigma}_+(t) =$$

$$\frac{\chi_{X_\varepsilon}(t)\cdot a_{12}(t)}{t} \oplus e_1 \cdot \left\| \frac{\chi_{X_\varepsilon}(t)\cdot a_{12}(t)}{t} \right\|^2_{L^2_{\vec{\sigma}_+}(\mathscr{E}_+)} \in E^A(X_\varepsilon)\mathscr{H}.$$

The rest is straightforward. □

1.4. Spectral functions with peculiarities

The following notion is a particular case of the notion introduced in [4] (see also [26] and [28]). Let $\mathfrak{R}_0$ be the family $\{X\}$ of all Borel subsets of $\mathbb{R}$ such that $\partial X \cap \{0\} = \emptyset$, where ∂X is the boundary of X in $\mathbb{R}$. Let $E\colon\ X \mapsto E(X)$ be a countably additive (with respect to weak topology) function, that maps $\mathfrak{R}_0$ to a commutative algebra of projections in a Hilbert space $\mathscr{H}$, where $E([-\varepsilon,+\varepsilon]) \neq 0$ for every $\varepsilon > 0$ and, moreover, $E(\mathbb{R}) = I$. $E(X)$ is called *a spectral function* (*on* $\mathbb{R}$) *with the peculiar spectral point* 0, the mention of 0 can be omitted. The symbol $\mathrm{Supp}(E)$ means the minimal closed subset S of $\mathbb{R}$, such that $E(X) = 0$ for every $X\colon X \subset \mathbb{R}\backslash S$ and $X \in \mathfrak{R}_0$. Note that always $0 \in \mathrm{Supp}(E)$. Besides the symbol E we shall use also as a notation for a spectral function the symbol E_λ, $\lambda \in \mathbb{R}$, where $E_\lambda = E((-\infty,\lambda))$. Note that the notion of peculiar point has no any direct connection with the behavior of the spectral function and it means only that point 0 on $\mathbb{R}$ is distinguished. See below Definition 10 for some explanations. In what follows the symbol let $\mathfrak{R}_0^{(0)}$ means the collection of all numerical subsets X such that $X \in \mathfrak{R}_0$ and $X \cap \Lambda = \emptyset$.

A spectral function E that acts in a Krein space, is said to be *J-orthogonal* (J-orth.sp.f.) if $E(X)$ is a J-ortho-projection for every $X \in \mathfrak{R}_0$.

Let us recall the definition of a *scalar spectral operator* with real spectrum ([10]). An operator A acting in a Hilbert space is said to be a scalar spectral operator if there exists a spectral function E which has not any peculiar spectral points and such that for every Borel set $X \in \mathbb{R}$: $E(X)A = AE(X)$, $\sigma(A|_{E(X)\mathscr{H}}) \subset \overline{X}$ and $AE(X) = \int_X \xi E(d\xi)$ in the weak sense.

Now let E be a spectral function with peculiar spectral point 0. A scalar function $f(\xi)$ is said to be *defined almost everywhere* (with respect to E), *to have finite value almost everywhere*, etc., if the corresponding property holds almost everywhere in the weak sense on an arbitrary set $X \in \mathfrak{R}_0^{(0)}$. We shall assume that the function $f(\xi)$ is not defined at 0. The following theorem represents a partial case of the theorem that was announced in [24] and proved in [4] (see also [2], § III.5).

THEOREM 6. *Let $\mathfrak{Y} \in D_1^+$ be a commutative family of J-s.a. operators with real spectra and at least one operator of $\mathfrak{Y}$ is not a scalar spectral operator. Then there exists a J-orth.sp.f. E with peculiar spectral point 0, such that the following conditions hold*

$$\left.\begin{array}{l}
a)\ E_\lambda \in \operatorname{Alg}\mathfrak{Y} \textit{ for all } \lambda \in \mathbb{R}\backslash\{0\};\\
b)\ \exists\, \mathscr{L}_+ :\ \mathscr{L}_+ \textit{ corresponds to Definition } 3,\ E(\Delta)\mathscr{L}_+[\dot{+}]E(\Delta)\mathscr{L}_- = \\
\quad E(\Delta)\mathscr{H},\ \Delta \textit{ being any closed segment satisfying } \Delta \in \mathfrak{R}_0^{(0)};\\
c)\ \forall A \in \mathfrak{Y},\ \exists \textit{ a defined almost everywhere function } \phi(\lambda), \textit{ such}\\
\quad \textit{that for every interval } \Delta \in \mathfrak{R}_0^{(0)} \textit{ the descomposition } AE(\Delta) = \\
\quad \int_\Delta \phi(\lambda)E(d\lambda) \textit{ is valid;}\\
d)\ \widetilde{\mathscr{H}} :=\ \operatorname*{CLin}_{\Delta \in \mathfrak{R}_0^{(0)}} \{E(\Delta)\mathscr{H}\} \textit{ is pseudo-regular or regular and its iso-}\\
\quad \textit{tropic part is one-dimensional or trivial;}\\
e)\ \forall A \in \mathfrak{A} \textit{ the set } \sigma(A|_{\mathscr{H}_r}), \textit{ where } \mathscr{H}_r = \bigcap_{0 \in \Delta \in \mathfrak{R}^{(0)}} E(\Delta)\mathscr{H}, \textit{ is a sin-}\\
\quad \textit{gletone } \{\lambda_A\}; \textit{ moreover, there is a natural number } n \leqslant 3 \textit{ (the}\\
\quad \textit{same for all } A\textit{) such that } (A - \lambda_A I)^n \mathscr{H}_r = \{0\};\\
f)\ \textit{neither } \limsup_{\lambda \to 0} \|E_\lambda\| = \infty \textit{ or at least for one } A \in \mathfrak{Y} \textit{ the operator}\\
\quad A|_{\mathscr{H}_r} \textit{ is not a scalar spectral operator.}
\end{array}\right\} \qquad (1.17)$$

A spectral function E with a peculiar spectral point 0 satisfying Conditions (1.17) are called *an eigen spectral function* (e.s.f.) of the operator family $\mathfrak{Y}$.

REMARK 7. E.s.f. of $\mathfrak{Y}$ is not uniquely determined but the space $\mathscr{H}_r$ depends only of $\mathfrak{Y}$ (see [29] for details).

Note that the restriction that all operators from $\mathfrak{Y}$ have real spectra is not very strong due to the following remark.

REMARK 8. If $\mathfrak{Y} \in D_1^+$ is a commutative family of J-s.a. operators and $\sigma(A_0)\setminus\mathbb{R} \neq \emptyset$ at least for one $A_0 \in \mathfrak{Y}$, then $\mathscr{H} = \mathscr{H}'_{max}[\dot{+}]\mathscr{H}''_{min}$, where $\mathscr{H}'_{max}$ and $\mathscr{H}''_{min}$ are $\mathfrak{Y}$-invariant subspaces, $\sigma(A|_{\mathscr{H}'_{max}}) \subset \mathbb{R}$ for all $A \in \mathfrak{Y}$, J-orthoprojection P'_{max} onto $\mathscr{H}'$ belongs to $\operatorname{Alg}\mathfrak{Y}$, $\dim(\mathscr{H}''_{min}) = 2$ and the family $\mathfrak{Y}|_{\mathscr{H}'_{max}}$ belongs to the class D_0^+.

DEFINITION 9. Let E_λ be an e.s.f. of an operator family $\mathfrak{Y}$ and let an operator $A \in \mathfrak{Y}$ and a function $\phi(\lambda)$ be connected by the system of equalities from (1.17c). Then the function $\phi(\lambda)$ is said to be *the portrait* of the operator A and the operator A is said to be *the original* of $\phi(\lambda)$ in $\mathfrak{Y}$ (with respect to E_λ).

DEFINITION 10. Let a sp. function E with a peculiar spectral point 0 be an e.s.f. of $\mathfrak{Y}$. The peculiarity is called *regular* if $\limsup\limits_{\lambda\to 0}\|E_\lambda\| < \infty$, otherwise it is called *singular*.

LEMMA 11. *Let a sp. function E with a peculiar spectral point 0 be an e.s.f. of a WJ^*-algebra $\mathfrak{A}$. Then there is an operator $D \in \mathfrak{A}$ and an increasing scalar function $\phi(t)$ such that:*

- $\mu(-1) = -1, \mu(0) = 0, \mu(1) = 1,$
- $E_\lambda = E^D_{\phi(\lambda)},$

where E^D_λ is the e.s.f. of D.

Proof. The statement of Lemma is trivial if 0 is regular peculiarity, because in this case it is enough to put $D\colon = \int_{-1}^1 \lambda\, dE_\lambda$, so let us consider the case $\limsup\limits_{\lambda\to 0}\|E_\lambda\| = \infty$. Due to the definition of sp. function with peculiar point at zero for every fixed closed interval $\Delta \in \mathfrak{R}_0^{(0)}$ and arbitrary function $f(t)$ continuous on Δ the integral $\int_\Delta f(\lambda)dE_\lambda$ is well defined and there is a constant $c_\Delta > 0$ such that $\|\int_\Delta f(\lambda)dE_\lambda\| \leqslant c_\Delta \cdot \max_{t\in\Delta}\{|f(t)|\}$ (see [10]). Thus, for the intervals $\Delta_j = [\frac{1}{j+1};\frac{1}{j}]$ we can find constants c_j, such that $\|\int_{\Delta_j} f(\lambda)dE_\lambda\| \leqslant c_j \cdot \max_{t\in\Delta_j}\{|f(t)|\}$, $j = 1,2,\ldots$. Let us choose a sequence $\{\gamma_j\}_{j=1}^\infty$ such that

- $\gamma_1 = 1,\ \gamma_j \geqslant \gamma_{j+1} > 0,\ j = 1,2,\ldots,$
- $\sum_{j=1}^\infty c_j\gamma_j < \infty.$

Put $\psi(t) = -\gamma_j \cdot j \cdot (1-(j+1)t) + \gamma_{j+1}\cdot(j+1)\cdot(1-tj)$ for $t \in \Delta_j$, $j = 1,2,\ldots$. It is evident that the integral $\int_0^1 \psi(\lambda)dE_\lambda$ is well defined as an improper integral with a singular point at zero. By a similar way we can introduce $\psi(t)$ for $t \in [-1;0)$ and finally put $\psi(0) = 0$. The operator $D\colon = \int_{-1}^1 \psi(\lambda)dE_\lambda$ and the function $\phi(\lambda)$ inverse to $\psi(\lambda)$ are as desired. $\square$

REMARK 12. Even in the case of of a WJ^*-algebra $\mathfrak{A}$ from the D_1^+-class the equality $E_\lambda = E^A_{\phi(\lambda)}$ does not mean that $\mathfrak{A} = \mathrm{Alg}(A)$ because A did not bring a main part of information concerning the nilpotent part of $\mathfrak{A}$.

REMARK 13. If a spectral function E_λ satisfies Conditions (1.17), it easy to show (using the corresponding result for spectral resolutions in Hilbert spaces [1]) that there is two vectors $u_\mu, v_\mu \in \widetilde{\mathscr{H}}$ such that the function $\sigma_{u_\mu,v_\mu}(\lambda): \ = [E_\lambda u_\mu, v_\mu]$ is real, bounded, non-decreasing and the property of functions with respect to Lebesgue-Stieltjes measure generated by $\sigma_{u_\mu,v_\mu}(\lambda)$ are the same that the corresponding properties with respect to E_λ. Note also that $\sigma_{u_\mu,v_\mu}(\lambda)$ can be defined by continuity at 0.

Now we discuss the spectrum multiplicity of the family E_λ. Recall that a subspace $\mathfrak{L}$ is said to be *cyclic* with respect to $\widetilde{E}_\lambda := E_\lambda|_{\widetilde{\mathscr{H}}}$ if $\underset{\lambda\in[-1;1]\setminus\{0\}}{\mathrm{CLin}}\{E_\lambda \mathfrak{L}\} = \widetilde{\mathscr{H}}$.

DEFINITION 14. In what follows a *non-peculiar multiplicity* of J-orth.sp.f. E_λ means the minimal dimension of all cyclic subspaces with respect to $\widetilde{E}_\lambda$.

2. Description of commutative D_1^+-families

2.1. Models for a J-orth.sp.f. with a singular peculiarity

We assume that J-oth.sp.f. $E_\lambda \in D_\kappa$ with the peculiar spectral point 0 satisfies the conditions

$$E_{-1} = 0, \quad E_{+1} = I, \quad E_{-1} = E_{-1+0} \tag{2.1}$$

and:

$$\sup_{\lambda\in[-1;1]\setminus\{0\}} \{\|E_\lambda\|\} = \infty. \tag{2.2}$$

Introduce some notation. Let (see (1.17))

$$\left.\begin{array}{l} \mathscr{H}_1 \ = \ \widetilde{\mathscr{H}} \cap \widetilde{\mathscr{H}}^{[\perp]}, \ \mathscr{H}_2 \ = \ \mathscr{H}_1^{\perp} \cap \widetilde{\mathscr{H}}, \ \ \mathscr{H}_0 \ = \ J\mathscr{H}_1, \\ \mathscr{H}_3 = \left(\widetilde{\mathscr{H}} \oplus \mathscr{H}_0\right)^{[\perp]}, \ P_j \text{ be an orthoprojection (in the} \\ \text{sense of Hilbert spaces) onto } \mathscr{H}_j, \ j = 0,1,2, \\ \widetilde{E}_\lambda: \ = E_\lambda|_{\widetilde{\mathscr{H}}}. \end{array}\right\} \tag{2.3}$$

REMARK 15. By Condition (2.2) the inequality $\mathscr{H}_1 \neq \{0\}$ holds. Moreover, Lemma 11 and Representation (1.16) yield $\mathscr{H}_1 = \mathscr{L}_1$, so $\dim(\mathscr{H}_1) = 1$.

In addition to (2.3) set

$$\widetilde{\mathscr{H}}^{\uparrow} = \mathscr{H}_0 \oplus \mathscr{H}_2, \ \widetilde{E}_\lambda = E_\lambda|_{\widetilde{\mathscr{H}}}, \ \widetilde{E}^{\uparrow}_\lambda = (P_0 + P_2)E_\lambda|_{\widetilde{\mathscr{H}}^{\uparrow}}. \tag{2.4}$$

It is necessary to take into account that, generally speaking, the subspace $\mathscr{H}_2$ is indefinite. Since J-orth.sp.f. E_λ belongs to the class D_κ^+, there is an E_λ-invariant pair of J-orthogonal maximal semi-definite pseudo-regular subspaces $\mathscr{L}_+$ and $\mathscr{L}_-$ with finite-dimensional isotropic part, moreover by Condition (1.17b) we can assume that for every closed interval $\Delta \subset [-1;1]\backslash\{0\}$ the subspace $(E(\Delta)\mathscr{H}) \cap \mathscr{L}_+$ is positive and the subspace $(E(\Delta)\mathscr{H}) \cap \mathscr{L}_-$ is negative. Thanks to the last hypothesis the following subspaces are well defined

$$\widetilde{\mathscr{H}_+} = \operatorname*{CLin}_{\Delta\subset[-1;1]\backslash\{0\}} \{E(\Delta)\mathscr{L}_+\}, \widetilde{\mathscr{H}_-} = \operatorname*{CLin}_{\Delta\subset[-1;1]\backslash\{0\}} \{E(\Delta)\mathscr{L}_-\}. \tag{2.5}$$

One of these subspaces can be trivial (for instance, $\widetilde{\mathscr{H}_+} = \{0\}$) or finite-dimensional. This case simplifies the main part of constructions below, so we usually will assume

$$\dim \widetilde{\mathscr{H}_+} = \infty \text{ and } \dim \widetilde{\mathscr{H}_-} = \infty. \tag{2.6}$$

REMARK 16. Let subspaces $\mathscr{H}_2^+$ and $\mathscr{H}_2^-$ be such that

$$\left.\begin{array}{l} \text{a) } \mathscr{H}_2^+ \text{ and } \mathscr{H}_2^- \text{ are, respectively, uniformly positive and} \\ \quad\text{uniformly negative subspaces;} \\ \text{b) the subspaces } \mathscr{H}_1 \dotplus \mathscr{H}_2^+ \text{ and } \mathscr{H}_1 \dotplus \mathscr{H}_2^- \text{ are } \mathfrak{A}\text{-invariant;} \\ \text{c) } \mathscr{H}_2 = \mathscr{H}_2^+[+]\mathscr{H}_2^-; \end{array}\right\} \tag{2.7}$$

where $\mathscr{H}_2$ is defined by (2.3). In particular, one can take

$$\mathscr{H}_2^+ = P_2\widetilde{\mathscr{H}_+}, \quad \mathscr{H}_2^- = P_2\widetilde{\mathscr{H}_-}. \tag{2.8}$$

In what follows we assume that a canonical scalar product on $\mathscr{H}$ is such that, first, $\mathscr{H}_3^{[\perp]} = \mathscr{H}_3^{\perp}$ and, second, on the subspace $\widetilde{\mathscr{H}} \oplus \mathscr{H}_0$ it is compatible (see Definition 2) with Decompositions (2.7). Thus, with respect to the decomposition

$$\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \mathscr{H}_3 \tag{2.9}$$

we have

$$J = \begin{pmatrix} 0 & V^{-1} & 0 & 0 \\ V & 0 & 0 & 0 \\ 0 & 0 & J_2 & 0 \\ 0 & 0 & 0 & J_3 \end{pmatrix}, \tag{2.10}$$

where the operator $V\colon \mathscr{H}_0 \mapsto \mathscr{H}_1$ is isometric, J_2 and J_3 are canonical symmetries of the form $[\cdot,\cdot]$ on $\mathscr{H}_2$ and $\mathscr{H}_3$ respectively. Since the subspace $\mathscr{H}_1$ is one-dimensional, we can fix a vector

$$e_1 \in \mathscr{H}_1 \text{ with } \|e_1\| = 1 \text{ and put } e_0 = V^{-1}e_1, \tag{2.11}$$

identify e_0 with e_1 and, finally, treat $\mathscr{H}_0$ and $\mathscr{H}_1$ as $\mathbb{C}$ and V as the identical operator.

Now let $E_\lambda^+ := P_2^+E_\lambda|_{\mathscr{H}_2^+}$ and $E_\lambda^- := P_2^-E_\lambda|_{\mathscr{H}_2^-}$, where P_2^+ and P_2^- are the ortho-projections onto the corresponding subspaces. Since the canonical scalar product is compatible with (2.8), $(\cdot,\cdot)|_{\mathscr{H}_2^+} = [\cdot,\cdot]|_{\mathscr{H}_2^+}$ and $(\cdot,\cdot)|_{\mathscr{H}_2^-} = -[\cdot,\cdot]|_{\mathscr{H}_2^-}$, thus E_λ^+ and E_λ^- are orthogonal spectral resolutions in the corresponding Hilbert spaces. So, there are Hilbert spaces $L^2_{\vec\sigma_+}(\mathscr{E}_+)$ and $L^2_{\vec\sigma_-}(\mathscr{E}_-)$ such that the operator-valued functions E^+_λ and E^-_λ are similar to the multiplication operator by indicator function $\chi_{[-1;\lambda)}(t)$ acting on $L^2_{\vec\sigma_+}(\mathscr{E}_+)$ and $L^2_{\vec\sigma_-}(\mathscr{E}_-)$ respectively with corresponding isometric operators of similarity $W_2^+ : L^2_{\vec\sigma_+}(\mathscr{E}_+) \mapsto \mathscr{H}_2^+$ and $W_2^- : L^2_{\vec\sigma_-}(\mathscr{E}_-) \mapsto \mathscr{H}_2^-$. Since the functions σ_+ and σ_- can be chosen within the class of functions generated equivalent measures, we can assume that for σ_+ and σ_- Conditions (1.6) are fulfilled. Now let us define the standard Krein space $\mathscr{J}\text{-}\widetilde{L}^2_{\vec\sigma}(\mathscr{E})$ as in (1.7). It is clear that by the construction the bloc

$$\begin{pmatrix} E^+_\lambda & 0 \\ 0 & E^-_\lambda \end{pmatrix}$$

is similar to the multiplication operator by indicator function $\chi_{[-1;\lambda)}(t)$ acting on $\mathscr{J}\text{-}\widetilde{L}^2_{\vec\sigma}(\mathscr{E})$, and the corresponding operator of similarity $W_2 : \mathscr{J}\text{-}L^2_{\vec\sigma}(\mathscr{E}) \mapsto \mathscr{H}_2$, $W_2 = W_2^+ \oplus W_2^-$ is simultaneously an isometry between Hilbert and Krein spaces. Next, if for an interval Δ: $\Delta \in \mathfrak{R}_0^{(0)}$, then with respect to the decomposition (2.9) the operator $E(\Delta)$ has the representation

$$E(\Delta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ E_{10}(\Delta) & 0 & E_{12}(\Delta) & 0 \\ E_{20}(\Delta) & 0 & E_{22}(\Delta) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.12}$$

Note that the operator $E_{12}(\Delta)$ can be treated as a boundary linear functional acting on the Krein space $\mathscr{H}_2$, so due to Riesz Theorem on bounded linear functionals and the equality $\big(E(\Delta)\big)^2 = E(\Delta)$ there is a function $\widetilde{g}(t) \in M_{\vec\sigma}(\mathscr{E})$, such that $E_{12}(\Delta)W_2^{-1}f(t) = \int_\Delta \big[f(t), \widetilde{g}(t)\big]_{\mathscr{E}} d\sigma(t)$. Note that the functional $E_{12}(\Delta)$ is bounded for a fixed interval $\Delta \in \mathfrak{R}_0^{(0)}$, but the whole family $\{E_{12}(\Delta)\}_{\Delta \in \mathfrak{R}_0^{(0)}}$ is unbounded, so $\chi_\Delta(t) \cdot \widetilde{g}(t) \in \mathscr{J}\text{-}\widetilde{L}^2_{\vec\sigma}(\mathscr{E})$, but $\widetilde{g}(t) \notin \mathscr{J}\text{-}\widetilde{L}^2_{\vec\sigma}(\mathscr{E})$. So, let us go to the following construction.

Let $\mathscr{J}\text{-}L^2_{\vec\sigma}(\mathscr{E})$ be a standard Krein space. Let $\widetilde{g}(t) \in M_{\vec\sigma}(\mathscr{E})$ be such that

$$\chi_\Delta(t) \cdot \widetilde{g}(t) \in \mathscr{J}\text{-}\widetilde{L}^2_{\vec\sigma}(\mathscr{E}) \text{ for every interval } \Delta \in \mathfrak{R}_0^{(0)}, \text{ but } \widetilde{g}(t) \notin \mathscr{J}\text{-}L^2_{\vec\sigma}(\mathscr{E}). \tag{2.13}$$

In what follows we say that $\widetilde{g}(t)$ is an *improper function* for $\mathscr{J}\text{-}L^2_{\vec\sigma}(\mathscr{E})$. Denote by $\mathscr{J}\text{-}\widetilde{L}^2_{\vec\sigma}(\mathscr{E}) \subset M_{\vec\sigma}(\mathscr{E})$ the linear span generated by the space $\mathscr{J}\text{-}L^2_{\vec\sigma}(\mathscr{E})$ and the function $\widetilde{g}(t)$. Define on $\mathscr{J}\text{-}\widetilde{L}^2_{\vec\sigma}(\mathscr{E})$ structures of Hilbert and Krein spaces in the following way: on $\mathscr{J}\text{-}L^2_{\vec\sigma}(\mathscr{E})$ both structures coincide with the original structures, the function $\widetilde{g}(t)$ is by definition positive (as an element of the Krein space), normalized and J-normalized, orthogonal and J-orthogonal to $\mathscr{J}\text{-}L^2_{\vec\sigma}(\mathscr{E})$. The space $\mathscr{J}\text{-}\widetilde{L}^2_{\vec\sigma}(\mathscr{E})$ is said to be *the expansion of* $\mathscr{J}\text{-}L^2_{\vec\sigma}(\mathscr{E})$ (*generated by the function* $\widetilde{g}(t)$). Thus, we proved above the following theorem (e.g. with Subsection 1.3)

THEOREM 17. *If a J-orth.sp.f. E_λ satisfies Condition* (2.2) *and a scalar product on $\mathscr{H}$ is compatible with* (2.7)*, then there are, first, a subspace $\mathscr{J}\text{-}L^2_{\vec\sigma}(\mathscr{E})$ and a* *function* $\widetilde{g}(t) \in M_{\vec{\sigma}}(\mathscr{E})$ *under Condition* (2.13) *forming together the space* $\mathscr{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathscr{E})$ *and, second, a* J*-isometric operator* $W|_{L^2_{\vec{\sigma}}(\mathscr{E})}$: $WL^2_{\vec{\sigma}}(\mathscr{E}) = \mathscr{H}_2$, *such that for every* $\Delta \in \mathfrak{R}_0^{(0)}$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ E_{10}(\Delta) & 0 & E_{12}(\Delta) & 0 \\ E_{20}(\Delta) & 0 & E_{22}(\Delta) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U \begin{pmatrix} 0 & 0 & 0 & 0 \\ E^{(f)}_{10}(\Delta) & 0 & E^{(f)}_{12}(\Delta) & 0 \\ E^{(f)}_{20}(\Delta) & 0 & E^{(f)}_{22}(\Delta) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} U^{-1},$$

where

$$U: \begin{pmatrix} \mathbb{C} \\ \mathbb{C} \\ \mathscr{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathscr{E}) \\ \mathscr{H}_3 \end{pmatrix} \mapsto \begin{pmatrix} \mathscr{H}_0 \\ \mathscr{H}_1 \\ \mathscr{H}_2 \\ \mathscr{H}_3 \end{pmatrix}, \quad U = \begin{pmatrix} U_0 & 0 & 0 & \\ 0 & U_1 & 0 & 0 \\ 0 & 0 & W_2 & 0 \\ 0 & 0 & 0 & I_3 \end{pmatrix},$$

$$U_0 : 1 \mapsto e_0, \quad U_1 : 1 \mapsto e_1, \quad E^{(f)}_{10} : 1 \mapsto \int_\Delta [\widetilde{g}(t), \widetilde{g}(t)]_{\mathscr{E}} d\sigma(t),\ E^{(f)}_{12} :$$

$$f(t) \mapsto \int_\Delta [f(t), \widetilde{g}(t)]_{\mathscr{E}} d\sigma(t),\ E^{(f)}_{20} : 1 \mapsto \chi_\Delta(t) \cdot \widetilde{g}(t),\ E^{(f)}_{22} : f(t) \mapsto \chi_\Delta(t) \cdot f(t).$$

REMARK 18. By the hypothesis in Theorem 17 the scalar product on $\mathscr{H}$ is compatible with (2.7), so $\widetilde{g}(t) = \widetilde{g}_+(t) \oplus \widetilde{g}_-(t)$, where:

- $\widetilde{g}_+(t) \in M_{\vec{\sigma}_+}(\mathscr{E}_+)$ and $\widetilde{g}_-(t) \in M_{\vec{\sigma}_+}(\mathscr{E}_-)$,
- the Hilbert spaces $L^2_{\vec{\sigma}_+}(\mathscr{E}_+)$ and $L^2_{\vec{\sigma}_-}(\mathscr{E}_-)$ are related with the standard Krein space $\mathscr{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathscr{E})$ as in (1.7),
- $W_2 L^2_{\vec{\sigma}_+}(\mathscr{E}_+) = \mathscr{H}_2^+$ and $W_2 L^2_{\vec{\sigma}_-}(\mathscr{E}_-) = \mathscr{H}_2^-$,
- at least $\widetilde{g}_+(t) \notin L^2_{\vec{\sigma}_+}(\mathscr{E}_+)$ or $\widetilde{g}_-(t) \notin L^2_{\vec{\sigma}_-}(\mathscr{E}_-)$,
- $E^{(f)}_{10} : 1 \mapsto \int_\Delta \|\widetilde{g}_+(t)\|^2_{\mathscr{E}_+} d\sigma_+(t) - \int_\Delta \|\widetilde{g}_-(t)\|^2_{\mathscr{E}_-} d\sigma_-(t)$.

Passing from $E(\Delta)$ to E_λ one can obtain a following result that is a particular case of Theorem 6.19 from [28].

THEOREM 19. *If a* J*-orth.sp.f.* E_λ *satisfies Condition* (2.2) *and a scalar product on* $\mathscr{H}$ *is compatible with* (2.7)*, then there are, first, a space* $\mathscr{J}\text{-}L^2_{\vec{\sigma}}(\mathscr{E})$ *and a function* $\widetilde{g}(t) \in M_{\vec{\sigma}}(\mathscr{E})$ *under Condition* (2.13) *forming together the space* $\mathscr{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathscr{E})$ *and, second, an isometric operator* $W: \mathscr{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathscr{E}) \mapsto \widetilde{\mathscr{H}}$ *with* J*-isometric restriction* $W_2 = W|_{\mathscr{J}\text{-}L^2_{\vec{\sigma}}(\mathscr{E})}$: $WL^2_{\vec{\sigma}}(\mathscr{E}) = \mathscr{H}_2$, *such that for every* $\lambda \in [-1;1]$

$$\widetilde{E}_\lambda = W \cdot \mathrm{X}^\#_\lambda \cdot (W)^{-1}, \quad W^\uparrow = (I_2 \oplus V)W, \quad \widetilde{E}^\uparrow_\lambda = W^\uparrow \cdot \mathrm{X}_\lambda \cdot (W^\uparrow)^{-1}, \tag{2.14}$$

where I_2 *is the identical operator on the space* $\mathscr{H}_2$ *and* $\mathrm{X}_\lambda = \{\mathrm{X}_\lambda, \mathscr{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathscr{E})\}$ *is the multiplication operator by the indicator* $\chi_{[-1,\lambda)}(t)$ *of the interval* $[-1,\lambda)$.

REMARK 20. If Condition (2.6) is not hold, the space $\mathscr{J}\text{-}L^2_{\vec{\sigma}}(\mathscr{E})$ in Theorems 17 and 19 must be replaced by a Hilbert or Pontryagin space.

DEFINITION 21. If for Decomposition (2.3), (2.8) a relation between a J-orth.sp.f. E_λ satisfying Condition (2.2) and a space $\mathscr{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathscr{E})$ is given by Formulae (2.14), then $\mathscr{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathscr{E})$ is said to be *a basic model space* for E_λ (compatible with (2.3), (2.4), (2.7)) and the operator W is said to be *an operator of similarity* corresponding to this space.

If a function $\gamma(t)$ is such that $\gamma(t)f(t) \in \mathscr{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathscr{E})$ for every $f(t) \in \mathscr{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathscr{E})$, then the multiplication operator $\Gamma = \{\Gamma, \mathscr{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathscr{E})\}$ by the function $\gamma(t)$ is well defined. Let us note the following fact.

PROPOSITION 22. ([28],[26]) The relation

$$\Gamma \mathscr{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathscr{E}) \subset \mathscr{J}\text{-}L^2_{\vec{\sigma}}(\mathscr{E}) \tag{2.15}$$

holds if and only if

$$\gamma(t) \in L^\infty_\sigma \cap L^2_\nu, \tag{2.16}$$

where ν is defined by (1.9) and

$$G(t) = 1 + \|\widetilde{g}(t)\|^2_{\mathscr{E}}. \tag{2.17}$$

THEOREM 23. ([27]) Assume that a J-orth.sp.f. E_λ satisfies Condition (2.2), a scalar product on $\mathscr{H}$ is compatible with (2.7) and $\mathscr{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathscr{E})$ is a basic model space for E_λ. If a J-s.a. operator C and a function $\gamma(t)$ are such that $CE(\Delta) = \int_\Delta \gamma(t)E(dt)$ for every interval $\Delta \in \mathfrak{R}_{\{0\}}$, $0 \notin \Delta$, then

- $a)$ a.e. $\gamma(t) = \overline{\gamma(t)}$;
- $b)$ $C\widetilde{\mathscr{H}} \subset \widetilde{\mathscr{H}}$;
- $c)$ for $\gamma(t)$ the condition $\Gamma \mathscr{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathscr{E}) \subset \mathscr{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathscr{E})$ holds;
- $d)$ $\widetilde{C} = W \cdot \Gamma^\# \cdot W^{-1}$, $\widetilde{C}^\uparrow = W^\uparrow \cdot \Gamma \cdot (W^\uparrow)^{-1}$,

where $\widetilde{C} = C|_{\widetilde{\mathscr{H}}}$, $\widetilde{C}^\uparrow = (P_0 + P_2)C|_{\widetilde{\mathscr{H}}^\uparrow}$, operators W and $W^\uparrow$ are from (2.14).

COROLLARY 24. *Assume that a J-orth.sp.f. E_λ satisfies Condition* (2.2)*, a scalar product on $\mathscr{H}$ is compatible with* (2.7) *and $\mathscr{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathscr{E})$ is a basic model space for E_λ. Let a J-s.a. operator C and a function $\gamma(t)$ be such that $CE(\Delta) = \int_\Delta \gamma(t)E(dt)$ for every interval $\Delta \in \mathfrak{R}_{\{0\}}$, $0 \notin \Delta$. Then $C|_{\mathscr{H}_1} = 0$ if and only if $\gamma(t) \in L^\infty_\sigma \cap L^2_\nu$.*

We start to consider the problem of the functional description of a commutative WJ^*-algebra $\mathfrak{A} \in D_1^+$. In this stage we assume that all J-self-adjoint operators in $\mathfrak{A}$ have real spectra and an e.s.f. E_λ of $\mathfrak{A}$ has zero as a singular peculiar point. An immediate consequence of this hypothesis is the relation $\mathscr{H}_1 := \widetilde{\mathscr{H}} \cap \widetilde{\mathscr{H}}^{[\perp]} \neq \{0\}$.

As a first step we describe an operator subalgebra of $\mathfrak{A}$ that can be directly calculated through E_λ. Let $\varphi(t)$ be a continuous scalar function vanishing near 0. Set

$$A_\varphi = \int_{-1}^{1} \varphi(t)dE_\lambda, \tag{2.18}$$

where the improper integral has the obvious meaning.

Denote $\mathfrak{A}^{(sc)}$ the weak closure of the operator set $\{A_\varphi\}$ generated by (2.18). The definition of $\mathfrak{A}^{(sc)}$ is valid independently of the fact if zero is a singular or regular peculiarity. We introduce the following notation: $\mathscr{G}_\varphi(E_\lambda)$ means the totality of operators from $\mathfrak{A}^{(sc)}$ which are originals of the function $\varphi(t)$.

PROPOSITION 25. ([26]) $\mathscr{G}_\varphi(E_\lambda) \neq \emptyset$ if and only if $\varphi(t) \in L^\infty_\sigma \cap L^2_\nu$.

THEOREM 26. *Let the J-self-adjoint operators of a family $\mathfrak{Y} \in D_1^+$ have real spectra, let an e.sp.f. E_λ of $\mathfrak{Y}$ be unbounded and let $\mathscr{J}$-$\widetilde{L}^2_{\vec{\sigma}}(\mathscr{E})$ be its basic model space. Then for every $A = A^\# \in \mathfrak{A} := \mathrm{Alg}(\mathfrak{Y})$ there is a real number α and a real function $\gamma_0(t) \in L^\infty_\sigma \cap L^2_\nu$ such that*

$$A = \alpha I + C + Q, \tag{2.19}$$

where $Q \in \mathfrak{A}$ is a nilpotent operator, $Q|_{\widetilde{\mathscr{H}}} = Q^\#|_{\widetilde{\mathscr{H}}} = 0$, $C \in \mathfrak{A}^{(sc)}$, $CE(\Delta) = \int_\Delta \gamma_0(\lambda)dE_\lambda$ for every interval $\Delta \in \mathfrak{R}_0^{(0)}$.

Proof. If $A = A^\# \in \mathfrak{Y}$, then the representation $AE(\Delta) = \int_\Delta \gamma(\lambda)dE_\lambda$ and the equality $Ae_1 = \alpha e_1$ with $\alpha = \overline{\alpha}$ follow directly from Properties (1.17c,e) and, since the algebra $\mathfrak{A}$ is generated by $\mathfrak{Y}$, the same is true for $A = A^\# \in \mathfrak{A}$. Next, Proposition 22 and Theorem 23 yield $\gamma_0(t) := (\gamma(t) - \alpha) \in L^\infty_\sigma \cap L^2_\nu$. Thus, we need to prove that there is $C \in \mathfrak{A}_0$ with the portrait $\gamma_0(t)$. Let us note (see Theorem 17) that the strong limit $\mathrm{s}-\lim_{\varepsilon\to+0}\left(\int_{-1}^{-\varepsilon} + \int_\varepsilon^1\right)\gamma_0(t)dE_t$ does not exist if and only if the improper integral $\int_{-1}^{1}\gamma_0(t)[\widetilde{g}(t),\widetilde{g}(t)]_{\mathscr{E}}d\sigma(t)$ diverges, so we need to consider the latter case, that (see Subsection 1.2) means

$$[\widetilde{g}(t),\widetilde{g}(t)]_{\mathscr{E}} \notin L^1_\sigma + L^2_\eta. \tag{2.20}$$

Let Ω be a set of all functions $\omega(t) \in L^\infty_\sigma \cap L^2_\nu$ vanishing near zero. Then the linear functional $\theta:\ \theta\omega := \int_{-1}^{1}\omega(t)[\widetilde{g}(t),\widetilde{g}(t)]_{\mathscr{E}}d\sigma(t)$ is well defined on Ω but at the same time unbounded on this linear manifold with respect to the norm of $L^\infty_\sigma \cap L^2_\nu$. Since the kernel of an unbounded linear functional is dense on its domain, there is a sequence $\{\omega_j(t)\}_{j=1}^\infty \subset \mathrm{Ker}(\theta)$, such that $\lim_{j\to\infty}\|\gamma_0(t) - \omega_j(t)\|_{L^\infty_\sigma\cap L^2_\nu} = 0$. It is evident that $\int_{-1}^{1}\omega_j(t)dE_t \in \mathfrak{A}_0$. Since the limit $\mathrm{s}-\lim_{j\to+0}\int_{-1}^{1}\omega_j(t)dE_t$ exists, we can put $C = \mathrm{s}-\lim_{j\to+0}\int_{-1}^{1}\omega_j(t)dE_t$. the rest is straightforward. $\square$

REMARK 27. If (2.20) holds, the operator C in (2.19) is defined by $\gamma_0(t)$ not uniquely but up to the summand $\xi \cdot S_0$, where $\xi = \overline{\xi}$ and the operator S_0 is defined as

$$S_0 x := [x, e_1] \cdot e_1 . \tag{2.21}$$

Indeed, one can replace the sequence $\{\omega_j(t)\}_{j=1}^{\infty} \subset \mathrm{Ker}(\theta)$ from the proof of Theorem 26 by a sequence $\{\psi_j(t)\}_{j=1}^{\infty} \subset \Omega$, $\theta\psi_j = \xi$ for all j. See [26] for details.

REMARK 28. Represenation (2.19) remains valid if $\mathscr{H}_1 = \{0\}$. It follows directly from the theory of spectral operators (see [10]).

Let us again consider Decomposition (2.9) together with Condition (2.2). Then for an operator $A = A^{\#} \in \mathfrak{A}$ with $A\,|_{\mathscr{H}_1} = 0$ the representation

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ A_{10} & 0 & A_{12} & A_{13} \\ A_{20} & 0 & A_{22} & 0 \\ A_{30} & 0 & 0 & A_{33} \end{pmatrix}$$

holds. Since $\widetilde{\mathscr{H}}^{[\perp]} = \mathscr{H}_1 \oplus \mathscr{H}_3$, we have

$$A|_{\widetilde{\mathscr{H}}^{[\perp]}} = \begin{pmatrix} 0 & A_{13} \\ 0 & A_{33} \end{pmatrix},$$

but $\sigma(A|_{\widetilde{\mathscr{H}}^{[\perp]}})$ is a singleton, so $\sigma(A_{33}) = \{0\}$. On the other hand due to the representation (see Lemma 5 and Lemma 11)

$$\mathscr{H}_3 = \left(\widetilde{\mathscr{L}_+} \cap \mathscr{H}_3\right) \oplus \left(\mathscr{L}_- \cap \mathscr{H}_3\right)$$

the operator A_{33} can be consider as a usual self-adjoint operator. So, $A_{33} = 0$ and

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ A_{10} & 0 & A_{12} & A_{13} \\ A_{20} & 0 & A_{22} & 0 \\ A_{30} & 0 & 0 & 0 \end{pmatrix}. \tag{2.22}$$

Finally, applying to A Theorem 26 we have

LEMMA 29. *Let the J-self-adjoint operators of a family $\mathfrak{Y} \in D_1^+$ have real spectra, let an e.sp.f. E_λ of $\mathfrak{Y}$ be unbounded and let $\mathscr{J}$-$\widetilde{L}^2_{\vec{\sigma}}(\mathscr{E})$ be its basic model space. Then for every $A = A^{\#} \in \mathfrak{A}$ with $A\,|_{\mathscr{H}_1} = 0$ and for C and Q from* (2.19) *Decomposition* (2.9) *yields the following representation*

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \widetilde{A}_{10} & 0 & A_{12} & 0 \\ A_{20} & 0 & A_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \widehat{A}_{10} & 0 & 0 & A_{13} \\ 0 & 0 & 0 & 0 \\ A_{30} & 0 & 0 & 0 \end{pmatrix},$$

where $\widetilde{A}_{10}\colon\ \mathbb{C} \mapsto \mathbb{C}, 1 \mapsto \zeta$, $\quad \zeta = \overline{\zeta}$ with arbitrary ζ if Condition (2.20) *holds and $\zeta = \int_{-1}^{1} \gamma_0(t)[\widetilde{g}(t), \widetilde{g}(t)]_{\mathscr{E}} d\sigma(t)$ in the opposite case.*

DEFINITION 30. In what follows the operators C and Q from (2.19) are called respectively scalar and nilpotent parts of an operator $A = A^\# \in \mathfrak{A}$ with $A\,|_{\mathscr{H}_1} = 0$. The subalgebra of all nilpotent parts of $\mathfrak{A}$ will be denote by $\mathfrak{A}^{(nil)}$.

REMARK 31. If Condition (2.20) holds, $\mathfrak{A}^{(nil)} \cap \mathfrak{A}^{(sc)} \neq \{0\}$ because the operator S_0 defined by (2.21) belongs to both subalgebras.

2.2. Some remarks on the nilpotent part of an algebra

Here we reproduce some constructions from [25].

Let $\mathfrak{A}$ be such that

$$\left.\begin{array}{l} \text{a) } \mathfrak{A} \text{ is a } WJ^*\text{-algebra (maybe non-commutative);} \\ \text{b) } \mathfrak{A} \in D_1^+; \\ \text{c) every } A \in \mathfrak{A} \text{ can be represented in the form } A = \alpha I + A_0, \\ \quad \text{where } A_0 \text{ is a nilpotent operator.} \end{array}\right\} \tag{2.23}$$

Let us recall that $\mathfrak{A}^{(nil)}$ means the subset of the algebra $\mathfrak{A}$, that contains all nilpotent operators and only them. In this subsection the codimension of $\mathfrak{A}^{(nil)}$ with respect to $\mathfrak{A}$ is equal one. Let us assume that our algebra is not trivial, i.e.

$$\mathfrak{A}^{(nil)} \neq \{0\}. \tag{2.24}$$

Let as before $\mathfrak{L}_+$ be a maximal non-negative invariant subspace of the algebra $\mathfrak{A}$, that is a direct sum of a uniformly positive subspace and a one-dimensional neutral subspace. Let $\mathfrak{L}_-$, $\mathfrak{L}_0$ and $\mathfrak{L}_1$ be as in (1.11). Then $A\mathfrak{L}_1 = \{0\}$ for every $A \in \mathfrak{A}^{(nil)}$ and, conversely, if $A \in \mathfrak{A}$ and $A\mathfrak{L}_1 = \{0\}$, then $A \in \mathfrak{A}^{(nil)}$. Let $e_1 \in \mathfrak{L}_1$ be a fixed vector with unit norm. Put

$$e_0 = Je_1,\ \mathscr{Q} = \mathfrak{L}_1^{[\perp]} \cap \mathfrak{L}_1^{\perp} = (\mathfrak{L}_0 \oplus \mathfrak{L}_1)^{[\perp]}. \tag{2.25}$$

Note that this definition does not contradict Formulae (2.11) because in Subsection 2.1 $\mathfrak{H}_1 = \mathfrak{L}_1$. Since $\mathrm{Lin}\{e_0, e_1\}$ is invariant with respect to J, the equality $J\mathscr{Q} = \mathscr{Q}$ holds.

Consider a structure of an arbitrary operator $A \in \mathfrak{A}^{(nil)}$. First, we have $A\mathfrak{L}_+ \subset \mathfrak{L}_1$ and $A\mathfrak{L}_0 \subset \mathfrak{L}_1^{[\perp]}$ (alongside with [25] see the reasoning related to Representation (2.22)). So for the operator A there are vectors $a, a^\# \in \mathscr{Q}$ and a number α, such that

$$Ae_0 = a + \alpha e_1;\ Ax = [x, a^\#]e_1, \text{ where } x \in \mathscr{Q};\ Ae_1 = 0. \tag{2.26}$$

Representation (2.26) implies that $\mathfrak{A}^{(nil)}$ is a subalgebra of $\mathfrak{A}$. Next, the direct calculations show that

$$A^\# e_0 = a^\# + \overline{\alpha} e_1;\ A^\# x = [x, a]e_1, \text{ where } x \in \mathscr{Q};\ A^\# e_1 = 0. \tag{2.27}$$

So, if $A = A^\#$, then $a = a^\#$ and $\alpha \in \mathbb{R}$.

Note that a choice of the subspaces $\mathfrak{L}_0$ and $\mathscr{Q}$ was based on a choice of the canonical symmetry J and therefore we can simplify (if necessary) the operator structure of $\mathfrak{A}^{(nil)}$ altering J.

Let S_0 be the same operator as in (2.21).

PROPOSITION 32. *If there is at least one definite vector a: $a = Ae_0$, where $A =\in \mathfrak{A}^{(nil)}$, then $S_0 \in \mathfrak{A}$.*

Let $\mathfrak{A}_{\mathscr{Q}}$ be a set of all operators $A_0 \in \mathfrak{A}^{(nil)}$, such that

$$Ae_0 \in \mathscr{Q}. \tag{2.28}$$

If $S_0 \in \mathfrak{A}$, then $\mathfrak{A}_{\mathscr{Q}}$ has linear co-dimension with respect to $\mathfrak{A}^{(nil)}$ equal one.

PROPOSITION 33. *If $S_0 \notin \mathfrak{A}^{(nil)}$, then there is a choice of a fundamental symmetry J such that $\mathfrak{A}^{(nil)}e_0 \subset \mathscr{Q}$.*

Let $a \in \mathscr{Q}$ be a vector, such that there exists an operator $A = A^{\#} \in \mathfrak{A}^{(nil)}$ related with a through Representations (2.26) and (2.27). The set of all a under this condition is said to be the *shadow* of e_0 (with respect to $\mathfrak{A}$) and is denoted by $sh_{\mathfrak{A}}(e_0)$, i.e.

$$sh_{\mathfrak{A}}(e_0) = \{x\colon\ x = Ae_0 - [Ae_0, e_0]e_1,\, A = A^{\#} \in \mathfrak{A}^{(nil)}\}. \tag{2.29}$$

Note that $sh_{\mathfrak{A}}(e_0)$ is a closed subset and for all vectors $a, b \in sh_{\mathfrak{A}}(e_0)$ and for all numbers $\alpha, \beta \in \mathbb{R}$ the relationship $\alpha a + \beta b \in sh_{\mathfrak{A}}(e_0)$ holds.

Recall that $\mathscr{Q}$ is a complex Hilbert space. Let $\mathscr{E}$ be its certain subset that is a closed real linear space, i.e. if $x, y \in \mathscr{E}$, $\alpha, \beta \in \mathbb{R}$, then $\alpha x + \beta y \in \mathscr{E}$ and, if $lim_{j\to\infty} x_j = x$, $x_j \in \mathscr{E}$, then $x \in \mathscr{E}$. In what follows a subset under this condition is said to be *real subspace* (with respect to $\mathscr{Q}$).

Let us note, that for $x, y \in \mathscr{E}$ the inequality $(x, y) \neq (y, x)$ is possible, i.e. a Hilbert structure, defined on $\mathscr{Q}$, may not induce on $\mathscr{E}$ a structure of a real Hilbert space. Indeed, one can define on $\mathscr{E}$ a structure of Euclidean space with the topology equal to the norm topology, generated on $\mathscr{E}$ by the topology of $\mathscr{Q}$, but, generally speaking, in this case a new scalar product would be defined on $\mathscr{E}$.

If $\mathscr{E}$ is a real subspace, then the subset $i\mathscr{E} = \{ix\}_{x\in\mathscr{E}}$ is a real subspace too. In general $i\mathscr{E} \neq \mathscr{E}$.

DEFINITION 34. A real subspace $\mathscr{E}$ is said to be *purely real*, if $\mathscr{Q} \cap i\mathscr{Q} = \{0\}$.

DEFINITION 35. Let $\mathscr{E}$ be a real subspace with respect to J-space $\mathscr{Q}$. Let us denote as $\mathscr{E}^{[b]}$ a real subspace, that is formed by all vectors $y \in \mathscr{Q}$ such that $[x, y] \in \mathbb{R}$ for every $x \in \mathscr{E}$. Then $\mathscr{E}^{[b]}$ is said to be the J-dual subspace to $\mathscr{E}$.

LEMMA 36. *Under Conditions* (2.23) *the algebra $\mathfrak{A}$ is commutative if and only if $sh_{\mathfrak{A}}(e_0) \subset \big(sh_{\mathfrak{A}}(e_0)\big)^{[b]}$.*

2.3. Commutative algebras of general type

Everywhere in this subsection the symbol $\mathfrak{A}$ means an arbitrary commutative WJ^*-algebra of D_1^+-class, i.e. Conditions (2.23a,c) for $\mathfrak{A}$ are not assumed.

In this context let us exclude some trivial cases. If $\mathfrak{A} \in D_0^+$, then $\mathfrak{A}$ can be consider as W^*-algebra, therefore $\mathfrak{A}'' = \mathfrak{A}$. Thus, this case doesn't need any special consideration. Next, if at least for one operator $A = A^\# \in \mathfrak{A}$ the condition $\sigma(A) \subset \mathbb{R}$ is not fulfilled, then $\mathscr{H} = \mathscr{H}_i[+]\mathscr{H}_r$, where $\mathscr{H}_i$ and $\mathscr{H}_r$ are invariant subspaces of the algebra $\mathfrak{A}$, the subspace $\mathscr{H}_i$ is two dimensional, and the algebra $\mathfrak{A}|_{\mathscr{H}_r}$ is an algebra of D_0^+-class, so $\mathfrak{A}'' = \mathfrak{A}$. Thus, we need to analyze only the algebras $\mathfrak{A}$, such that $\mathfrak{A} \notin D_0^+$ and for every $A = A^\# \in \mathfrak{A}$ the relation $\sigma(A) \subset \mathbb{R}$ fulfilled. Below in this subsection we assume that these conditions are fulfilled without any additional remarks. Let us assume also that E_λ (i.s.f. of $\mathfrak{A}$) has a unique spectral singularity in zero. In that follows we will maintain Notations (1.14), (2.3) and (2.25). Since the vectors e_1 and e_0 will play some important role, we need to consider an ambiguousness in the choice of them. Since the subspace $\mathfrak{L}_1$ is one-dimensional, the vectors e_1 and e_0 are defined up to a scalar multiple with the absolute value equal to one, but the subspace $\mathfrak{L}_1$ is not, generally speaking, uniquely determined. The following example was given in [25] for a different case.

EXAMPLE 37. Assume that the space $\mathscr{H}$ is formed by an orthonormalized basis $\{e_j\}_1^4$, the fundamental symmetry J is given by the equalities $Je_0 = e_1, Je_1 = e_0, Je_2 = e_3, Je_3 = e_2$, and a WJ^*-algebra $\mathfrak{A}$ is generated by the identical operator and the following operators

$$A_1\colon A_1e_0 = e_2, A_1e_1 = 0, A_1e_2 = 0, A_1e_3 = e_1;$$

$$A_2\colon A_2e_0 = ie_2, A_2e_1 = 0, A_2e_2 = 0, A_2e_3 = -ie_1;$$

The operators A_1 and A_2 are J-s.a., $A_1^2 = A_2^2 = A_1A_2 = 0$. As a first non-negative invariant subspace for this family we can take $\mathrm{Lin}\{e_1, e_2 + e_3\}$ and as a second one it can be used $\mathrm{Lin}\{e_0 + e_1, e_2\}$.

Let us pass to cases of uniqueness for $\mathfrak{L}_1$. The operator S_0 is defined by (2.21).

PROPOSITION 38. *If e.s.f. E_λ of $\mathfrak{A}$ is unbounded, then $\mathfrak{L}_1 = \mathscr{H}_1$.*

This propositions follows from Lemmas 5 and 11. Let us note that $\mathscr{H}_1$ depends directly of $\mathfrak{A}$ and doesn't depend of the choice of E_λ (see [29] for details).

PROPOSITION 39. *If $S_0 \in \mathfrak{A}$, then the subspace $\mathfrak{L}_1$ is uniquely defined.*

Proof. Let $\widehat{\mathfrak{L}}_+$ be some another non-negative pseudo-regular subspace with one-dimensional isotropic part invariant with respect to $\mathfrak{A}$ and $\widehat{\mathfrak{L}}_1 = \widehat{\mathfrak{L}}_+ \cap \widehat{\mathfrak{L}}_- \neq \mathfrak{L}_1$. Under the hypothesis $S_0\widehat{\mathfrak{L}}_+ \subset \widehat{\mathfrak{L}}_+$ and $S_0\widehat{\mathfrak{L}}_+^{[\perp]} \subset \widehat{\mathfrak{L}}_+^{[\perp]}$. Since simultaneously $S_0\mathscr{H} = \mathfrak{L}_1$ and $\mathfrak{L}_1 \cap \widehat{\mathfrak{L}}_+ = \mathfrak{L}_1 \cap \widehat{\mathfrak{L}}_+^{[\perp]} = \{0\}$, we have $S_0\widehat{\mathfrak{L}}_+ = S_0\widehat{\mathfrak{L}}_+^{[\perp]} = \{0\}$. The latter brings $S_0\mathscr{H} \subset (\mathrm{CLin}\{\mathfrak{L}_+^{[\perp]}, \widehat{\mathfrak{L}}_+\})^{[\perp]} = \widehat{\mathfrak{L}}_+ \cap \widehat{\mathfrak{L}}_+^{[\perp]}$. It is a contradiction. $\square$

Here by the symbol $\mathfrak{A}^{(nil)}$ we denote the set of all nilpotent operators $A \in \mathfrak{A}$. Note that

$$\text{if } A \in \mathfrak{A}^{(nil)} \text{ then } A|_{\widetilde{\mathscr{H}}} = 0. \tag{2.30}$$

PROPOSITION 40. *If* $\mathfrak{A}^{(nil)} \neq \{0\}$, *then* $\mathfrak{L}_1 \subset \bigcap_{0\neq A=A^\# \in \mathfrak{A}^{(nil)}} A\mathscr{H}$.

Instead of a proof we can note that operators from $\mathfrak{A}^{(nil)}$ have Representation (2.26), where (e.g. with Definition 30)

$$\left.\begin{array}{l}\text{a) if } \mathscr{H}_1 \neq \{0\}, \text{then } \mathscr{Q} = \widetilde{\mathscr{H}}^{[\perp]} \cap \mathscr{H}_1^{\perp}; \\ \text{b) if } \mathscr{H}_1 = \{0\}, \text{then } \mathscr{Q} = \widetilde{\mathscr{H}}^{[\perp]} \cap (\mathfrak{L}_1 \oplus \mathfrak{L}_0)^{\perp}.\end{array}\right\} \tag{2.31}$$

REMARK 41. Example 37 represents in some sense an exceptional case. Indeed, the subspace $\mathscr{L}_1$ must belong to the range of every operator $A = A^\# \in \mathfrak{A}^{(nil)}$, $A \neq 0$, from the other side this range is one- or two-dimensional and if $S_0 \notin \mathfrak{A}^{(nil)}$ there are no more then two J-s.a. linear independent operators with the same range. Thus, a non-uniqueness of $\mathfrak{L}_1$ can occur only if the sub-algebra $\mathfrak{A}^{(nil)}$ is spanned by one or two J-s.a. operators with the same range.

Let us re-define $\mathfrak{A}^{(sc)}$ by a manner which is not connected directly with a choice of e.sp.f. of $\mathfrak{A}$. So, $\mathfrak{A}^{(sc)}$ is the weak closure of all scalar spectral operators from $\mathfrak{A}$ annulated on $\mathscr{L}_1$. Remark 41 shows that the re-definition is correct. See also [29].

Let us go to a simplification option.

LEMMA 42. *Let e.s.f. of the algebra* $\mathfrak{A}$ *be unbounded and let its basic model space* $\mathscr{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E})$ *compatible with the given scalar product and spanned by a standard* J*-space* $\mathscr{J} - L^2_{\vec{\sigma}}(\mathfrak{E})$ *and an improper function* $\widetilde{g}(t)$ *be such that*

$$[\widetilde{g}(t), \widetilde{g}(t)] \in L^1_\sigma + L^2_\eta. \tag{2.32}$$

Then one can define on $\mathscr{H}$ *a new canonical scalar product with a new canonical symmetry* J', *such that for* $\mathfrak{A}$ *there is a new basic model space* $\mathscr{J}\text{-}\check{L}^2_{\vec{\sigma}}(\mathfrak{E})$ *spanned by the same space* $\mathscr{J}\text{-}L^2_{\vec{\sigma}}(\mathfrak{E})$ *and an unbounded element (compatible with the new scalar product)* $\check{g}(t)$ *with the properties*

$$\left.\begin{array}{l}a)\ \text{Hilbert structure on the subspace } \widetilde{\mathscr{H}}^{[\perp]} \text{ is the same;} \\ b)\ Ae_1 = AJ'e_1 \text{ for every } A \in \mathfrak{A}^{(nil)}; \\ c)\ [\check{g}(t), \check{g}(t)]_{\mathfrak{E}} \equiv 0.\end{array}\right\} \tag{2.33}$$

Proof. First, let us note that Relation (2.32) is equivalent the following property (see Remark 31)

$$\mathfrak{A}^{(nil)} \cap \mathfrak{A}^{(sc)} = \{0\} \tag{2.34}$$

and therefore isn't connected with any choice of a basic model space. Moreover, Property (2.33) also can be reformulated in a form independent of the choice of a basic model space. For instance, (2.33) is equivalent to the following condition

$$[E(\Delta)e_0, e_0] = 0 \text{ for every segment } \Delta \subset \mathfrak{R}_0^{(0)}. \tag{2.35}$$

Fulfillment of Condition (2.35) depends (in difference with Condition (2.34)) of the choice of a canonical scalar product. Indeed, $e_0 = Je_1$.

The above reasoning and Proposition 6.24 from [28] show that if a transition from a space $\mathscr{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E})$ to a space $\mathscr{J}\text{-}\check{L}^2_{\vec{\sigma}}(\mathfrak{E})$ with Property (2.33) is realized for some basic standard J-space $\mathscr{J}\text{-}L^2_{\vec{\sigma}}(\mathfrak{E})$, then there is a transition for every basic model space of J-orth.sp.f. E_λ. Taking into account this remark we choose a standard J-space $\mathscr{J}\text{-}L^2_{\vec{\sigma}}(\mathfrak{E})$ by a special way.

Let semi-definite subspaces $\widetilde{\mathscr{H}_+}$ and $\widetilde{\mathscr{H}_-}$ are the same, that in (2.5). At least one of these subspaces contains the isotropic part that must be the same as $\mathscr{H}_1$. Let, for instance, $\mathscr{H}_1 \subset \widetilde{\mathscr{H}_+}$. Then (e.g.(2.8)) $\mathscr{H}_2^+ = \widetilde{\mathscr{H}_+} \cap \mathscr{H}_1^\perp$. With no loss of generality we can assume that the canonic scalar on $\mathscr{H}_2^+$ product is equal $[\cdot,\cdot]$. Next, the spectral function $E_\lambda|_{\widetilde{\mathscr{H}_+}}$ can be consider as a restriction of J-orth.sp.f defined in a Pontryagin space Π_1 (in our case it is the space $\widetilde{\mathscr{H}_+} \oplus \mathscr{H}_0$), so we can apply to it Proposition 5.3 from [28]. Due to this proposition the non-peculiar multiplicity of $E_\lambda|_{\widetilde{\mathscr{H}_+}}$ (Definition 14) is equal one or there is a decomposition $\mathscr{H}_2^+ = \mathscr{H}_{2,1}^+ \oplus \mathscr{H}_{2,2}^+$, where the subspace $\mathscr{H}_{2,2}^+$ is invariant with respect to E_λ and the non-critical multiplicity of the spectral function $E_\lambda|_{\mathscr{H}_{2,1}^+ \oplus \mathscr{H}_1}$ is equal one (if the first case takes place, we put for a simplicity in formulae below $\mathscr{H}_{2,1}^+ = \mathscr{H}_2^+$).

Let us pass to $E_\lambda|_{\widetilde{\mathscr{H}_-}}$. First, $\mathscr{H}_1 \subset \widetilde{\mathscr{H}_-}$. Indeed, Remark 18 jointly with Condition (2.32) show that if $\int_\Delta \|\widetilde{g}_+(t)\|^2_{\mathscr{E}_+} d\sigma_+(t) = \infty$ then $\int_\Delta \|\widetilde{g}_-(t)\|^2_{\mathscr{E}_-} d\sigma_-(t) = \infty$ too, but in terms of Theorem 17 and Remark 18 the equality $\int_\Delta \|\widetilde{g}_+(t)\|^2_{\mathscr{E}_+} d\sigma_+(t) = \infty$ yields $\mathscr{H}_1 \subset \widetilde{\mathscr{H}_+}$ and the equality $\int_\Delta \|\widetilde{g}_-(t)\|^2_{\mathscr{E}_-} d\sigma_-(t) = \infty$ yields $\mathscr{H}_1 \subset \widetilde{\mathscr{H}_-}$. Next, a reasoning, similar to the reasoning above, shows that or the non-critical multiplicity of $E_\lambda|_{\widetilde{\mathscr{H}_-}}$ or there is a representation $\mathscr{H}_2^- = \mathscr{H}_{2,1}^- \oplus \mathscr{H}_{2,2}^-$, where $\mathscr{H}_{2,2}^-$ is an invariant subspace for E_λ and the non-peculiar multiplicity of the spectral function $E_\lambda|_{\mathscr{H}_{2,1}^- \oplus \mathscr{H}_1}$ is equal one (in the first case we put $\mathscr{H}_{2,1}^- = \mathscr{H}_2^-$).

Let us pass to a construction of $\mathscr{J} - L^2_{\vec{\sigma}}(\mathfrak{E})$. Let $\sigma(t)$ be a non-decreasing function defined on $[-1;1]$ and satisfying the conditions of Remark 13 (i.e. $\sigma(t) = \sigma_{u_\mu,v_\mu}(t)$). Let $L^2_{\vec{\sigma}_+}(\mathfrak{E}_+)$ be a model space for $P_2^+ E_\lambda|_{\mathscr{H}_2^+}$, where P_2^+ is the orthogonal projection on $\mathscr{H}_2^+$, where the scalar function $\sigma_+(t)$ has the form $\sigma_+(t) = \int_{-1}^t \rho_+(\lambda) d\sigma(\lambda)$ with $\rho_+(\lambda) = \rho_+^2(\lambda)$. We denote by W_2^+ the corresponding operator of similarity assuming that for $x \in \mathscr{H}_{2,1}^+$ the representation $(W_2^+)^{-1}x = \alpha(t) \cdot d_+$, takes place, where d_+ is a fixed basis vector from $\mathfrak{E}_+$ and $\alpha(t)$ is some function. By the analogous way for the spectral function $P_2^- E_\lambda|_{\mathscr{H}_2^-}$ we introduce a model space $L^2_{\vec{\sigma}_-}(\mathfrak{E}_-)$ together with an operator of similarity W_2^- satisfying the condition $(W_2^-)^{-1}x = \alpha(t)d_-$ for $x \in \mathscr{H}_{2,1}^-$. As a next step we put $\mathscr{J} - L^2_{\vec{\sigma}}(\mathfrak{E}) = L^2_{\vec{\sigma}_+}(\mathfrak{E}_+) \oplus L^2_{\vec{\sigma}_-}(\mathfrak{E}_-)$, $W_2 = W_2^+ \oplus W_2^-$, $\mathscr{J} f(t) = f_+(t) - f_-(t)$, $f_+(t) \in L^2_{\vec{\sigma}_+}(\mathfrak{E}_+)$, $f_-(t) \in L^2_{\vec{\sigma}_-}(\mathfrak{E}_-)$ and define as a basic model space for E_λ, compatible with the decomposition $\widetilde{\mathscr{H}} = \mathscr{H}_1 \oplus \mathscr{H}_2^+ \oplus \mathscr{H}_2^-$ the space $\mathscr{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E})$ that is the linear span of $\mathscr{J}\text{-}L^2_{\vec{\sigma}}(\mathfrak{E})$ and the corresponding improper function $\widetilde{g}(t)$. It is clear that $\widetilde{g}(t)$ has the form $\widetilde{g}(t) =$

$\beta_+(t)d_+ + \beta_-(t)d_-$. With no loss of generality we can assume that $\beta_+(t) \geqslant 0$, $\beta_-(t) \geqslant 0$, because in the opposite case one can substitute the operators W_2^+ and W_2^- by the operators $e^{i\cdot arg\beta^+(t)} \cdot W_2^+$ and $e^{i\cdot arg\beta^-(t)} \cdot W_2^-$ respectively. The above assumptions give $G(t) = 1 + \beta_+^2(t) + \beta_-^2(t)$, $[\widetilde{g}(t), \widetilde{g}(t)] = \beta_+^2(t) - \beta_-^2(t)$. Thus, by virtue of (2.32) we have

$$\beta_+^2(t) - \beta_-^2(t) \in L_\sigma^1 + L_\eta^2. \tag{2.36}$$

Since, evidently, $|\beta_+^2(t) - \beta_-^2(t)| < G(t)$, then by Remark 4 Condition 2.36 is equivalent to the condition $\beta_+^2(t) - \beta_-^2(t) \in L_\eta^2$, i.e.

$$\int_{-1}^{1} \frac{(\beta_+^2(t) - \beta_-^2(t))^2}{1 + \beta_+^2(t) + \beta_-^2(t)} d\sigma(t) < \infty.$$

The latter gives

$$\int_{-1}^{1} (\beta_+(t) - \beta_-(t))^2 d\sigma(t) < \infty. \tag{2.37}$$

Next, $d\vec{\sigma}_+(t) = d_+ \cdot \rho_+(t) d\sigma(t) + \ldots$; $d\vec{\sigma}_-(t) = d_- \cdot \rho_-(t) d\sigma(t)$ and without loss of generality (see the proof of Proposition 6.3. from [28]) we can assume that

$$\rho_+(t) = \begin{cases} 1, & \beta_+(t) \neq 0, \\ 0, & \beta_+(t) = 0, \end{cases} \qquad \rho_-(t) = \begin{cases} 1, & \beta_-(t) \neq 0, \\ 0, & \beta_-(t) = 0. \end{cases}$$

Set

$$\delta_+(t) = \frac{(2 - \rho_-(t))\beta_+(t) - \rho_+(t)\beta_-(t)}{2},$$
$$\delta_-(t) = \frac{(2 - \rho_+(t))\beta_-(t) - \rho_-(t)\beta_+(t)}{2}.$$

It is easy to check that

$$\delta_+(t), \delta_-(t) \in L_\sigma^2(\mathbb{C}). \tag{2.38}$$

Indeed, $(1 - \rho_-(t))\beta_-(t) \equiv 0$, so by (2.37)

$$\int_{-1}^{1} (1 - \rho_-(t))^2 \beta_+^2(t) d\sigma(t) = \int_{-1}^{1} (1 - \rho_-(t))^2 (\beta_+(t) - \beta_-(t))^2 d\sigma(t) < \infty,$$

$\int_{-1}^{1} (\beta_+(t) - \rho_+(t)\beta_-(t))^2 d\sigma(t) = \int_{-1}^{1} \rho_+(t)(\beta_+(t) - \beta_-(t))^2 d\sigma(t) < \infty$ etc. Moreover, the equalities

$$\delta_+(t)\rho_+(t) = \delta_+(t),\ \delta_-(t)\rho_-(t) = \delta_-(t). \tag{2.39}$$

are true. Now, let $\check{g}(t) = \frac{\beta_+(t) + \beta_-(t)}{2} \rho_-(t)\rho_+(t)(d_+ + d_-)$. A direct verification shows that $\widetilde{g}(t) = h(t) + \check{g}(t)$, where $h(t) = \delta_+(t)d_+ + \delta_-(t)d_-$. By conditions (2.38) and (2.39) we have $h(t) \in \mathscr{J} - L_{\vec{\sigma}}^2(\mathfrak{E})$, so the spaces $\mathscr{J} - \widetilde{L}_{\vec{\sigma}}^2(\mathfrak{E})$ and $\mathscr{J} - \check{L}_{\vec{\sigma}}^2(\mathfrak{E})$, formed by joining to the same space $\mathscr{J} - L_{\vec{\sigma}}^2(\mathfrak{E})$ the unbounded elements $\widetilde{g}(t)$ and $\check{g}(t)$ respectively, coincide as linear manifolds in $M_{\vec{\sigma}}(\mathfrak{E})$. This fact and Theorem 5.28 from

[28] yield that not only the space $\mathscr{J}-\widetilde{L}^2_{\vec{\sigma}}(\mathfrak{E})$ but also the space $\mathscr{J}-\check{L}^2_{\vec{\sigma}}(\mathfrak{E})$ are basic model spaces for E_λ. Let us find directly the new operator of similarity $\check{W}$. Let $f(t)\in\mathscr{J}-L^2_{\vec{\sigma}}(\mathfrak{E})$, $x=Wf(t)\in\mathscr{H}_2$. Then by Definition 21

$$\widetilde{E}(\Delta)x=\widetilde{E}(\Delta)Wf(t)=W\Big(f(t)\cdot\chi_\Delta(t)+\widetilde{g}(t)\cdot\big\{\int_\Delta[f(t),h(t)+\check{g}(t)]_{\mathscr{E}}d\sigma(t)\big\}\Big)=$$

$$\Big(E_{22}(\Delta)x\oplus e_1\cdot\big\{\int_{-1}^{1}[f(t)\chi_\Delta(t),h(t)]_{\mathscr{E}}d\sigma(t)\big\}\Big)\oplus e_1\cdot\big\{\int_{-1}^{1}[f(t)\chi_\Delta(t),\check{g}(t)]_{\mathscr{E}}d\sigma(t)\big\}.$$

Thus, $\check{W}$ can be introduce by the following way:

$$\check{W}_2f(t)=\Big(x\oplus e_1\cdot\big\{\int_{-1}^{1}[f(t),h(t)]_{\mathscr{E}}d\sigma(t)\big\}\Big),\quad \check{W}\check{g}(t)=e_1.$$

If we redefine the operator of similarity by the above way, the subspace $\mathscr{H}_2$ must be replaced by the space $\check{\mathscr{H}}_2=\Big\{x\oplus e_1\cdot\big\{\int_{-1}^{1}[f(t),h(t)]_{\mathscr{E}}d\sigma(t)\big\}\Big\}_{x=Wf(t)\,f(t)\in\mathscr{J}-L^2_{\vec{\sigma}}(\mathfrak{E})}$ and the vector e_0 must be replaced, for instance, by the vector $\check{e}_0=e_0-W_2h(t)-\frac{1}{2}\int_{-1}^{1}[h(t),(t)]_{\mathscr{E}}d\sigma(t)\cdot e_1$. Note that by construction $[\check{g}(t),\check{g}(t)]=0$ and $Ae_0=A\check{e}_0$ for every $A\in\mathfrak{A}^{(nil)}$. $\square$

REMARK 43. Reasonings used during the proof of Lemma 42 show that the behavior of J-orth.sp.f. E_λ from D_1^+-class can be more or less completely analyzed on the base of spectral functions with non-peculiar multiplicity equal two. At the same time the case (2.32) cannot be modeled by the non-peculiar multiplicity equal one.

Summarizing the above results one can say that there are three types of commutative WJ^*-algebras of D_1^+-class.

THEOREM 44. *If a commutative WJ^*-algebra $\mathfrak{A}\in D_1^+$ is such that $\sigma(A)\not\subset\mathbb{R}$ at least for one operator $A=A^\#\in\mathfrak{A}$, then $\mathfrak{A}$ is similar to the algebra acting in a Krein space that is a J-orthogonal sum of two $\mathfrak{A}$-invariant subspaces. First of them is two-dimensional subspace, say $\mathbb{C}^2$, with $J|_{\mathbb{C}^2}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $\mathfrak{A}|_{\mathbb{C}^2}=\Big\{\begin{pmatrix}\alpha&0\\0&\beta\end{pmatrix}\Big\}_{\alpha,\beta\in\mathbb{C}}$. The second one is a standard Krein space $\mathscr{J}-L^2_{\vec{\sigma}}(\mathfrak{E})$ and $\mathfrak{A}|_{\mathscr{J}-L^2_{\vec{\sigma}}(\mathfrak{E})}$ is the algebra of multiplication operators by functions from L^∞_σ.*

THEOREM 45. *If a commutative WJ^*-algebra $\mathfrak{A}\in D_1^+$ is such that $\sigma(A)\subset\mathbb{R}$ for every operator $A=A^\#\in\mathfrak{A}$ and all operators of $\mathfrak{A}$ are spectral, then $\mathfrak{A}$ is similar to the algebra acting in a Krein space that is a J-orthogonal sum of two $\mathfrak{A}$-invariant subspaces $\mathscr{H}_r$ and $\widetilde{\mathscr{H}}$. First of them $\mathscr{H}_r$ is in turn the J-orthogonal sum of two subspaces: a two-dimensional subspace, say $\mathbb{C}^2$ and a Krein space $\mathscr{Q}$ with a fundamental symmetry $J_{\mathscr{Q}}$ and a real linear manifold $sh_{\mathfrak{A}}\subset\mathscr{Q}$ such that $sh_{\mathfrak{A}}\subset\big(sh_{\mathfrak{A}}\big)^{[b]}$. With respect to the sum $\mathbb{C}\oplus\mathbb{C}\oplus\mathscr{Q}$ the fundamental symmetry J has the form $J=$*

$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & J_{\mathscr{Q}} \end{pmatrix}$ *and the algebra* $\mathfrak{A}|_{\mathscr{H}_r}$ *is generated by the identical operator and operators* $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & [\cdot, a] \\ a & 0 & 0 \end{pmatrix}$, *where* $a \in sh_{\mathfrak{A}}$ *and, maybe, operator* $S_0 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & \\ 0 & 0 & 0 \end{pmatrix}$. *The second space* $\widetilde{\mathscr{H}}$ *is a standard Krein space* $\mathscr{J} - L^2_{\vec{\sigma}}(\mathfrak{E})$ *and* $\mathfrak{A}|_{\mathscr{J} - L^2_{\vec{\sigma}}(\mathfrak{E})}$ *is the algebra of multiplication operators by functions from* $L^\infty_{\vec{\sigma}}$.

THEOREM 46. *If a commutative* WJ^**-algebra* $\mathfrak{A} \in D_1^+$ *is such that* $\sigma(A) \subset \mathbb{R}$ *for every operator* $A = A^\# \in \mathfrak{A}$ *and at least one operator of* $\mathfrak{A}$ *is not spectral, then the subspace* $\mathscr{L}_1$ *from* (1.11) *is uniquely determined and* $\mathfrak{A}$ *is similar to an algebra acting in a Krein space that is a* J*-orthogonal sum of three* J*-invariant subspaces: a two-dimensional subspace, say* $\mathbb{C}^2$, *with* $J|_{\mathbb{C}^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *a Krein space* $\mathscr{Q}$ *with a fundamental symmetry* $J_{\mathscr{Q}}$ *and a real linear manifold* $sh_{\mathfrak{A}} \subset \mathscr{Q}$ *such that* $sh_{\mathfrak{A}} \subset \left(sh_{\mathfrak{A}}\right)^{[b]}$, *and a standard Krein space* $\mathscr{J} - L^2_{\vec{\sigma}}(\mathfrak{E})$, *moreover within this model the subspace* $\mathscr{L}_1 = \mathscr{H}_1$ *corresponds to the subspace* $\{0\} \times \mathbb{C} \subset \mathbb{C}^2$. *The algebra* $\mathfrak{A}$ *contains the identical operator,* $\mathfrak{A}^{(nil)}$ *and* $\mathfrak{A}^{(sc)}$. *The subspace* $\mathbb{C}^2 \oplus \mathscr{Q}$ *is invariant to the subalgebra corresponding to* $\mathfrak{A}^{(nil)}$ *and is organized on this subspace as in subsection* 2.2, *the same subalgebra is annulated on* $\mathscr{J} - L^2_{\vec{\sigma}}(\mathfrak{E})$. *The subspace* $\mathbb{C}^2 \oplus \mathscr{J} - L^2_{\vec{\sigma}}(\mathfrak{E})$ *is invariant to the subalgebra corresponding to* $\mathfrak{A}^{(sc)}$ *and is organized on this subspace as in Theorem* 19 *including the creation of an improper function* $\widetilde{g}(t)$, *the same subalgebra is annulated on the subspace* $\mathscr{Q}$.

3. On a bicommutant structure

3.1. Some commutant properties

Everywhere in this subsection the symbol $\mathfrak{A}$ means an arbitrary commutative WJ^*-algebra of D_1^+-class. Here we start to analyze only the algebras $\mathfrak{A}$, such that $\mathfrak{A} \notin D_0^+$ and for every $A = A^\# \in \mathfrak{A}$ the relation $\sigma(A) \subset \mathbb{R}$ fulfilled. Below in the subsection we assume that these conditions are fulfilled without any additional remarks. Let us recall also that E_λ (e.s.f. of $\mathfrak{A}$) has a unique spectral singularity in zero. In that follows we will maintain Notations (2.3). We assume that $\mathscr{H}_3 \perp \left(\widetilde{H} \oplus \mathscr{H}_0\right)$. In this case $\mathscr{H}_3$ is an invariant subspace for J and with respect to Decomposition (2.9) Representations (2.10) and (2.12) hold. Now let $A = A^\# \in \mathfrak{A}$ and $A\,|_{\mathscr{H}_1} = 0$. Then A has the representation (2.22). Now let us go to a description for the commutant $\mathfrak{A}'$ of the algebra $\mathfrak{A}$. Let $B = B^\# \in \mathfrak{A}'$. Since $E(X) \in \mathfrak{A}$, $E(X)$ commutes with B, so the subspace $E(X)\mathscr{H}$ is invariant with respect to B. Thus, the subspaces $\widetilde{\mathscr{H}}$, $\widetilde{\mathscr{H}^{[\perp]}}$ and $\mathscr{H}_1$ are also invariant with respect to B. If $B\,|_{\mathscr{H}_1} = 0$, then B has the representation similar to (2.22):

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ B_{10} & 0 & B_{12} & B_{13} \\ B_{20} & 0 & B_{22} & 0 \\ B_{30} & 0 & 0 & B_{33} \end{pmatrix}. \tag{3.1}$$

REMARK 47. Due to Theorem 26 one can study the structure of $B \in \mathfrak{A}'$ using only scalar and nilpotent parts of operators from $\mathfrak{A}$. If $B\,|_{\mathscr{H}_1} = 0$, a simple calculation shows that commutation relations for B in the case of scalar parts involve only the blocs B_{12}, B_{22} and B_{22} of (3.1) and the commutation relations in the case of nilpotent parts involve the blocs B_{13}, B_{30} and B_{33}. Thus, there are no conditions for the bloc B_{10} and bloc-operators

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ B_{10} & 0 & B_{12} & 0 \\ B_{20} & 0 & B_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{13} \\ 0 & 0 & 0 & 0 \\ B_{30} & 0 & 0 & B_{33} \end{pmatrix}$$

also belong to $\mathfrak{A}'$ and can be studied separately.

This Remark and Lemma 2.19 from [25] yield the following result.

LEMMA 48. *Let $B = B^{\#}$ and $Be_1 = 0$. Put $b = Be_0 - [Be_0, e_0]e_1$. Then $B \in \mathfrak{A}'$ if and only if the following conditions*

$$\left.\begin{array}{l} a)\ \text{there is a } J\text{-self-adjoint operator } B_{\mathscr{Q}}: \mathscr{Q} \to \mathscr{Q}, \text{ such that} \\ \quad Bx = [x,b]e_1 + B_{\mathscr{Q}}x \text{ for all } x \in \mathscr{Q}; \\ b)\ b \in (sh_{\mathfrak{A}}(e_0))^{[b]}; \\ c)\ B_{\mathscr{Q}}(sh_{\mathfrak{A}}(e_0)) = \{0\},\ B_{\mathscr{Q}}\mathscr{Q} \subset sh_{\mathfrak{A}}(e_0)^{[\perp]}; \end{array}\right\} \tag{3.2}$$

hold.

3.2. Function representation of the bicommutant

Within this subsection WJ^*-algebra $\mathfrak{A} \in D_1^+$ is commutative, E_λ is its e.s.f. with a peculiar point in zero and there are no more restrictions on $\mathfrak{A}$.

LEMMA 49. *In Properties* (1.17) *one can replace $\mathfrak{A}$ by $\mathfrak{A}''$.*

Proof. Since $\mathfrak{A} \subset \mathfrak{A}''$, we need only to check that the corresponding modification of Properties (1.17c,e) is valid for E_λ and $\mathfrak{A}''$. By virtue of (1.17a) $(\mathfrak{A}|_{E(\Delta)\mathscr{H}})'' = (\mathfrak{A})''|_{E(\Delta)\mathscr{H}}$ for every interval $\Delta \in \mathfrak{R}_0^{(0)}$, so the modified Property (1.17c) follows from the corresponding theorem of von Neuman ([1]).

Now let us pass to Property (1.17e). The condition of the type (1.17e) can be transformed to the following condition:

$$\forall B \in \mathfrak{A}'' \text{ the representation } B = \beta I + B_0 \text{ holds}, \tag{3.3}$$

where β is a scalar (depended of B) and $B_0|_{\widetilde{\mathscr{H}}^{[\perp]}}$ is a nilpotent operator.

Let $\mathfrak{A}_0$ be the collection of the operators $A\in\mathfrak{A}$, such that $\sigma(A|_{\widetilde{\mathscr{H}}^{[\perp]}})=\{0\}$. Note that by Theorem 26 the codimension of $\mathfrak{A}_0$ in $\mathfrak{A}$ is equal one. Take $\operatorname{Ker}\mathfrak{A}_0=\bigcap_{A\in\mathfrak{A}_0}\operatorname{Ker}A\supset\mathscr{H}_1$. It is clear that $\operatorname{Ker}\mathfrak{A}_0$ is an invariant subspace for $\mathfrak{A}''$.

In order to prove (3.3) let us show, first, that for every $B\in\mathfrak{A}''$ there is $\beta\in\mathbb{C}$, such that

$$B|_{\operatorname{Ker}\mathfrak{A}_0}=\beta I|_{\operatorname{Ker}\mathfrak{A}_0} \tag{3.4}$$

Let us assume the contrary. Then there is a vector $x\in\operatorname{Ker}\mathfrak{A}_0$, such that the vectors $y:=Bx$ and x are linearly independent. Let us set $Z\colon\mathscr{H}\mapsto\mathscr{H}$, $Zu=[u,x]\cdot x$. Since $\mathscr{H}$, there is a vector $v\in\mathscr{H}$, such that $[v,x]=1$, $Z\neq 0$. On the other hand $A\mathscr{H}[\perp]\operatorname{Ker}\mathfrak{A}_0$ if $A\in\mathfrak{A}_0$. Thus, for every $A\in\mathfrak{A}_0$ the equalities $ZAu=[Au,x]\cdot x=0$ and $AZu=[u,x]Ax=0$ hold, $Z\in\mathfrak{A}'$ but, from the other hand, $y=BZv\neq ZBv=[Bv,x]\cdot x$. It is the contradiction.

Thus, (3.4) is proved and with no loss of generality we can assume

$$B|_{\operatorname{Ker}\mathfrak{A}_0}=0. \tag{3.5}$$

Next, $\operatorname{Ker}(\mathfrak{A}_0\times\mathfrak{A}_0)$ is a B-invariant subspace and Equality (3.5) yields $B\operatorname{Ker}(\mathfrak{A}_0\times\mathfrak{A}_0)\subset\operatorname{Ker}\mathfrak{A}_0$. The analogous reasoning gives $B\operatorname{Ker}(\mathfrak{A}_0\times\mathfrak{A}_0\times\mathfrak{A}_0)\subset\operatorname{Ker}(\mathfrak{A}_0\times\mathfrak{A}_0)$, but by Lemma 29 (the matrix representation for Q) $\operatorname{Ker}(\mathfrak{A}_0\times\mathfrak{A}_0\times\mathfrak{A}_0)=\widetilde{\mathscr{H}}^{[\perp]}$. □

COROLLARY 50. *Let $\mathfrak{A}\in D_1^+$ be a commutative WJ^*-algebra and let E_λ be its unbounded e.s.f. Then for every operator $B=B^\#\in\mathfrak{A}''$ the representation*

$$B=\beta I+Q+C \tag{3.6}$$

holds, where $Q=Q^\#\in\mathfrak{A}''$ is a nilpotent operator, $Q|_{\widetilde{\mathscr{H}}}=0$, $C\in\mathfrak{A}^{sc}$, $CE(\Delta)=\int_\Delta\gamma_0(\lambda)dE_\lambda$ for every interval $\Delta\in\mathfrak{R}_0^{(0)}$, $\beta=\overline{\beta}$, $\gamma_0(t)=\overline{\gamma_0(t)}$.

Proof. By Property (1.17c), Theorem 19 and 23 a basic model space $\mathscr{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathscr{E})$ for E_λ is simultaneously a basic model space for $\widetilde{\mathfrak{A}}''$. Since $\mathscr{H}_1$ is an invariant subspace for B, we need only to prove that $\beta=\overline{\beta}$ and $\gamma_0(t)\colon=(\gamma(t)-\beta)\in L^\infty_\sigma\cap L^2_\nu$. Let us take $\widetilde{g}(t)$ from $\mathscr{J}\text{-}\widetilde{L}^2_{\vec{\sigma}}(\mathscr{E})$ for E_λ. Thus $\int_{-1}^1\|\widetilde{g}(t)\|^2d\sigma(t)=\infty$. We assume that the scalar product on $\mathscr{H}$ is compatible with Representations (2.7) end (2.9). Then the operator J_2 from (2.10) commutes with the projection $E_{22}(\Delta)$ from (2.12), with the spectral function $P_2E_\lambda|_{\mathscr{H}_2}$ and with the operator $B_{22}\colon=P_2B|_{\mathscr{H}_2}$. Thanks to these relations of commutativity for the functions $h_\Delta(t)\colon=W_2^{-1}J_2W_2\chi_\Delta\cdot\widetilde{g}(t)$, where $\Delta\in\mathfrak{R}_0^{(0)}$, there is the function $\widetilde{h}(t)\in M_{\vec{\sigma}}(\mathscr{E})$, such that for every $\Delta\in\mathfrak{R}_0^{(0)}$: $h_\Delta(t)=\chi_\Delta\cdot\widetilde{h}(t)$ and $\int_\Delta\|\widetilde{g}(t)\|^2_{\mathscr{E}}d\sigma(t)=\int_\Delta\|\widetilde{h}(t)\|^2_{\mathscr{E}}d\sigma(t)=\int_\Delta[\widetilde{g}(t),\widetilde{h}(t)]_{\mathscr{E}}d\sigma(t)$. Taking into account Theorem 19 we have $E(\Delta)Wh_\Delta(t)=Wh_\Delta(t)\oplus(\int_\Delta[\widetilde{h}(t),\widetilde{g}(t)]_{\mathscr{E}}d\sigma(t))\cdot e_1=Wh_\Delta(t)\oplus(\int_\Delta\|\widetilde{g}(t)\|^2_{\mathscr{E}}d\sigma(t))\cdot e_1$. Let $x_\varepsilon=$

$$\Big(\int_{[-1;-\varepsilon]\cup[\varepsilon;1]}\|\widetilde{g}(t)\|^2_{\mathscr{E}}d\sigma(t)\Big)^{-1}\Big\{E([-1;-\varepsilon])Wh_{[-1;-\varepsilon]}(t)+E([\varepsilon;1])Wh_{[\varepsilon;1]}(t)\Big\},$$

where $\varepsilon \in (0;1)$. Then $\lim_{\varepsilon\to 0} x_\varepsilon = e_1$, so $\lim_{\varepsilon\to 0} Bx_\varepsilon = \beta \cdot e_1$. On the other hand $Bx_\varepsilon =$

$$\Big(\int_{[-1;-\varepsilon]\cup[\varepsilon;1]} \|\widetilde{g}(t)\|^2_{\mathscr{E}} d\sigma(t)\Big)^{-1} \Big\{W\big(\gamma(t)\cdot(h_{[-1;-\varepsilon]}(t)+h_{[\varepsilon;1]}(t))\big)\oplus$$

$$(\int_{[-1;-\varepsilon]\cup[\varepsilon;1]} \gamma(t)\cdot \|\widetilde{g}(t)\|^2_{\mathscr{E}} d\sigma(t))\cdot e_1\Big\}.$$

But $(\int_{[-1;-\varepsilon]\cup[\varepsilon;1]} \gamma(t)\cdot \|\widetilde{g}(t)\|^2_{\mathscr{E}} d\sigma(t)) \in \mathbb{R}$, so $\beta = \overline{\beta}$. $\square$

REMARK 51. Representation (3.6) shows that the bicommutant of a commutative WJ^*-algebra of D_1^+-class has a structure like a structure of the original algebra, but it does not mean that $\mathfrak{A}$ and $\mathfrak{A}''$ coincide or even $\mathfrak{A}'' \in D_\kappa^+$ for some $\kappa < \infty$. The corresponding example was given in [25] (Remark 2.24). At the same time Representation (3.6) means that scalar parts of $\mathfrak{A}$ and $\mathfrak{A}''$ coincide, so only $(\mathfrak{A}'')^{(nil)}$ can be larger than $\mathfrak{A}^{(nil)}$.

3.3. The bicommutant for J-symmetric nilpotent algebras

This Subsection is a continuation of Subsection 2.2, so the algebra $\mathfrak{A}$ is under Conditions (2.23).

PROPOSITION 52. ([25]) If $S_0 \notin \mathfrak{A}$, then the linear codimension of $(\mathfrak{A}')_0$ with respect to $\mathfrak{A}'$ is equal two, and if $S_0 \in \mathfrak{A}$, then the same codimension is equal one.

Let us only mentione a non-identical operator that is not in $(\mathfrak{A}')_0$ in the case $S_0 \notin \mathfrak{A}$. By Proposition 32 the real linear subspace $sh_{\mathfrak{A}}(e_0)$ is neutral, so its complexification $csh_{\mathfrak{A}}(e_0) := \mathrm{CLin}\{sh_{\mathfrak{A}}(e_0), ish_{\mathfrak{A}}(e_0)\}$ is neutral too. Next, since $csh_{\mathfrak{A}}(e_0)$ is neutral, we have $csh_{\mathfrak{A}}(e_0) \subset \mathrm{Ker}\,\mathfrak{A}^{(nil)}$, therefore the subspaces $\mathfrak{L}_0 \oplus csh_{\mathfrak{A}}(e_0)$ and $J(csh_{\mathfrak{A}}(e_0)) \oplus \mathscr{L}_1$ are invariant with respect to the algebra $\mathfrak{A}$. Thus for an C, described by the conditions

$$\left.\begin{aligned} &Cx = -ix \text{ for } x \in \mathfrak{L}_0 \oplus csh_{\mathfrak{A}}(e_0);\\ &Cx = ix \text{ for } x \in J(csh_{\mathfrak{A}}(e_0)) \oplus \mathscr{L}_1;\\ &Cx = 0 \text{ for } x[\perp]\mathfrak{L}_0 \oplus csh_{\mathfrak{A}}(e_0) \oplus J(csh_{\mathfrak{A}}(e_0)) \oplus \mathscr{L}_1; \end{aligned}\right\} \tag{3.7}$$

we have $C \in \mathfrak{A}'$. Now let $B = B^\# \in \mathfrak{A}'$ and $Be_1 = (\alpha+\beta i)e_1$. Then $B - \alpha I - \beta C \in (\mathfrak{A}')_0$. This proves what we wanted for the first part.

THEOREM 53. ([25]) Let an algebra $\mathfrak{A}$ satisfy Conditions (2.23), $\mathfrak{L}_+$ be the corresponding invariant subspace of $\mathfrak{A}$ and let $0 \neq e_1 \in \mathfrak{L}_+ \cap \mathfrak{L}_+^{[\perp]}$ be an arbitrary fixed vector. Let e_0 be a arbitrary fixed neutral vector such that $[e_1, e_0] = 1$, and let the operator S_0 and the set $sh_{\mathfrak{A}}(e_0)$ correspond Formulae (2.25), (2.21) and (2.29). If $S_0 \notin \mathfrak{A}$, then $\mathfrak{A} = \mathfrak{A}''$. If $S_0 \in \mathfrak{A}$, then $\mathfrak{A} = \mathfrak{A}''$ if and only if the set $sh_{\mathfrak{A}}(e_0)$ is a purely real subspace.

REMARK 54. Now we can describe the characteristic of $sh_{\mathfrak{A}}(e_0)$, that defines the structure of $\mathfrak{A}''$: it is a property of $sh_{\mathfrak{A}}(e_0)$ to be or not to be a purely real linear subspace.

Theorem 53 shows not only a criteria for the reflexivity of the corresponding algebra but a possibility of an extension of the initial algebra within the same class. See [25] for details.

3.4. A pass to the general case

LEMMA 55. *Let a canonical symmetry J be compatible with the decomposition (see (2.8))*

$$\widetilde{\mathscr{H}} = \mathscr{H}_1 \oplus \mathscr{H}_2^+ \oplus \mathscr{H}_2^- \tag{3.8}$$

and let WJ^*-algebra $\mathfrak{A}$ be such that $S_0 \notin \mathfrak{A}$ and $sh(e_0) \subset \mathscr{Q}$, where the subspace $\mathscr{Q}$ is defined by (2.31). Then there is a J-normal projection $P \in \mathscr{H}'$, such that

$$a)]Pe_0 = e_0;\ b)Psh(e_0) = sh(e_0);\ c)P(Jsh(e_0) \oplus \mathscr{H}_1)^{[\perp]} = \{0\}. \tag{3.9}$$

Proof. Proposition 52 and Equality (2.30) cover in fact the case of bounded J-orth.sp.f. E_λ, so we need to consider only the case $\sup\{\|E_\lambda\|\} = \infty$. Additionally let us assume that the canonical symmetry on $\mathscr{H}$ is such that (2.35) holds. The latter is possible by Lemma 42. Set $\mathscr{H}_4 := \mathrm{CLin}_{\Delta\subset[-1;1]\setminus\{0\}}\{E(\Delta)e_0\}$, $\mathscr{H}_5 = J\mathscr{H}_4$. By Theorem 17, Remark 18 and Condition (2.35) the subspace $\mathscr{H}_4$ is neutral, $\mathscr{H}_4 \subset \mathscr{H}_2$ and $\mathscr{H}_4$ is invariant with respect to $\mathfrak{A}$. Next, since the canonical symmetry J is compatible with (3.8), in terms of Theorem 17, Remark 18 we have $E(\Delta_1)JE(\Delta_2)e_0 = E(\Delta_1)JU\Big(\chi_{\Delta_2}(t)\cdot\big(\widetilde{g}_+(t)\oplus\widetilde{g}_-(t)\big)\Big) = E(\Delta_1)U\Big(\chi_{\Delta_2}(t)\cdot\big(\widetilde{g}_+(t)\oplus(-\widetilde{g}_-(t))\big)\Big) = U\Big(\chi_{\Delta_1}(t)\cdot\chi_{\Delta_2}(t)\cdot\big(\widetilde{g}_+(t)\oplus(-\widetilde{g}_-(t))\big)\oplus e_1\cdot\int_{\Delta_1\cap\Delta_2}(\|\widetilde{g}_+(t)\|^2_{\mathscr{E}_+}+\|\widetilde{g}_-(t)\|^2_{\mathscr{E}_-})d\sigma(t)\Big) = J\big(E(\Delta_1\cap\Delta_2)e_0\big)\oplus\big(\int_{\Delta_1\cap\Delta_2}(\|\widetilde{g}_+(t)\|^2_{\mathscr{E}_+}+\|\widetilde{g}_-(t)\|^2_{\mathscr{E}_-})d\sigma(t)\big)\cdot e_1$. Thus, the subspace $\mathscr{H}_5\oplus\mathscr{H}_1$ is invariant with respect to $E(\Delta)$ and, hence, to $\mathfrak{A}$. So, the subspace $\mathscr{H}_0\oplus csh(e_0)\oplus Jcsh(e_0)\oplus\mathscr{H}_1\oplus\mathscr{H}_4\oplus\mathscr{H}_5$ is projectionally complete and invariant with respect to $\mathfrak{A}$, therefore the J-orthogonal projection onto this subspace belongs to $\mathfrak{A}'$ and without loss of generality we can assume that

$$\mathscr{H} = \mathscr{H}_0\oplus csh(e_0)\oplus Jcsh(e_0)\oplus\mathscr{H}_1\oplus\mathscr{H}_4\oplus\mathscr{H}_5.$$

Next, the projection P, that maps $\mathscr{H}$ onto the subspace $\mathscr{H}_0\oplus csh(e_0)\oplus\mathscr{H}_4$ and annulates the subspace $\mathscr{H}_1\oplus Jcsh(e_0)\oplus\mathscr{H}_5$, commutes with $\mathfrak{A}$. A simple calculation shows that $P^\#$ is the projection which maps $\mathscr{H}$ onto the subspace $\mathscr{H}_1\oplus Jcsh(e_0)\oplus\mathscr{H}_5$ and annulates the subspace $\mathscr{H}_0\oplus csh(e_0)\oplus\mathscr{H}_4$. Since $PP^\# = P^\#P = 0$, the operator P (and $P^\#$ also) is J-normal. $\square$

REMARK 56. The operator P given in the proof of Lemma 55 is such that $C = C^\# := iP - iP^\# \in \mathfrak{A}'$. E.g this result with Proposition 52 and Formulae (3.7).

3.5. Main Theorem for the bicommutant.

THEOREM 57. *Let $\mathfrak{A}$ be commutative and $\mathfrak{A} \in D_1^+$. Then the equality $\mathfrak{A} = \mathfrak{A}''$ holds if and only if at least one of the following conditions*

a) there is at least one operator $A = A^\# \in \mathfrak{A}$ with $\sigma(A) \backslash \mathbb{R} \neq \emptyset$;

b) $S_0 \notin \mathfrak{A}$;

c) $sh(e_0)$ is purely real subspace

is fulfilled.

Proof. If $A = A^\# \in \mathfrak{A}$ and $\sigma(A) \backslash \mathbb{R} \neq \emptyset$, the space $\mathscr{H}$ can be presented in the form $\mathscr{H} = \mathscr{H}_{im}[+]\mathscr{H}_{re}$, where $\mathscr{H}_{im}$ is a two-dimensional subspace, the J-orthogonal projection on $\mathscr{H}_{im}$ belongs to $\mathfrak{A}'$ and the restriction $\mathfrak{A}|_{\mathscr{H}_{re}}$ is similar to a commutative W^*-algebra, thus this case is trivial. So, let $\sigma(A) \subset \mathbb{R}$ for all $A = A^\# \in \mathfrak{A}$. By Corollary 50 and Remark 51 the algebra $\mathfrak{A}''$ can be larger than $\mathfrak{A}$ in its nilpotent part only. Simultaneously Proposition 52, Lemma 55 and Remark 56 mean that any operator from $(\mathfrak{A}^{(nil)}|_{\mathscr{H} \cap \mathscr{H}_2^\perp})'$ can be extended as an operator belonging to $\mathfrak{A}'$, so the nilpotent part of $\mathfrak{A}''$ restricted on $\mathscr{H} \cap \mathscr{H}_2^\perp$ coincides with the nilpotent part of the algebra $(\mathfrak{A}^{(nil)}|_{\mathscr{H} \cap \mathscr{H}_2^\perp})''$. The rest follows from Theorem 53. $\square$

COROLLARY 58. *If $\mathfrak{A} = \mathrm{Alg} A, A = A^\# \in D_1^+$, then $\mathfrak{A} = \mathfrak{A}''$.*

Proof. If $\sigma(A) \subset \mathbb{R}$ and $A \notin D_0^+$, then the subset $sh(e_0)$ is a zero-dimensional or one-dimensional real subspace, so it is a pure real subspace. Another cases, as it was noted in above, are trivial. $\square$

COROLLARY 59. *Let $\mathscr{H}$ be a space Π_1. Then $\mathfrak{A} = \mathfrak{A}''$.*

Proof. As above we consider the non-trivial case only. Let the subspace $\mathscr{Q}$ be defined by (2.31). Then $\mathscr{Q}$ is positive or trivial subspace. Thus, it is a pure real subspace. $\square$

4. Closing remarks

A complete model for a commutative WJ^*-algebra of D_1^+-class is given here for the first time. At the same time there are some works on model representations for self-adjoint operators and algebras in Pontryagin spaces (the majority of them consider the case with the rank of indefiniteness 1) [19], [18], [30], [17], [14] (see also [12] for more references), [16], [13]. A theorem on the equality $\mathfrak{A}'' = \mathfrak{A}$ for an algebra generated by a single J-s.a. operator in a space Π_1 was announced by author during IX School on Operator Theory in Functional Spaces (Ternopol, Ukraine, 1984), the same result with a complete proof was published in [23]. A generalization of the theorem for a case of an algebra generated by a single J-s.a. operator of the class D_1^+ contains in [22]. Next, S.N.Litvinov and co-authors ([17] and [5],[6]) proved the corresponding theorem for an arbitrary commutative WJ^*-algebra in Π_1. Theorem 53 was announced by author in 1990 during XV School on Operator Theory in Functional Spaces (Uliyanovsk, Russia), its proof was published in [25].

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