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NON–DEFINITE STURM–LIOUVILLE PROBLEMS FOR THE p–LAPLACIAN

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Abstract. For a weighted Sturm-Liouville-type problem of the form

$$-\Delta_p y = (p-1)(\lambda r - q)\operatorname{sgn} y|y|^{p-1}, \quad \text{on } (0,1)$$

with Sturmian-type boundary conditions (Δ_p being the p-Laplacian), we examine the structure, asymptotics and parametric dependence of the eigenvalues, together with properties of the eigenfunctions such as oscillation and interlacing of zeros. We discuss definitions and consequences of left and right (semi-) definiteness, and also the fully indefinite case.

1. Introduction

We shall discuss a weighted Sturm-Liouville-type problem of the form

$$-\Delta_p y = (p-1)(\lambda r - q)[y]^{p-1}, \quad \text{on } (0,1) \tag{1.1}$$

where $p > 1, [y]^{p-1} = |y|^{p-1}\operatorname{sgn} y, \Delta_p$ is the p-Laplacian given by $\Delta_p y = ([y']^{p-1})'$ and r,q are in $L_1(0,1)$ so (1.1) is taken a.e. (in the Carathéodory sense). Equation (1.1) will be subjected to boundary conditions

$$y'(j)\sin\alpha_j = y(j)\cos\alpha_j, \quad j = 0,1, \tag{1.2}$$

where $\alpha_0 \in [0,\pi)$, $\alpha_1 \in (0,\pi]$ and λ is an *eigenvalue* if (1.1), (1.2) admit an *eigenfunction* y which is not identically zero.

Sturmian properties for problems of this type which are right definite (RD), i.e., with a positive (or negative) weight function r, have been studied in several publications, e.g., [5, 13, 22]. Indeed the work here can be considered as a sequel to parts of [5]. On the other hand "weighted" p-Laplacian problems have usually referred to those which are not RD, and we shall concentrate on such situations here. Our primary aims are to discuss the structure, asymptotics and parametric dependence of the eigenvalues, together with properties of the eigenfunctions such as oscillation and interlacing of zeros.

Multiplying (1.1) by y, integrating over $(0,1)$ and applying (1.2), we obtain an equation of the form

$$\ell_q[y] = \lambda r[y] \tag{1.3}$$

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where $r[y]=\int_0^1 r|y|^p$ and $\ell_q[y]$ (which is also a pth power form) will be specified in Section 2. If $\ell_q[y]>0$ for all nonzero $y\in W_p^1(0,1)$, then (1.1), (1.2) is termed left definite (LD) while left semidefinite (LSD) means $\ell_q[y]\geqslant 0$. Also (1.1) is right semidefinite (RSD) if r does not take both signs.

Several authors have discussed the special case $q=0$, often with a leading term of the form $\left([sy']^{p-1}\right)'$ (involving a positive coefficient s) instead of $\Delta_p y$ in (1.1). Since the modified Sturm transformation $t(x)=\int_0^x s^{\frac{1}{1-p}}$ effectively converts s to 1 (cf. [5, 12]) we shall continue to use equation (1.1) as stated. Most authors studying $q=0$ have also imposed boundary conditions of Dirichlet or Neumann type. In the Dirichlet case $(\alpha_0=0,\alpha_1=\pi)$, it is easily seen that the problem is LD, and in the Neumann case $(\alpha_j=\pi/2)$ it is LSD, but we emphasize that for general boundary conditions, LSD fails even for $q=0$. Similarly even Dirichlet and Neumann problems may fail LSD if there is a potential q. Such problems have been discussed in, e.g., [6, 10, 11].

The first investigation of Sturmian properties for (1.1), (1.2) that we know was by Elbert [13] who treated the case of $q=0$ and continuous $r\geqslant 0$ with Dirichlet conditions, so his problem was both LD and RSD. Elbert obtained a sequence of positive eigenvalues $\lambda_0<\lambda_1<\lambda_2\cdots$ accumulating at $+\infty$, λ_n having *oscillation count n*, i.e., with eigenfunctions vanishing n times in $(0,1)$. Similar results (but without positivity of λ_n) were obtained for (1.1) with continuous q and $r>0$ in [22] and for L_1 coefficients in [5]. Several authors have obtained two eigenvalue sequences $\lambda_n^{\pm}\to\pm\infty$ as $n\to\infty$ for $q=0$, indefinite r and Dirichlet conditions, e.g., in [14] for r piecewise differentiable (and satisfying other conditions) and in [1] for $r\in L_\infty$.

Our principal tool is the Prüfer angle $\theta(x,\lambda)$ as extended by Elbert [13]. The main properties we need are developed in Section 2. For example, (1.2) can be written in the form

$$(\sin_p\beta_j)\,y'(j)=\left(\sin_p'\beta_j\right)y(j),\quad j=0,1, \tag{1.4}$$

for angles β_j explicitly dependent on α_j, $\sin_p$ being Elbert's p-trigonometric function [13]. Here $\beta_0\in[0,\pi_p)$, $\beta_1\in(0,\pi_p]$ and $\pi_p=2\left(\frac{\pi}{p}\right)/\sin\left(\frac{\pi}{p}\right)$. Eigenvalues λ_n with oscillation count n are then characterized via $\theta(0,\lambda)=\beta_0$, $\theta(1,\lambda)=n\pi_p+\beta_1$. Thus as for the standard RD case with $p=2$, they are easily determined once enough information about $\theta(1,\lambda)$ is available, but some of that information is less clear when $p\neq 2$. We note that $\theta(1,\lambda)$ also relates to the R(S)D, L(S)D classification as follows. If $\pm r>0$ on a positive measure set $E_\pm$, then $\theta(1,\lambda)\to+\infty$ as $\lambda\to\pm\infty$. Conversely if RSD holds, i.e., if no such E_+ (or E_-) exist, then $\theta(1,\lambda)$ has a nonnegative limit (which is zero under RD) as $\lambda\to+\infty$ (or $-\infty$). Moreover LD (resp. LSD) holds if and only if $\theta(1,0)<$ (resp. $\leqslant$) β_1. There is also a useful generalisation TL(S)D involving a translation of λ to $\lambda-\mu$, i.e., of $\ell_q[y]$ in (1.3) to $\ell_{q-\mu r}[y]$ for certain μ, and this will be explored below.

The study of fully indefinite (i.e., neither LSD nor RSD) problems goes back at least to Richardson (e.g., in [20], for analytic coefficients and Dirichlet conditions) for $p=2$. Since then such problems have been generalised in several directions, but there seems little for $p\neq 2$. In Section 3 (if RSD fails) we shall establish the existence of two sequences $\lambda_n^{\pm}$, $n\geqslant m$, where m is the minimal oscillation count which is connected with translation of λ to $\lambda-\mu$, $\lambda=\mu$ being a minimiser of $\theta(1,\lambda)$. We also obtain an asymptotic for $\theta(1,\lambda)$ with leading term $\int_0^1(\lambda r_\pm)^{1/p}$ as $\lambda\to\pm\infty$, providing eigenvalue asymptotics generalising those of [5, 13] for definite cases, and of [3] for $p=2$. More accurate asymptotics, under stronger conditions on the coefficients, can be found in [7, 9, 14]. Finally, monotonic dependence of the eigenvalues on the problem data is also examined. As far as we know, this has been discussed previously only for definite cases with $p=2$.

RSD problems also have a long history for $p=2$, but some of the results in print are incorrect. We cite [2, 4, 7, 15] for correct results, all proved by different methods. In Section 4 we obtain a single eigenvalue sequence, say λ_n for $n\geqslant m$, accumulating at $+\infty$ if $r\geqslant 0$, and at $-\infty$ if $r\leqslant 0$, provided that r is non-zero on a set of positive measure. This corresponds to one of the two sequences from Section 3, but now one can say more. The minimal oscillation number m corresponds to the infimum of $\theta(1,\lambda)$ (this is no longer attained, but can be calculated explicitly in terms of the problem data). Moreover the λ_n are unique for each $n\geqslant m$, they increase strictly with n and are algebraically simple in the sense that $\partial\theta_\lambda(1,\lambda_n)/\partial\lambda\neq 0$. (It is well known, and easily proved via the Elbert-Prüfer transformation, that the λ_n are geometrically simple, i.e., all eigenfunctions y are proportional, regardless of definiteness conditions). Continuous and differentiable dependence of eigenvalues on the problem data is also established. This was studied in several papers by Zettl and colleagues (see [24] for references) for $p=2$, but their methods used analyticity of the characteristic function in λ and the Lagrange identity, neither of which seem clear for $p\neq 2$. We have instead adapted an earlier approach in [4] based on the Prüfer angle for $p=2$. Finally, we show that the zeros of the eigenfunctions interlace for different eigenvalues.

In Section 5 we treat LD and LSD cases, together with translated versions via $\lambda\to\lambda-\mu$. In such cases the minimal oscillation number m is zero (in fact this is equivalent to TLSD). For brevity we outline the right indefinite case here. Then μ is the minimiser (in this case attained and unique) of $\theta(1,\lambda)$. There are two sequences $\lambda_n^\pm$, each with properties similar to those in Section 4. For example, they are unique for each n, and $\pm\lambda_n^\pm$ increase with n. In the TLD case, they are algebraically simple, while if TLD fails but TLSD holds, then $\lambda_0^-=\lambda_0^+$ is nonsimple but the other eigenvalues are simple (algebraically). Finally, the *classical* L(S)D cases occur when $q\geqslant 0$ a.e. and $\alpha_0\leqslant\frac{\pi}{2}\leqslant\alpha_1$. These include the Dirichlet/Neumann cases with $q=0$ referenced earlier. Then one can use a modified Elbert-Prüfer angle, simplifying the arguments and also yielding additional properties such as parametric dependence of the eigenvalues and interlacing of eigenfunction zeros. In fact we have not seen our interlacing results before even for $p=2$.

We should point out that some of the citations above deal with more general problems, e.g., non-definite s, Fučík spectra, and higher dimensional cases. Also we have concentrated on topics for which the Elbert-Prüfer transformation seems well suited. There is a number of extra Sturmian properties for $p=2$ (particularly in the fully indefinite case) that our methods do not recover, and we hope to use alternative (e.g., eigencurve and variational) methods elsewhere to discuss such questions.

Before proceeding, we note that if r=0 a.e. on $(0,1)$ then (1.1), (1.2) either has a nontrivial solution y (in which case every λ is an eigenvalue) or there is no such y (in which case there are no eigenvalues λ). In what follows we avoid these trivial cases by assuming that $\int_0^1 |r| > 0$.

2. Generalized Prüfer Angle

We start with Elbert's trigonometric functions. For further details see [5, 12]. We define $y = \sin_p x$ as the solution of (1.1) with $q = 0, \lambda r = 1$ and the initial conditions $y(0) = 0, y'(0) = 1$. It follows that $\sin'_p(0) = 1$, and $\sin_p(x) = 0$ if and only if $x = k\pi_p, k \in \mathbb{Z}$, where $\pi_p = 2\left(\frac{\pi}{p}\right) / \sin\left(\frac{\pi}{p}\right)$. Moreover

$$|\sin_p x|^p + |\sin'_p x|^p = 1, \tag{2.1}$$

for all x.

We write $\cot_p = \sin'_p / \sin_p$, noting that $\cot_p$ decreases strictly over $(0, \pi_p)$ from $+\infty$ to $-\infty$. Then we define $\beta_j \in [0, \pi_p]$ by $\cot_p \beta_j = \cot \alpha_j$, with $\beta_0 = 0$ if $\alpha_0 = 0$ and $\beta_1 = \pi_p$ if $\alpha_1 = \pi$.

Next we define Elbert's modification of Prüfer's transformation via

$$y = \rho \sin_p \theta, \quad y' = \rho \sin'_p \theta. \tag{2.2}$$

For any non-zero solution y of (1.1)-(1.2), θ and ρ are functions of (x, λ, β_0) although some or all of these arguments may be suppressed as appropriate.

Proceeding in a well known fashion, one obtains

$$\theta' = 1 + (\lambda r - q - 1)|\sin_p \theta|^p := f(\theta, \lambda), \tag{2.3}$$

$$\rho' = \rho \sin'_p \theta [\sin_p \theta]^{p-1}(1 + q - r\lambda), \tag{2.4}$$

from (1.1), and θ and ρ are Carathéodory solutions of the initial value problem given by $\theta(0) = \beta_0$, $\rho(0) = (|y(0)|^p + |y'(0)|^p)^{1/p}$ which follow from (1.2) and (2.1). From (2.2) we see that $\theta(1, \lambda)$ is finite, and so the same is true of each oscillation count. From (2.4) it follows that $\rho(x)$ is either zero for all x or nonzero for all x, and in particular, we can scale the eigenfunctions y of (1.1)-(1.2) so that

$$\rho(1) = 1. \tag{2.5}$$

We next discuss the dependence of $\theta(x, \lambda, \beta_0)$ on x and β_0 for an arbitrary solution of (2.3) with initial value β_0, not necessarily corresponding to an eigenpair λ, y. For continuous r, these results are well known but perhaps less so for general $r \in L_1$, so we present explicit statements and some details of proofs.

LEMMA 2.1. *For each* λ, $\theta(x,\lambda,\beta_0)$ *is*

(i) absolutely continuous, and strictly increasing through non-negative integer multiples of π_p, *in* x,

(ii) C^1 *and strictly increasing in* β_0.

Proof. (i) This follows from [5, Lemma 2.1]; the requisite "Lipschitz differential inequality theory" can be obtained from, e.g., [23, Theorem XXI].

(ii) The first contention follows from standard theory of Carathéodory solution dependence – see [21, Section II.4]. If the second fails then we violate solution uniqueness (with x reversed) and this follows from (2.3). $\square$

We turn now to the dependence of θ on λ and a further parameter z, say. Let $\tilde{\theta}$ correspond to (2.3) with λ, r replaced by $\tilde{\lambda}, \tilde{r}$ (but with the same q).

LEMMA 2.2. *(i) Assume that* $(a,b) \subset (0,1)$ *and* $\theta(a) = \tilde{\theta}(a)$. *If* $\lambda r \geqslant \tilde{\lambda}\tilde{r}$ *on* (a,b) *then* $\theta \geqslant \tilde{\theta}$ *on* $[a,b]$, *and if*

$$\int_a^b \lambda r > \int_a^b \tilde{\lambda}\tilde{r} \tag{2.6}$$

then $\theta(b) > \tilde{\theta}(b)$.

(ii) For each x, $\theta(x,\lambda)$ *is* C^1 *in* λ *and the derivative* $\theta_\lambda := \partial\theta/\partial\lambda$ *satisfies the initial value problem*

$$\theta_\lambda' = p(\lambda r - q - 1)\sin_p' \theta[\sin_p \theta]^{p-1}\theta_\lambda + r|\sin_p \theta|^p, \quad \theta_\lambda(0) = 0. \tag{2.7}$$

(iii) If f *of* (2.3) *is* C^j *dependent on a further parameter* z, *where* $j = 0$ *or* 1, *then* $\theta(x,\lambda)$ *is also* C^j *in* z.

Proof. (i) The first contention follows from [23, Theorem XXI]. For the second, assume that $\theta(b) = \tilde{\theta}(b)$. Then $f(\theta,\lambda) = f(\tilde{\theta},\tilde{\lambda})$ so $\lambda r = \tilde{\lambda}\tilde{r}$ a.e. on (a,b) by (2.3) and Lemma 2.1(i). Integrating over (a,b), we contradict (2.6), so we must have $\theta(b) > \tilde{\theta}(b)$.

(ii) This follows from standard theory of Carathéodory solution dependence – see [21, Section II.4].

(iii) For $j = 0$, see [21, Theorem II.3.9]; the case $j = 1$ follows as for (ii). $\square$

In view of Lemma 2.2, we can make the following

DEFINITION 2.3. An eigenvalue λ of (1.1), (1.2) is said to be algebraically simple if $y \neq 0$ and $\theta_\lambda(1,\lambda) \neq 0$.

It is well known and follows from the initial condition $\theta(0) = \beta_0$, (2.3) and (2.4) that every eigenvalue is geometrically simple, i.e., if (1.1), (1.2) has a solution $y = y_0 \neq 0$ for given λ, then all other solutions y of (1.1), (1.2) are proportional to y_0.

Next we connect θ_λ with the p^{th} power forms of (1.3). For the remainder of this section, we assume that θ corresponds to an eigenpair λ, y.

Define $\cot_p^* \gamma = 0$ if γ is an integer multiple of π_p and $\cot_p^* \gamma = \sin_p' \gamma / \sin_p \gamma$ otherwise. Multiplying (1.1) by y and integrating, we obtain the formulae

$$\begin{aligned} l_q[y] &= \int_0^1 (|y'|^p + q|y|^p) - |y(1)|^p [\cot^* \alpha_1]^{p-1} + |y(0)|^p [\cot^* \alpha_0]^{p-1}, \\ r[y] &= \int_0^1 r|y|^p. \end{aligned} \tag{2.8}$$

LEMMA 2.4. *For all λ and y satisfying (1.1)-(1.2),*

$$\theta_\lambda(1) = r[y], \quad \lambda\theta_\lambda(1) = l_q[y]. \tag{2.9}$$

Proof. From (2.4) and (2.7) we have

$$(\rho^p \theta_\lambda)' = \rho^p r |\sin_p \theta|^p, \quad \rho^p \theta_\lambda(0) = 0.$$

Then (2.2) gives

$$(\rho^p \theta_\lambda)(1) = \int_0^1 r|y|^p,$$

and (2.9) now follows from (2.5) and (1.3). $\square$

3. Right indefinite case

Our basic asymptotic result, which does not in fact require right indefiniteness, is as follows.

THEOREM 3.1. *For $r, q \in L_1(0,1)$ with $\int_0^1 r_+ \, dt > 0$, we have*

$$\theta(1,\lambda) - \theta(0,\lambda) = \lambda^{1/p} \int_0^1 r_+^{1/p} + o(\lambda^{1/p}) \tag{3.1}$$

as $\lambda \to +\infty$.

Proof. Let $\delta > 0$ and u be a polynomial with $\int_0^1 |r - u| < \delta$. Set

$$\begin{aligned} H &= \{t \in [0,1] | u(t) \geqslant 0\}, \\ I &= \{t \in [0,1] | u(t) < 0\}, \end{aligned}$$

Then H is a disjoint union of closed intervals $H_1 = [a_1, b_1], \ldots, H_n = [a_n, b_n]$ and I is a disjoint union of intervals $I_1, \ldots, I_m$, open in $[0,1]$, where I_j has endpoints $c_j < d_j$. Note that m and n depend on δ.

By Lemma 2.1, $\theta(d_j,\lambda)-\theta(c_j,\lambda) > -\hat{\pi}_p$, giving

$$\theta(1,\lambda)-\theta(0,\lambda) > -m\hat{\pi}_p + \sum_{j=1}^{n}(\theta(b_j,\lambda)-\theta(a_j,\lambda)). \tag{3.2}$$

For $t \in I$, $-u(t) > 0$, so $|r-u| \geqslant r_+$ on I, and, by Hölder's inequality, we have

$$\int_I r_+^{1/p} \leqslant \int_0^1 |r-u|^{1/p} \leqslant \delta^{1/p}. \tag{3.3}$$

Thus it remains only to consider the behaviour of θ on each interval H_j.

Let $h = (u+\delta)^{1/p}$ so $\int_H |r-h^p| < 2\delta$. Moreover $h \geqslant \delta^{1/p}$ and so

$$\int_H \left|\frac{r}{h^{p-1}} - h\right| = \int_H \left|\frac{r-h^p}{h^{p-1}}\right| \leqslant \int_H \frac{|r-h^p|}{\delta^{(p-1)/p}} \leqslant 2\delta^{1/p}. \tag{3.4}$$

On H, since $h^p \geqslant 0$, $|r-h^p| \geqslant |r_+ - h^p|$, giving

$$\int_H |r_+ - h^p| < 2\delta. \tag{3.5}$$

We define a modified Prüfer angle φ on each H_j by

$$\frac{\sin_p \varphi}{\sin_p' \varphi} = \frac{\sin_p \theta}{h\lambda^{1/p}\sin_p' \theta}.$$

From [5] with $f = 1/(h\lambda^{1/p})$, φ satisfies the first order differential equation

$$\varphi' - \lambda^{1/p} r_+^{1/p} = \lambda^{1/p}(h - r_+^{1/p}) + \lambda^{1/p}\left(\frac{r}{h^{p-1}} - h\right)|\sin_p \varphi|^p + Q, \tag{3.6}$$

where

$$Q = \frac{h'}{h}[\sin_p' \varphi]^{p-1}\sin_p \varphi - q h^{1-p}\lambda^{(1-p)/p}|\sin_p \varphi|^p,$$

and there is a constant $K(\delta) > 0$ such that

$$\int_H |Q| \leqslant K(\delta) \quad \text{for all} \quad \lambda > 1. \tag{3.7}$$

Now $|a^{1/p} - b^{1/p}| \leqslant |a-b|^{1/p}$ for $a,b \geqslant 0$, so by Hölder's inequality and (3.5),

$$\int_H |r_+^{1/p} - h| \leqslant \int_H |r_+ - h^p|^{1/p} \leqslant \left(\int_H |r_+ - h^p|\right)^{1/p} \leqslant 2\delta^{1/p}. \tag{3.8}$$

Integrating (3.6) over each of the intervals H_j, summing and using inequalities (3.3), (3.4), (3.7) and (3.8) we obtain

$$\left|\sum_{j=1}^{n}\lambda^{-1/p}(\varphi(b_j)-\varphi(a_j)) - \int_0^1 r_+^{1/p}\right| \leqslant 5\delta^{1/p} + \lambda^{-1/p}K(\delta). \tag{3.9}$$

Now $|\varphi(t)-\theta(t)|<\pi_p/2$ so from the above bound and (3.2),

$$\lambda^{-1/p}\theta(1,\lambda)-\int_0^1 r_+^{1/p}\geqslant -5\delta^{1/p}-\lambda^{-1/p}(K(\delta)+(m+n+1)\pi_p+\alpha_0).$$

Let $\varepsilon>0$. Choosing $\delta>0$ so that $5\delta^{1/p}<\varepsilon/2$ and subsequently taking $\lambda>1$ so large that $\lambda^{-1/p}(K(\delta)+(m+n+1)\pi_p+\alpha_0)<\varepsilon/2$ we find that

$$\lambda^{-1/p}\theta(1,\lambda)>\int_0^1 r_+^{1/p}-\varepsilon.$$

We now repeat the argument with r replaced by r_+, θ by θ_+, and with $u\geqslant 0$ as a C^1 function satisfying $\int_0^1|r_+-u|<\delta$. Then $H=[0,1]$, so $m=0,n=1$ and we can treat (3.9) as before, leading to

$$\lambda^{-1/p}\theta_+(1,\lambda)\leqslant\int_0^1 r_+^{1/p}+\varepsilon$$

for sufficiently large $\lambda>1$. The result now follows because $\theta\leqslant\theta_+$, by Lemma 2.2. □

For the remainder of this section we assume that r is indefinite, i.e., $\int_0^1 r_\pm>0$. Then Theorem 3.1 implies that $\theta(1,\lambda)\to+\infty$ as $\lambda\to\pm\infty$, while Lemma 2.2 shows that $\theta(1,\lambda)$ is continuous (even C^1) in λ. Thus $\theta(1,\lambda)$ attains its global minimum θ_* at μ, say (θ_* is unique but μ may not be). Note that θ_* is positive by Lemma 2.1(i) and define M as the least integer such that $\theta_*\leqslant M\pi_p+\beta_1$. Let S_n be the set of eigenvalues of (1.1)-(1.2) with oscillation count n (see Section 1). Recalling Definition 2.3, we have the following

THEOREM 3.2. *Assume that $r,q\in L_1(0,1)$ with $\int_0^1 r_\pm>0$.*

(i) If $\theta_*<M\pi_p+\beta_1$, then S_n is empty for each $n<M$, and for each $n\geqslant M$, there exist (perhaps non-unique) eigenvalues $\lambda_n^+>\mu$ and $\lambda_n^-<\mu$ in S_n.

(ii) If $\theta_*=M\pi_p+\beta_1$, then the conclusions of (i) hold for each $n\neq M$. Moreover, $\mu\in S_M$ and any $\lambda\in S_M$ is algebraically non-simple.

(iii) For any choice of $\lambda_n^\pm$ as above,

$$(\pm\lambda_n^\pm)^{1/p}=\frac{n\hat{\pi}_p}{\int_0^1 r_\pm^{1/p}}+o(n)$$

as $n\to\infty$.

Proof. Most of the conclusions of (i) and (ii) follow from the condition $\lambda\in S_n\Leftrightarrow\theta(1,\lambda)=n\pi_p+\beta_1$ and the remarks in the previous paragraph. Minimality of M and Lemma 2.2(ii) give the non-simplicity contention. For (iii), we write

$$(\pm\lambda_n^\pm)^{1/p}\int_0^1 r_\pm^{1/p}=n\pi_p+\varepsilon_n$$

and deduce from Theorem 3.1 and $\lambda r = (-\lambda)(-r)$ that $\varepsilon_n = o(n)$. $\square$

It should be noted that the above theorem makes no claims about the number of possible $\lambda_n^{\pm}$ nor about the extent of possible interlacing of the λ_n^+ or the λ_n^- with respect to n (although it is possible to choose such sequences which are monotonic in n).

We conclude this section with a monotonic dependence result. We compare the eigenvalues of (1.1-1.2) with those of a similar problem distinguished by tildes, and we regard λ_n^+ as increasing in q if for $\tilde{q} \geqslant q$ (resp. $\leqslant q$) and $\lambda_n^+ \in S_n$ there is $\tilde{\lambda}_n^+ \geqslant \lambda_n^+$ (resp. $\leqslant \lambda_n^+$) with $\tilde{\lambda}_n^+ \in \tilde{S}_n$, etc.

COROLLARY 3.3. *Assume the above notation and convention. Then, provided that $\theta_* \leqslant n\pi_p + \beta_1$ throughout, $\pm\lambda_n^{\pm}$ increases in q and also increases (resp. decreases) in r, if $\pm\lambda_n^{\pm} \leqslant 0$ (resp. $\geqslant 0$).*

Proof. Assume that $\lambda_n^+ \geqslant 0, r \leqslant \tilde{r}$ and $\tilde{\theta}_* \leqslant n\pi_p + \beta_1$. Then

$$\tilde{\theta}(1,\tilde{\mu}) \leqslant n\pi_p + \beta_1 = \theta(1,\lambda_n^+) \leqslant \tilde{\theta}(1,\lambda_n^+)$$

from Lemma 2.2(i) and Theorem 3.2. Thus $\tilde{\theta}(1,\tilde{\lambda}) = n\pi_p + \beta_1$ for some $\tilde{\lambda} \in [\tilde{\mu}, \lambda_n^+]$. The other cases are similar. $\square$

There are similar results if the monotonicity is strict, or with respect to β_j ($j = 0$ or 1) or s of Section 1. C^j dependence results are possible at simple eigenvalues, but not in general, and we shall return to this below.

4. Right semi-definite cases

LEMMA 4.1. *Assume $r \geqslant 0, r \in L_1(0,1)$ with $\int_0^1 r > 0$. Then $\theta(1,\lambda)$ increases strictly with $\lambda \in \mathbb{R}$ and, at an eigenvalue λ, $\theta_\lambda(1,\lambda) > 0$.*

Proof. Let $\lambda < \tilde{\lambda}, \theta' = f(\theta,\lambda), \tilde{\theta}' = f(\tilde{\theta},\tilde{\lambda})$ and $\theta(0) = \tilde{\theta}(0) = \alpha_0$. Thus, by Lemma 2.2(i) with $a = 0, b = 1$ and $\tilde{r} = r$, $\theta(1) < \tilde{\theta}(1)$.

Thus $\theta(1,\lambda)$ increases with λ, so $\theta_\lambda(1,\lambda) \geqslant 0$ for all λ. If $\theta_\lambda(1,\lambda) = 0$ at an eigenvalue λ then $r[y] = 0$ by (2.9), so y vanishes on a set of positive measure. By Lemma 2.1(i), every zero of y is isolated, so y can vanish only on a set of measure zero. This contradiction completes the proof. $\square$

It follows from Lemmas 2.1 and 4.1 that the structure of the set of eigenvalues of (1.1), (1.2) depends entirely on the behaviour of $\theta(1,\lambda)$ as $\lambda \to \pm\infty$.

The case $\lambda \to +\infty$ is covered by Theorem 3.1, so we turn to the case $\lambda \to -\infty$. Let S be the set of maximal intervals $I \subset [0,1]$ for which $\int_I r = 0$, i.e., $r = 0$ a.e. on I, and let U be the union of all points in such intervals. On each interval $I = [a,b] \in S$ we define a function θ_∞ satisfying the same differential equation (2.3) (which is independent of λ since $r = 0$) as θ but with initial condition given by $\theta_\infty(a) = \beta_0$ if $a = 0$ and $\theta_\infty(a) = 0$ if $a > 0$. Then $\theta_\infty(b) \in [k\pi_p, (k+1)\pi_p)$ for some integer k, and we write $\theta_\infty(I) = k\pi_p$ if $b < 1$ and $\theta_\infty(I) = \theta_\infty(1)$ if $b = 1$.

We also denote by S^* the set of those $I \in S$ such that $\theta_\infty(I) \geqslant \pi_p$, and by U^* the corresponding union. Note that $|\theta_\infty'| \leqslant 1+|q|$ on each $I \in S$, so S^* is a finite set.

LEMMA 4.2. *Under the above conditions*

$$\lim_{\lambda \to -\infty} \theta(1,\lambda) = L := \sum_{I \in S^*} \theta_\infty(I).$$

Proof. This result is given (in different notation) in [8, Theorem 3.2] for $p=2$. Even though the argument carries over to the present case, we shall present a simplified version for the convenience of the reader.

From Lemma 4.1 we can define

$$l(x) := \lim_{\lambda \to -\infty} \theta(x,\lambda)$$

for all $x \in [0,1]$.

Evidently the complement C^* of U^* in $[0,1]$ also consists of finitely many (say N) intervals. If $N=0$ then $[0,1]$ is the only interval in S^* and clearly $l(1) = \theta_\infty(1) = \theta_\infty(I)$. If $N>0$, let J be the left hand interval in C^*, with endpoints $c<d$, and let $\theta_\infty(c) \in [k\pi_p, (k+1)\pi_p)$. (If $c=0$ then $k=0$ since $\beta_0 < \pi_p$, but if $c>0$ then $k>0$ since by construction $[0,c] \in S^*$).

Then it can be shown that for all $x \in J$, $l(x) = \theta_\infty(x)$ if $x \in U$, and $l(x) = k\pi_p$ otherwise. If $d=1$ then we set $x=d$ to establish the result in this case. If $d<1$ then we repeat the above argument over $[d,1]$ instead of $[0,1]$, and so on. $\square$

We are now ready to discuss the eigenvalue structure in right semidefinite cases. Recall Definition 2.3 and let M be the least integer satisfying

$$L < M\pi_p + \beta_1$$

with L from Lemma 4.2. We remark that L, and hence M, depend only on β_0 and the values of q where $r=0$.

THEOREM 4.3. *If $r \geqslant 0$ and $\int_0^1 r > 0$, then M is the minimal oscillation number for* (1.1)*,* (1.2)*. Indeed, there is exactly one eigenvalue λ_m for each $m \geqslant M$, and it is simple (algebraically). Moreover*

$$\lambda_m^{1/p} = \frac{m\pi_p}{\int_0^1 r^{1/p}} + o(m)$$

as $m \to \infty$.

Proof. By Lemmas 4.1 and 4.2, $\theta(1,\lambda)$ increases strictly and continuously from an unattained asymptote L as $\lambda \to -\infty$ and to $+\infty$ as $\lambda \to +\infty$, and thus takes the value $m\pi_p + \beta_1$ for exactly one λ, for each $m \geqslant M$. From Lemma 4.1, we see that every eigenvalue of (1.1), (1.2) is (algebraically) simple. The asymptotic estimate follows as for Theorem 3.2. $\square$

It follows that the set of eigenvalues λ_m accumulates exactly at $+\infty$ under the above semidefiniteness condition. If instead $r \leqslant 0$ with $\int_0^1 r < 0$ then the eigenvalues accumulate at $-\infty$. The eigenvalues accumulate at both $\pm\infty$ only in the right indefinite situation of Theorem 3.2.

For a given m, uniqueness of λ_m permits a sharper statement of monotonic dependence than in Corollary 3.3, and using Lemmas 2.1 and 2.2 as well we can also give some continuous and C^1 dependence results. For example, we could take r of the form $r_1 + zr_2$ for a parameter z.

COROLLARY 4.4. *Under the conditions of Theorem* 4.3*, the eigenvalue* λ_m *increases strictly with* q, β_0 *and* $-\beta_1$*, provided that the asymptote* L *satisfies the restriction* $L < m\pi_p + \beta_1$ *throughout. Under the same restriction,* λ_m*, if* $\leqslant 0$ *(resp.* $\geqslant 0$*), increases (resp. decreases) strictly in* r*, and if the coefficients and boundary data are* C^j *in a parameter (for* $j = 0$ *or* 1*), then so are the eigenvalues.*

Proof. Monotonicity follows as for Corollary 3.3, and C^1 dependence comes from the eigenvalue equation

$$\theta(1,\lambda) = n\pi_p + \beta_1, \tag{4.1}$$

Lemmas 2.1 and 2.2 and the implicit function theorem. A similar argument holds for the case $j = 0$. $\square$

To conclude this section, we discuss interlacing of the zeros of the eigenfunctions y_m corresponding to λ_m.

THEOREM 4.5. *Under the conditions of Theorem* 4.3*, if* $m < n$ *and* s,t *are consecutive zeros of* y_m *then there is at least one zero of* y_n *in* (s,t) *provided* $\int_s^t r > 0$.

Proof. First note that

$$\theta(x,\lambda,\beta_0 + k\pi_p) = \theta(x,\lambda,\beta_0) + k\pi_p \tag{4.2}$$

for all x, λ and nonnegative integers k. Thus the change in θ over an interval (e.g., $[s,t]$) is independent of the addition of an integer multiple of π_p to θ at any fixed $x \in [s,t]$.

Since s is a zero of y_m, $\theta(s,\lambda_m)$ is an integer multiple of π_p. Suppose first that this is also true for $\theta(s,\lambda_n)$, so we can set

$$\tilde{\theta}(x,\lambda_m) := \theta(x,\lambda_m) + k\pi_p \tag{4.3}$$

to give $\tilde{\theta}(s,\lambda_m) = \theta(s,\lambda_n)$ for some integer k. Then we can apply (4.2) and Lemma 2.2(i) with $a = s, b = t$ and $\tilde{r} = r$ to give $\theta(t,\lambda_n) > \theta(s,\lambda_n) + \pi_p$ whence $\theta(x,\lambda_n) = \theta(s,\lambda_n) + \pi_p$ for some $x \in (s,t)$.

A similar argument holds with x traversing $[s,t]$ in reverse if $\theta(t,\lambda_n)$ is an integer multiple of π_p. In the remaining case, we argue by contradiction, assuming that

$$l\pi_p < \theta(s,\lambda_n),\ \theta(t,\lambda_n) < (l+1)\pi_p$$

for some integer l. Thus for some integer k, (4.3) gives $\theta(u,\lambda_n) = \tilde{\theta}(u,\lambda_m)$ for some $u \in (s,t)$. We then contradict Lemma 2.2(i) with $a = u, b = t$ if $\int_u^t r > 0$, and with $a = s, b = u$ (with x traversing $[s,u]$ in reverse) otherwise. $\square$

5. Translatable to left semi-definite cases

We define the various types of left definiteness in terms of the form $l_q[y]$ of (2.8). In particular (1.1)-(1.2) is left definite, denoted LD, (resp. left semidefinite, denoted LSD) if $l_q[y] > 0$ (resp. $\geqslant 0$) for all non-zero $y \in W_p^1(0,1)$. If these conditions hold after a translation of the eigenparameter λ, say by τ, (and hence replacement of $l_q[y]$ by $l_{q-\tau r}[y]$) then we shall use the acronym TLD (resp. TLSD).

These conditions can be described in terms of $\theta(1,\lambda)$. Consider the right definite problem, with (1.1) replaced by

$$-\Delta_p y = (\mu - q)[y]^{p-1}. \tag{5.1}$$

From the analogue of (1.3) with λ and r replaced by μ and 1, we see that (1.1)-(1.2) is LD if and only if (5.1) has minimal eigenvalue $\mu_0 > 0$. The corresponding Elbert-Prüfer angle increases strictly in μ, so at $\mu = 0$ it is less than β_1 and coincides with $\theta(x,0)$ for (1.1)-(1.2). Thus the LD condition for (1.1)-(1.2) corresponds to $\theta(1,0) < \beta_1$ of (1.2). Similarly LSD (resp. TLD, TLSD) corresponds to $\theta(1,0) \leqslant \beta_1$ (resp. $\theta(1,\tau) < \beta_1, \theta(1,\tau) \leqslant \beta_1$).

If RSD holds (say with $r \geqslant 0$), then $\theta(1,\lambda)$ is strictly increasing by Lemma 4.1 and TLSD corresponds to $M = 0$ in Theorem 4.3 with all $\lambda_n \geqslant \tau$. Then the preceding discussion leads to the following.

THEOREM 5.1. *Suppose $r \geqslant 0$ a.e. Then the following are equivalent: $\theta(1,\tau_0) \leqslant \beta_1$; TLSD (for $\tau = \tau_0$); $M = 0$ in Theorem 4.3; and $\lambda_0 \geqslant \tau_0$ in Theorem 4.3. Moreover, TLSD (for $\tau = \tau_0$) $\Rightarrow$ TLD (for all $\tau < \tau_0$) $\Rightarrow$ TLSD (for all $\tau \leqslant \tau_0$).*

The LD case corresponds to $\tau = 0$, for example when $q = 0$ and Dirichlet conditions $\beta_0 = 0, \beta_1 = \pi_p$, are imposed. This situation has been discussed by many authors, in particular by Elbert [13] under RSD as well.

We turn now to cases which are not RSD, so by Theorem 3.2, $\theta(1,\lambda) \to +\infty$ as $\lambda \to \pm\infty$. As for Theorem 3.2, $\theta(1,\lambda)$ has a global minimum value θ_*, say, achieved at λ_*, say. We shall see later that λ_* is unique under TLSD. Let us analyse the TLD case first.

THEOREM 5.2. *Suppose that RSD fails, i.e., $\int_0^1 r_\pm$ are both positive, and let $\theta_* = \theta(1,\lambda_*)$, as above. Then the following are equivalent: $\theta_* < \beta_1$; TLD (with $\tau = \lambda_*$); $M = 0$ in Theorem 3.2; and $\lambda_0^- < \lambda_* < \lambda_0^+$ in Theorem 3.2.*

If the above conditions hold then for all $n \geqslant 0$ there are precisely two eigenvalues $\lambda_n^\pm$ with oscillation count n. They are simple, they satisfy $\pm(\lambda_n^\pm - \tau) > 0$ for all $n \geqslant 0$, and they obey the asymptotics of Theorem 3.2. Moreover the $\lambda_n^\pm$ satisfy the continuity and monotonicity conclusions of Corollary 4.4 (with monotonicity reversed for the λ_n^-), provided that TLD holds throughout.

Finally, LD holds if and only if $\pm\lambda_n^{\pm} > 0$ *for all* $n \geqslant 0$*, so we may take* $\tau = 0$ *in the above.*

Proof. The equivalences follow as for Theorem 5.1 but with Theorem 3.2 substituted for Theorem 4.3. For simplicity of notation, we now translate the λ axis so that $\lambda_* = 0$. Then uniqueness and simplicity follow from (2.9) which gives

$$\lambda_n^{\pm}\theta_\lambda(1,\lambda_n^{\pm}) > 0 \tag{5.2}$$

for each $n \geqslant 0$. Continuity and monotonicity may be established via (4.1) and the implicit function theorem, cf. Corollary 4.4. □

The TLSD case is slightly different.

THEOREM 5.3. *Suppose that RSD fails. Then* (1.1)*-*(1.2) *is TLSD but not TLD if and only if* $\theta_* = \beta_1$*. In this case,* λ_* *is the unique eigenvalue with oscillation count* 0*, and it is algebraically non-simple. Also the conclusion of Theorem* 5.2 *holds for all* $\lambda_n^{\pm}$ *with* $n \geqslant 1$.

Proof. The proof of the first contention follows from Theorem 5.2 and the first paragraph of this section.

For the rest of the proof, we again translate the λ axis to $\lambda_* = 0$, which is evidently an eigenvalue with oscillation count 0. Moreover, since β_1 is the minimum of $\theta(1,\lambda)$,

$$\theta_\lambda(1,0) = 0 \tag{5.3}$$

and so $\lambda = 0$ is non-simple.

Suppose that there is another eigenvalue $\tilde{\lambda} \neq 0$, say, with oscillation count 0. Then $\theta(1,\tilde{\lambda}) = \beta_1$ so we also have

$$\theta_\lambda(1,\tilde{\lambda}) = 0. \tag{5.4}$$

Let y and $\tilde{y}$ be eigenfunctions corresponding to $\lambda = 0$ and $\lambda = \tilde{\lambda}$, respectively. Then (2.9), (5.3) and (5.4) give $r[y] = r[\tilde{y}] = 0$, so

$$l_q[y] = 0 \tag{5.5}$$

and $l_{q-\tilde{\lambda}r}[\tilde{y}] = 0$, whence $l_q[\tilde{y}] = \tilde{\lambda}r[\tilde{y}] = 0$, follow from (2.9).

It follows that y and $\tilde{y}$ are both minimisers of l_q, and hence are eigenfunctions for (5.1), (1.2) with eigenvalue $\mu = 0$, see [5, Section 5]. From geometric simplicity of this eigenvalue we have $y = t\tilde{y}$ for some $t \neq 0$. Thus y and $\tilde{y}$ generate the same Elbert-Prüfer angle θ, and we obtain

$$1 + (0r - q - 1)|\sin_p\theta|^p = 1 + (\tilde{\lambda}r - q - 1)|\sin_p\theta|^p.$$

Then $\tilde{\lambda}r = 0$ except where θ is an integer multiple of π_p, i.e., almost everywhere, by Lemma 2.1. Since $\tilde{\lambda} \neq 0$, we have the contradiction $r = 0$ a.e.

This proves uniqueness of the eigenvalue $\lambda_* = 0$. Finally, for $n \geqslant 1$, $\lambda_n^\pm \neq 0$ so if $l_q[y_n^\pm] = 0$ for any corresponding eigenfunction $y_n^\pm$, then we obtain a contradiction as above. Thus in fact $l_q[y_n^\pm] > 0$ and so (5.2) must hold. □

We can now deduce the following for when LSD holds but LD fails. This is the case, for example, with $q = 0$ and Neumann boundary conditions $\alpha_0 = \alpha_1 = \pi_p/2$, a situation discussed by several authors.

COROLLARY 5.4. *If (1.1)-(1.2) is LSD but not LD, i.e.,* $\theta(1,0) = \beta_1$, *then Theorem 5.2 holds with* $\tau = 0$ *and all* $n > 1$. *For* $n = 0$, *either Theorem 5.3 holds with* $\lambda_* = 0 = \theta_\lambda(1,0)$, *or else* $\theta_\lambda(1,0) \neq 0, \lambda_0^\sigma = 0$ *and* $\sigma\lambda_0^{-\sigma} < 0$, *where* $\sigma = \operatorname{sgn}\theta_\lambda(1,0)$.

Proof. The only case needing comment is $n = 0$ with $\theta_\lambda(1,0) \neq 0$, so TLD holds. Evidently $\theta(1,0) = 0$ gives $\lambda_0^\sigma = 0$ with $\sigma = \operatorname{sgn}\theta_\lambda(1,0)$ and the result then follows from Theorem 5.2. □

We next discuss some properties of the graph of $\theta(1,\lambda)$ under TLSD. Note that if $\beta_1 = \theta_*(\leqslant \pi_p)$ then we are in the situation of Theorem 5.3, and so the minimiser λ_* of $\theta(1,\lambda)$ (which is independent of β_1) is unique, as claimed earlier.

Now suppose that $\beta_1 = \pi_p$. As noted above $\theta_\lambda(1,\lambda_*) = 0$ and, by (5.2), $\theta_\lambda(1,\lambda)$ takes the sign of $\lambda - \lambda_*$ at any eigenvalue $\lambda \neq \lambda_*$. Elsewhere, however, our methods give no information about $\theta_\lambda(1,\lambda)$. If $\beta_1 < \pi_p$, then one can apply the previous reasoning to the related problem (1.1)-(1.2) with β_1 replaced by some $\beta \in (\beta_1, \pi_p]$ to give regions of monotonicity, but there remain infinitely many other regions where we do not know how the sign of $\theta_\lambda(1,\lambda)$ varies. Thus even for variation of β_1, we need an implicit condition (TLD in Theorem 5.2, cf. the condition on L in Corollary 4.4) on the coefficients to obtain our dependence and interlacing results.

One situation where explicit conditions suffice is the special case $q \geqslant 0$ a.e. and $\beta_0 \leqslant \frac{\pi_p}{2} \leqslant \beta_1$, (which covers most of the literature for $p \neq 2$ in one dimension, but which we call "classical" LSD since it was already examined for $p = 2$ around 1900 by Bôcher and others – see [16, Section 10.61]). Indeed Bôcher gave a transformation of the problem leading to λ dependent boundary conditions. This gives rise to an analogous modified Prüfer angle, but since there are related ideas in [4, 17], we shall be brief. We define an angle ψ via $\cot_p \psi = |\lambda|^{\frac{1}{1-p}} \cot_p \theta$ and derive the equation

$$\psi' = |\lambda|^{\frac{1}{p-1}} |\sin' \psi|^p - (|\lambda|^{-1} q - r)|\sin \psi|^p$$

(cf. [5, equation (2.4)]) together with boundary conditions

$$\cot_p \psi(j,\lambda) = |\lambda|^{\frac{1}{1-p}} \cot_p \beta_j, \quad j = 0, 1.$$

Then for $\lambda > 0$, it is easily seen that $\psi(1,\lambda)$ increases strictly, while the branch of $\cot_p^{-1}(\lambda^{\frac{1}{1-p}} \cot_p \beta_1)$ in $(n\pi_p, (n+1)\pi_p]$ is nonincreasing, in λ. Similar results hold (with monotonicities reversed) for $\lambda < 0$.

Together with the estimate $|\psi - \theta| < \frac{\pi_p}{2}$, this leads to an alternative derivation of Theorem 5.2, but the important point from our present perspective is the global nature (for λ of fixed sign) of the monotonicities observed above. Using the fact that for an eigenfunction y, $y(x,\lambda)=0$ precisely when the corresponding $\psi(x,\lambda)$ is an integer multiple of π_p, we have the following analogue of Corollary 4.4 and Theorem 4.5 for the λ_n^+ (with similar statements for the λ_n^-).

COROLLARY 5.5. *In the classical LSD case, the positive eigenvalues increase strictly with $q, -r, \beta_0$ and $-\beta_1$. If the coefficients and boundary data are C^j in a parameter (for $j=0$ or 1), then so are the positive eigenvalues. Finally, the eigenfunction zeros interlace for different positive eigenvalues.*

REMARK. After our work was submitted, reference [18] appeared, and a referee has kindly drawn our attention to this paper. It deals with the case $q=0$ under Dirichlet and Neumann conditions. These are covered by Theorem 5.2 (LD case) and Corollary 5.5 respectively, but [18] also treats further topics including eigenvalue dependence on the weight function r in topologies not considered here.

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