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DIRICHLET FORMS FOR SINGULAR DIFFUSION ON GRAPHS

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Abstract. We describe operators driving the time evolution of singular diffusion on finite graphs whose vertices are allowed to carry masses. The operators are defined by the method of quadratic forms on suitable Hilbert spaces. The model also covers quantum graphs and discrete Laplace operators.

Introduction

The present paper is a continuation and extension of [2]. We present suitable boundary or glueing conditions on graphs (quantum graphs) with singular second order differential operators on the edges. In particular, we describe those boundary conditions leading to positive and submarkovian C_0-semigroups.

The graph consists of finitely many bounded intervals, the edges, whose end points are connected with the vertices of the graph. On each of the edges e a finite Borel measure μ_e is given, determining where particles may be located. The particles move according to "Brownian motion" but are slowed down or accelerated by the "speed measure" μ_e. Further, each of the vertices v is provided with a weight $\mu_v \geqslant 0$, and particles may also be located at those vertices v with $\mu_v > 0$.

The motivations for the treatment in [2] were twofold. The first issue was to treat singular diffusion, including gap diffusion, on the edges of the graph, in the framework of Dirichlet forms. The second aim was to describe glueing conditions on the vertices, in the spirit of [4], and investigate conditions under which the associated self-adjoint operator gives rise to a positive or submarkovian C_0-semigroup.

In the present paper, the extension with respect to [2] consists in two issues. On the one hand, the boundary conditions we describe are more general than glueing conditions. By glueing conditions or "local boundary conditions", we understand conditions where, for a given vertex, only the values of a function on adjacent edges and on the vertex itself can interact. In our treatment in Sections 2 and 3, however, the graph structure does not intervene at all, and we only specify later the case of local boundary conditions, in Section 5. On the other hand, we include the general case of vertices with masses, whereas in [2, Section 4] only a special case was treated. These results have been obtained in [7].

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The ultimate objective of the treatment is to obtain a semi-bounded (below) self-adjoint operator H on a Hilbert space $\mathscr{H}_\Gamma$ over the graph Γ which can then be used in the initial value problem for the diffusion equation or heat equation

$$u' = -Hu, \tag{0.1}$$

thus governing the time evolution of a process, i.e., giving rise to a C_0-semigroup on $\mathscr{H}_\Gamma$. For this equation it is of interest to obtain H in such a way that the associated C_0-semigroup is positive or submarkovian. The self-adjointness of H is also of interest for the initial value problem for the Schrödinger equation

$$u' = -\mathrm{i}Hu.$$

The part of the operator H acting on an edge e is of the form $(Hf)_e = -\partial_{\mu_e}\partial f_e$, where ∂_{μ_e} is the derivative with respect to μ_e; cf. Section 1. The domain of H is restricted by conditions on the boundary values of the functions on the edges and the values at the vertices.

The Hilbert space $\mathscr{H}_\Gamma$ is given by

$$\mathscr{H}_\Gamma = \bigoplus_{e\in E} L_2([a_e,b_e],\mu_e) \oplus \mathbb{K}^V,$$

where E is the set of edges, the interval $[a_e,b_e]$ corresponds to the edge e, and V is the set of vertices; cf. Section 2 for more details. The operator H is obtained by the method of forms. Avoiding all technicalities (which will be given in Section 2), the form τ giving rise to H is of the form

$$\tau(f,g) = \sum_{e\in E} \int_{a_e}^{b_e} f_e'(x)\overline{g_e'(x)}\,dx + (L\operatorname{tr} f \mid \operatorname{tr} g),$$

with domain

$$D(\tau) = \big\{f \in \ldots;\ \operatorname{tr} f \in X\big\}.$$

Here, $\operatorname{tr} f$ denotes the boundary values of f on the edges and the values of f on the vertices, X is a subspace of the set of possible boundary values and values on the vertices, and L is a self-adjoint operator (matrix) on X. The boundary conditions for functions in the domain of H are encoded in the space X as well as in the operator L; cf. Theorem 3.1. Our treatment includes the case that some of the edges or vertices may have weight zero.

For the discussion of positivity and the submarkovian property in connection with equation (0.1) we use the Beurling-Deny criteria for τ. These yield the result that the subspace X should satisfy lattice properties and L should satisfy positivity properties; cf. Theorem 4.1.

The investigations mentioned so far did not take into account the graph structure of Γ. In the description of glueing conditions, allowing only interactions between vertices and adjacent edges, the space X and the operator L decompose into parts corresponding to single vertices; cf. Corollaries 5.1 and 5.2.

In Section 1 we recall some notation and facts from the one-dimensional case on an interval. In Section 2 we define the form in the Hilbert space $\mathscr{H}_\Gamma$ on the graph which then defines the operator driving the evolution. We show that the defined form τ constitutes a form that is bounded below and closed. Let us point out that our definition of the form looks somewhat different from the one given in [2, Section 3]. In fact, looking at the definition of τ in [2, Section 3], one realises that there is some interpretation needed in order to understand $D(\tau)$ as a subset of the Hilbert space $\mathscr{H}_\Gamma$. This interpreation is made explicit in the present paper by the use of the mapping ι introduced in Sections 1 and 2. In Section 3 we describe the operator H associated with the form τ (Theorem 3.1). In Section 4 we indicate conditions for the C_0-semigroup $(e^{-tH})_{t\geqslant 0}$ to be positive and submarkovian. In Section 5 we describe the case of local boundary conditions.

1. One-dimensional prerequisites

In order to define the classical Dirichlet form we have to recall some notation and facts for a single interval $[a,b]\subseteq\mathbb{R}$, where $a,b\in\mathbb{R}$, $a<b$. Let μ be a finite Borel measure on $[a,b]$, $a,b\in\operatorname{spt}\mu$, $\mu(\{a,b\})=0$. Our function spaces will consist of $\mathbb{K}$-valued functions, where $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$. We define

$$C_\mu[a,b] := \big\{f\in C[a,b];\, f \text{ affine linear on the components of } [a,b]\setminus\operatorname{spt}\mu\big\},$$
$$W^1_{2,\mu}(a,b) := W^1_2(a,b)\cap C_\mu[a,b].$$

For later use we recall the following inequalities. There exists a constant $C>0$ such that

$$\|f\|_\infty \leqslant C\big(\|f'\|^2_{L_2(a,b)} + \|f\|^2_{L_2([a,b],\mu)}\big)^{1/2} \tag{1.1}$$

for all $f\in W^1_2(a,b)\cap C[a,b]$, and for all $r\in(0,b-a]$ one has

$$|f(a)| \leqslant r^{1/2}\|f'\|_{L_2(a,a+r)} + \|f\|_{L_2([a,a+r],\mu)}\,\mu([a,a+r])^{-1/2}, \tag{1.2}$$

and correspondingly for b; cf. [2, Lemma 1.4 and Remark 3.2(b)].

Let $\kappa\colon W^1_2(a,b)\cap C[a,b]\to L_2([a,b],\mu)$ be defined by $\kappa f := f$. Then it can be shown that $R(\kappa) = R(\kappa|_{W^1_{2,\mu}(a,b)})$ (cf. [2, Lemma 1.2]), and that $\kappa|_{W^1_{2,\mu}(a,b)}$ is injective (cf. [2, lower part of p. 639]). We define $\iota := \big(\kappa|_{W^1_{2,\mu}(a,b)}\big)^{-1}$. Thus, ι is an operator from $L_2([a,b],\mu)$ to $W^1_{2,\mu}(a,b)$,

$$D(\iota) = \big\{f\in L_2([a,b],\mu);\text{ there exists } g\in W^1_2(a,b)\cap C[a,b] \text{ such that } g=f\ \mu\text{-a.e.}\big\},$$

and ιf is the unique element $g\in W^1_{2,\mu}(a,b)$ such that $g=f$ μ-a.e.

In order to describe the operator associated with the form defined in the following section we need some additional notions and facts concerning derivatives with respect to μ.

If $f \in L_{1,\mathrm{loc}}(a,b)$, $g \in L_1([a,b],\mu)$ are such that $f' = g\mu$ (where $f' = \partial f$ denotes the distributional derivative of f), then we call g *distributional derivative of f with respect to μ*, and we write

$$\partial_\mu f := g.$$

Note that then necessarily $f' = 0$ on $[a,b] \setminus \operatorname{spt}\mu$, i.e., f is constant on each of the components of $[a,b] \setminus \operatorname{spt}\mu$. It is easy to see that this definition is equivalent to

$$f(x) = c + \int_{(a,x)} g(y)\, d\mu(y) \quad \text{a.e.,} \tag{1.3}$$

with some $c \in \mathbb{K}$. Thus, the function f has representatives of bounded variation and these have one-sided limits (not depending on the representative) at all points of $[a,b]$.

2. The form on the graph

Let $\Gamma = (V,E,\gamma)$ be a finite directed graph. This means that V and E are finite sets, $V \cap E = \varnothing$, V is the set of *vertices* (or *nodes*) of Γ, E the set of *edges*, and $\gamma = (\gamma_0,\gamma_1)\colon E \to V \times V$ associates with each edge e a "starting vertex" $\gamma_0(e)$, and an "end vertex" $\gamma_1(e)$.

We assume that each edge $e \in E$ corresponds to an interval $[a_e,b_e] \subseteq \mathbb{R}$ (where $a_e, b_e \in \mathbb{R}$, $a_e < b_e$), and we assume that μ_e is a finite Borel measure on $[a_e,b_e]$ satisfying either $\mu_e = 0$ or else $a_e, b_e \in \operatorname{spt}\mu_e$, $\mu_e(\{a_e,b_e\}) = 0$. We denote

$$E_0 := \{e \in E;\, \mu_e = 0\}, \qquad E_1 := E \setminus E_0.$$

We further assume that, for each $v \in V$, we are given a weight $\mu_v \geqslant 0$, and we define

$$V_0 := \big\{v \in V;\, \mu_v = 0\big\}, \qquad V_1 := V \setminus V_0.$$

REMARK 2.1. The sets E_1 and V_1 encode the parts of the graph Γ, where a particle driven by the diffusion can be localised. In the present section we describe general glueing conditions which do not take into account the correspondence of the edges to the vertices. In the case $E_1 = E$, $V_1 = \varnothing$ and μ_e the Lebesgue measure on $[a_e,b_e]$, the model will describe quantum graphs; cf. [3], [4], [5]. In the case $E_1 = \varnothing$ we obtain (weighted) discrete diffusion on the vertices; cf. [1].

We are going to describe the self-adjoint operator driving the evolution in the Hilbert space

$$\mathscr{H}_\Gamma := \mathscr{H}_E \oplus \mathbb{K}^{V_1},$$

where on

$$\mathscr{H}_E := \bigoplus_{e \in E_1} L_2([a_e,b_e],\mu_e)$$

we use the scalar product

$$\big((f_e)_{e\in E_1} \,\big|\, (g_e)_{e \in E_1}\big)_{\mathscr{H}_\Gamma} := \sum_{e \in E_1} (f_e \,|\, g_e)_{L_2([a_e,b_e],\mu_e)},$$

and on $\mathbb{K}^{V_1}$ we use the scalar product

$$\big((f_v)_{v\in V_1} \,\big|\, (g_v)_{v\in V_1}\big)_{\mathscr{H}_\Gamma} := \sum_{v\in V_1} f_v \overline{g_v}\, \mu_v$$

(for $f = \big((f_e)_{e\in E_1}, (f_v)_{v\in V_1}\big), g = \big((g_e)_{e\in E_1}, (g_v)_{v\in V_1}\big) \in \mathscr{H}_\Gamma$).

In the following, the mapping ι defined in Section 1 will be applied in the situation of the edges $e \in E_1$, and will then be denoted by ι_e. We then define the operator ι from $\mathscr{H}_\Gamma$ to $\prod_{e\in E_1} W^1_{2,\mu_e}(a_e,b_e) \times \mathbb{K}^{V_1}$, by

$$\begin{aligned}
D(\iota) &:= \big\{f \in \mathscr{H}_\Gamma;\, f_e \in D(\iota_e)\ (e \in E_1)\big\},\\
(\iota f)_e &:= \iota_e f_e \qquad (e \in E_1),\\
(\iota f)_v &:= f_v \qquad (v \in V_1).
\end{aligned}$$

We define the *trace mapping* (or *boundary value mapping*) $\mathrm{tr}\colon \prod_{e\in E_1} C[a_e,b_e] \times \mathbb{K}^{V_1} \to \mathbb{K}^{E_1'\cup V_1}$, where $E_1' := E_1 \times \{0,1\}$, by

$$\begin{aligned}
\mathrm{tr} f(e,j) &:= \begin{cases} f_e(a_e) & \text{if } e \in E_1,\ j = 0,\\ f_e(b_e) & \text{if } e \in E_1,\ j = 1,\end{cases}\\
\mathrm{tr} f(v) &:= f_v \qquad (v \in V_1).
\end{aligned}$$

The space $\mathbb{K}^{E_1'\cup V_1}$ will be provided with the scalar product

$$(\xi \,|\, \eta) := \sum_{(e,j)\in E_1'} \xi(e,j)\overline{\eta(e,j)} + \sum_{v\in V_1} \xi(v)\overline{\eta(v)}\, \mu_v.$$

For the definition of the form we assume that X is a subspace of $\mathbb{K}^{E_1'\cup V_1}$ and that L is a self-adjoint operator in X. Then we define the form τ by

$$\begin{aligned}
D(\tau) &:= \big\{f \in D(\iota);\ \mathrm{tr}(\iota f) \in X\big\},\\
\tau(f,g) &:= \sum_{e\in E_1} \int_{a_e}^{b_e} (\iota_e f_e)'(x)\overline{(\iota_e g_e)'(x)}\, dx + \big(L\mathrm{tr}(\iota f) \,\big|\, \mathrm{tr}(\iota g)\big).
\end{aligned}$$

REMARK 2.2. The subspace X encodes boundary conditions for the elements of $D(\tau)$. One would expect boundary conditions to be in the form of some equation for $\mathrm{tr}(\iota f)$. Of course, if P denotes the orthogonal projection from $\mathbb{K}^{E_1'\cup V_1}$ onto $X^\perp$, then $D(\tau) = \big\{f \in D(\iota); P\mathrm{tr}(\iota f) = 0\big\}$.

Further boundary conditions for the elements of the associated operator H are encoded in the operator L; we refer to the description of H in Theorem 3.1.

LEMMA 2.3. *The form τ defined above is symmetric. $D(\tau)$ is dense if and only if*

$$\mathrm{pr}_{V_1}(X) = \mathbb{K}^{V_1}, \tag{2.1}$$

where pr_{V_1} denotes the canonical projection $\mathrm{pr}_{V_1}\colon \mathbb{K}^{E_1'\cup V_1} \to \mathbb{K}^{V_1}$.

Proof. The symmetry of τ is obvious.

Assume that $D(\tau)$ is dense. The image of the dense set $D(\tau)$ under the orthogonal projection

$$\mathrm{pr}_2 \colon \mathscr{H}_\Gamma \to \mathbb{K}^{V_1}$$

is dense in $\mathbb{K}^{V_1}$, and therefore is equal to $\mathbb{K}^{V_1}$. From the definition of $D(\tau)$ it follows that $\mathrm{pr}_2(D(\tau))$ is contained in $\mathrm{pr}_{V_1}(X)$, and therefore $\mathrm{pr}_{V_1}(X) = \mathbb{K}^{V_1}$.

Now assume that (2.1) holds. For $v \in V_1$ let $\xi^v \in X$ be such that $\xi^v(v) = 1$ and $\xi^v(w) = 0$ for all $w \in V_1 \setminus \{v\}$. Let $g^v \in D(\iota)$ be defined by $\mathrm{tr}(\iota g^v) = \xi^v$, and g^v affine linear on the edges. The affine linear interpolation of the prescribed boundary values evidently yields an element of $g^v \in D(\tau)$.

Let $f \in \mathscr{H}_\Gamma$, and define

$$\tilde{f} := f - \sum_{v \in V_1} f_v g^v.$$

Then $\tilde{f}_v = 0$ for all $v \in V_1$. Because $C_c^1(a_e, b_e)$ is dense in $L_2([a_e, b_e], \mu_e)$ $(e \in E_1)$, the function $\tilde{f}$ can be approximated by functions in

$$D_c := \big\{ f \in D(\tau);\, f_e \in C_c^1(a_e, b_e)\ (e \in E_1),\ f_v = 0\ (v \in V_1) \big\}.$$

Therefore f can be approximated by functions in

$$D_c + \sum_{v \in V_1} f_v g^v \subseteq D(\tau). \quad \square$$

REMARKS 2.4. (a) For the special case that $X = \mathbb{K}^{E_1' \cup V_1}$ and $L = 0$ we denote the corresponding form by τ_{N} (the index N indicating Neumann boundary conditions). The form τ_{N} decomposes as the sum of the Neumann forms on each of the edges and the null form on $\mathbb{K}^{V_1}$. Therefore the closedness of τ_{N} follows from the closedness in the one-dimensional cases; cf. [2, Section 1 and Remark 3.2].

(b) Condition (2.1) did not occur in the previous treatment [2]. The reason is that it is obviously satisfied if the vertices do not have masses, i.e. $V_1 = \varnothing$. Also, in the case of vertices with masses, but with local boundary conditions of continuity (see Example 5.3), condition (2.1) is automatically satisfied.

THEOREM 2.5. *The form τ defined above is bounded below and closed.*

Proof. For $f \in D(\tau)$ we obtain the estimate

$$\tau(f) = \tau_{\mathrm{N}}(f) + \big(L\mathrm{tr}(\iota f) \,\big|\, \mathrm{tr}(\iota f)\big) \geqslant \tau_{\mathrm{N}}(f) - \|L\| |\mathrm{tr}(\iota f)|^2$$

(with τ_{N} defined in Remark 2.4(a)). From inequality (1.2) we obtain that the mapping $f \mapsto \mathrm{tr}(\iota f)|_{E_1'}$ is infinitesimally form small with respect to τ_{N}. The remaining part of the trace, $f \mapsto \mathrm{tr}(\iota f)|_{V_1}$, is bounded. These observations imply that τ is bounded below and the that the embedding $D_{\tau_{\mathrm{N}}} \ni f \mapsto \iota f \in (\prod_{e \in E_1} C[a_e, b_e] \times \mathbb{K}^{V_1}, \|\cdot\|_\infty)$ is continuous. (Here, $D_{\tau_{\mathrm{N}}}$ denotes $D(\tau_{\mathrm{N}})$, provided with the form norm.)

In order to obtain that τ is closed it is sufficient to show that $D(\tau)$ is a closed subset of $D_{\tau_{\mathrm{N}}}$. This, however, is immediate from the continuity of the mapping $D_{\tau_{\mathrm{N}}} \ni f \mapsto \mathrm{tr}(\iota f) \in \mathbb{K}^{E_1' \cup V_1}$ (and the fact that X is a closed subspace of $\mathbb{K}^{E_1' \cup V_1}$). $\square$

3. The operator H associated with the form τ

We assume that the notation and the hypotheses are as in the previous section, and that (2.1) holds.

Besides the trace mapping defined in the previous section we also need the *signed trace* (or *signed boundary values*)

$$\operatorname{str}\colon \prod_{e\in E_1} BV(a_e,b_e) \to \mathbb{K}^{E_1'} \subseteq \mathbb{K}^{E_1'\cup V_1}$$

(where $BV(a_e,b_e)$ denotes the set of functions of bounded variation, with equivalence of functions coinciding a.e.), defined by

$$\operatorname{str} f(e,j) := \begin{cases} f_e(a_e+) & \text{if } e\in E_1,\ j=0,\\ -f_e(b_e-) & \text{if } e\in E_1,\ j=1.\end{cases}$$

The inclusion $\mathbb{K}^{E_1'} \subseteq \mathbb{K}^{E_1'\cup V_1}$ is to be understood in the canonical sense; we want to be able to use $\operatorname{str} f$ also as an element of $\mathbb{K}^{E_1'\cup V_1}$.

For the description of the self-adjoint operator H associated with the form τ we use a maximal operator $\hat{H}$ for the differential part of the form. With the notation described in Section 1, we define

$$\begin{aligned} D(\hat{H}) &:= \Big\{ f\in \prod_{e\in E_1} D(\iota_e);\, (\iota_e f_e)'\in L_1(a_e,b_e),\ \partial_{\mu_e}(\iota_e f_e)' \text{ exists},\\ &\qquad\qquad\qquad\qquad \partial_{\mu_e}(\iota_e f_e)'\in L_2([a_e,b_e],\mu_e)\ (e\in E_1)\Big\},\\ \hat{H} f &:= (-\partial_{\mu_e}(\iota_e f_e)')_{e\in E_1} \qquad (f\in D(\hat{H})). \end{aligned}$$

Thus, for $f\in D(\hat{H})$, the signed trace $\operatorname{str}((\iota_e f_e)')_{e\in E_1}$ exists, and it describes the "ingoing derivatives" from the endpoints of the intervals. It is to be understood that for $(\iota_e f_e)'$ we choose representatives of bounded variation (which exist by the explanation given at the end of Section 1), in order to be able to apply the signed trace mapping.

Let

$$X_0 := \big\{x\in X;\ \operatorname{pr}_{V_1} x = 0\big\},$$

which could also be expressed as $X_0 := X\cap \mathbb{K}^{E_1'}$ (with our understanding of $\mathbb{K}^{E_1'}$ as a subspace of $\mathbb{K}^{E_1'\cup V_1}$), and let Q_0 be the orthogonal projection from $\mathbb{K}^{E_1'\cup V_1}$ onto X_0. Also, for $v\in V_1$, let $\xi^v\in X$ be such that $\xi^v|_{V_1} = \mathbf{1}_{\{v\}}$ (see the proof of Lemma 2.3). In the following, for $f\in D(\iota)$ we will use the shorthand notation $(\iota f)' := \big((\iota_e f_e)'\big)_{e\in E_1}$.

THEOREM 3.1. *The operator H associated with the form τ is given by*

$$\begin{aligned} D(H) &= \Big\{ f\in \mathscr{H}_\Gamma;\, (f_e)_{e\in E_1}\in D(\hat{H}),\ \operatorname{tr}(\iota f)\in X,\\ &\qquad\qquad\qquad\qquad Q_0 \operatorname{str}(\iota f)' = Q_0 L \operatorname{tr}(\iota f)\Big\},\\ ((Hf)_e)_{e\in E_1} &= \hat{H}(f_e)_{e\in E_1},\\ (Hf)_v &= \frac{1}{\mu_v}\big(L\operatorname{tr}(\iota f) - \operatorname{str}(\iota f)' \,\big|\, \xi^v\big) \qquad (v\in V_1). \end{aligned}$$

Proof. (i) A preliminary step: Let $f \in D(\hat{H})$, $g \in D(\tau)$. For all $e \in E_1$ one has

$$\int_{a_e}^{b_e} (\iota_e f_e)'(x)\overline{(\iota_e g_e)'(x)}\,dx = -\int_{(a_e,b_e)} \partial_{\mu_e}(\iota_e f_e)'(x)\overline{g_e(x)}\,d\mu_e(x) + (\iota_e f_e)'(b_e-)\overline{\iota_e g_e(b_e)} - (\iota_e f_e)'(a_e+)\overline{\iota_e g_e(a_e)};$$

cf. [2, equ. (1.2)]. Summing this equation over all the edges in E_1 we obtain

$$\sum_{e\in E_1} \int_{a_e}^{b_e} (\iota_e f_e)'(x)\overline{(\iota_e g_e)'(x)}\,dx = (\hat{H} f \,|\, g)_{\mathscr{H}_E} - \big(\mathrm{str}((\iota_e f_e)')_{e\in E_1} \,\big|\, \mathrm{tr}\,(\iota g)\big)_{\mathbb{K}^{E_1'}}.$$

(ii) Let $f \in D(H)$, $g \in D(\tau)$. From $D(H) \subseteq D(\tau)$ we conclude that $\mathrm{tr}(\iota f) \in X$. As in [2, proof of Theorem 1.9] one obtains that $(f_e)_{e\in E_1} \in D(\hat{H})$, $\hat{H}(f_e)_{e\in E_1} = \big((Hf)_e\big)_{e\in E_1}$. Using part (i) above we obtain

$$\begin{aligned}
&(Hf\,|\,g)_{\mathscr{H}_\Gamma} \\
&= -\sum_{e\in E_1} \int_{(a_e,b_e)} \partial_{\mu_e}(\iota_e f_e)'(x)\overline{g_e(x)}\,d\mu_e(x) + \sum_{v\in V_1} (Hf)_v \overline{g_v}\,\mu_v \\
&= \sum_{e\in E_1} \int_{a_e}^{b_e} (\iota_e f_e)'(x)\overline{(\iota_e g_e)'(x)}\,dx + \big(\mathrm{str}(\iota f)' \,\big|\, \mathrm{tr}\,(\iota g)\big)_{\mathbb{K}^{E_1'}} + \sum_{v\in V_1} (Hf)_v \overline{g_v}\,\mu_v.
\end{aligned}$$

Because of

$$(Hf\,|\,g)_{\mathscr{H}_\Gamma} = \sum_{e\in E_1} \int_{a_e}^{b_e} (\iota_e f_e)'(x)\overline{(\iota_e g_e)'(x)}\,dx + \big(L\mathrm{tr}(\iota f) \,\big|\, \mathrm{tr}(\iota g)\big)$$

we therefore obtain

$$\sum_{v\in V_1} (Hf)_v \overline{g_v}\,\mu_v = \big(L\mathrm{tr}(\iota f) - \mathrm{str}(\iota f)' \,\big|\, \mathrm{tr}(\iota g)\big). \tag{3.1}$$

For $\xi \in X_0$ choose $g \in D(\tau)$ satisfying $\mathrm{tr}(\iota g) = \xi$, and g affine linear on the edges $e \in E_1$. Then equation (3.1) implies

$$0 = \big(L\mathrm{tr}(\iota f) - \mathrm{str}(\iota f)' \,\big|\, \xi\big).$$

This shows that $Q_0 L\mathrm{tr}(\iota f) = Q_0\,\mathrm{str}(\iota f)'$.

Let $v \in V_1$, and choose $g \in D(\tau)$ satisfying $\mathrm{tr}(\iota g) = \xi^v$, and g affine linear on the edges $e \in E_1$. Then equation (3.1) yields

$$(Hf)_v\,\mu_v = \sum_{w\in V_1} (Hf)_w \overline{\xi^v(w)}\,\mu_w = \big(L\mathrm{tr}(\iota f) - \mathrm{str}(\iota f)' \,\big|\, \xi^v\big).$$

This shows the second part of the formula for Hf.

(iii) Now let $\tilde{H}$ denote the operator indicated on the right hand side of the assertion, and let $f \in D(\tilde{H})$. Then $f \in D(\tau)$. Let $g \in D(\tau)$. Then $\xi := \mathrm{tr}(\iota g) - \sum_{v\in V_1} g_v \xi^v \in X_0$, and therefore

$$\big(L\mathrm{tr}(\iota f) - \mathrm{str}(\iota f)' \,\big|\, \xi\big) = \big(Q_0(L\mathrm{tr}(\iota f) - \mathrm{str}(\iota f)') \,\big|\, \xi\big) = 0.$$

Using part (i) as well as the previous equality we obtain

$$\begin{aligned}(\tilde{H}f\,|\,g)_{\mathscr{H}_\Gamma} &= \big(\hat{H}(f_e)_{e\in E_1}\,\big|\,(g_e)_{e\in E_1}\big)_{\mathscr{H}_E} + \sum_{v\in V_1}\frac{1}{\mu_v}(L\operatorname{tr}(\iota f) - \operatorname{str}(\iota f)'\,|\,\xi^v)\overline{g_v}\,\mu_v \\ &= \sum_{e\in E_1}\int_{a_e}^{b_e}(\iota_e f_e)'(x)\overline{(\iota_e g_e)'(x)}\,dx + \big(\operatorname{str}(\iota f)'\,\big|\,\operatorname{tr}(\iota g)\big) \\ &\qquad\qquad + (L\operatorname{tr}(\iota f) - \operatorname{str}(\iota f)'\,|\,\operatorname{tr}(\iota g)) \\ &= \tau(f,g).\end{aligned}$$

The definition of H then yields that $f\in D(H)$ and $Hf = \tilde{H}f$. $\square$

REMARKS 3.2. (a) For $f\in D(H)$ and $v\in V_1$, the expression given for $(Hf)_v$ given in Theorem 3.1 does not depend on the choice of ξ^v.

(b) The case of a weight $\mu_v > 0$ at a vertex leads to a case of Wentzell boundary condition at v. The expression of $(Hf)_v$ in Theorem 3.1 generalises the expression obtained at a boundary point in the case of a single interval; cf. [8, Poposition 4.3].

4. Positivity and contractivity

In this section we indicate conditions for the C_0-semigroup $(e^{-tH})_{t\geqslant 0}$ to be positive or submarkovian. We assume that the hypotheses are as in Section 2 and that (2.1) holds.

In the following we need the notion of a (Stonean) sublattice of $\mathbb{K}^n$. We consider $\mathbb{K}^n$ as the function space $C(\{1,\dots,n\})$, and accordingly use the notation $|x| = (|x_1|,\dots,|x_n|)$, for $x\in\mathbb{K}^n$, and $x\wedge y = (x_1\wedge y_1,\dots,x_n\wedge y_n)$, for $x,y\in\mathbb{R}^n$. A *sublattice* X of $\mathbb{K}^n$ is a subspace for which $x\in X$ implies that $|x|\in X$. A sublattice X is called *Stonean* if additionally $x\wedge\mathbf{1}\in X$ for all real $x\in X$.

We refer to [2, Appendix] for the description of (Stonean) sublattices of $\mathbb{K}^n$ and of generators for positive (submarkovian) C_0-semigroups on these sublattices.

THEOREM 4.1. (a) *Assume that X is a sublattice of $\mathbb{K}^{E_1'\cup V_1}$ and that the semigroup $(e^{-tL})_{t\geqslant 0}$ is positivity preserving. Then $(e^{-tH})_{t\geqslant 0}$ is positivity preserving.*

(b) *Assume that X is a Stonean sublattice of $\mathbb{K}^{E_1'\cup V_1}$ and that the semigroup $(e^{-tL})_{t\geqslant 0}$ is a submarkovian semigroup. Then $(e^{-tH})_{t\geqslant 0}$ is submarkovian.*

This result was proved for the case of local boundary conditions (cf. Section 5) and no vertex masses in [2, Theorem 3.5], and for the case of vertices with masses and local boundary conditions of continuity (cf. Example 5.3) in [2, Theorem 4.2]. Its proof is completely analogous to [2, proof of Theorem 3.5]; so we refrain from giving a complete proof but rather only mention the main ingredients. The proof consists in an application of the Beurling-Deny criteria (cf. [6, Corollary 2.18]; see also [2, Remarks 1.6]). So, in order to prove part (a), it is equivalent to prove that the normal contraction $f\mapsto|f|$ acts on $D(\tau)$, and that $\tau(|f|)\leqslant\tau(f)$ for all $f\in D(\tau)$. That the inequality works on the differential part is a one-dimensional issue which is taken care of in [2, Theorem 1.7]. For the trace part, the main observation is the equation $\operatorname{tr}\iota|f| = |\operatorname{tr}\iota f|$. This is less obvious than it might appear at the first glance since, in general, one does not have $\iota|f| = |\iota f|$. However, this equality holds on $\operatorname{spt}\mu_e$, and therefore at the end points of the intervals $[a_e,b_e]$, for all $e \in E_1$. The reasoning for part (b) is analogous.

5. Local boundary conditions

So far, the structure of the graph did not enter the considerations; in fact the function γ linking the edges to the vertices was not used at all. In order to explain what we understand by local boundary conditions, we need the following definitions.

For $v \in V$, the sets

$$E_{1,v,j} := \{e \in E_1;\, \gamma_j(e) = v\} \qquad (j = 0,1)$$

describe the sets of all edges having mass and starting or ending at v, respectively, and the set

$$E_{1,v} := \big(E_{1,v,0} \times \{0\}\big) \cup \big(E_{1,v,1} \times \{1\}\big)$$

is the set of all edges having mass connected with v (and where loops starting and ending at v yield two contributions). Note that then $E_1' = \bigcup_{v\in V} E_{1,v}$.

Recall that the boundary conditions are specified by the choice of a subspace $X \subseteq \mathbb{K}^{E_1'\cup V_1}$ and a self-adjoint operator L in X. The boundary conditions will be called *local* if for each $v \in V$ there exists a subspace

$$X_v \subseteq \mathbb{K}^{E_{1,v}} \quad \text{if } v \in V_0, \qquad X_v \subseteq \mathbb{K}^{E_{1,v}\cup\{v\}} \quad \text{if } v \in V_1,$$

and a selfadjoint operator L_v in X_v, such that

$$X = \bigoplus_{v\in V} X_v, \qquad L = \bigoplus_{v\in V} L_v.$$

For $v \in V$, we define the "local trace mapping"

$$\operatorname{tr}_v\colon \prod_{e\in E_1} C[a_e,b_e] \times \mathbb{K}^{V_1} \to \begin{cases} \mathbb{K}^{E_{1,v}} & \text{if } v \in V_0, \\ \mathbb{K}^{E_{1,v}\cup\{v\}} & \text{if } v \in V_1 \end{cases}$$

by

$$\operatorname{tr}_v f := \begin{cases} \operatorname{tr} f|_{E_{1,v}} & \text{if } v \in V_0, \\ \operatorname{tr} f|_{E_{1,v}\cup\{v\}} & \text{if } v \in V_1. \end{cases}$$

Then for the form τ we obtain

$$D(\tau) = \big\{f \in D(\iota);\ \operatorname{tr}_v(\iota f) \in X_v\ (v \in V)\big\},$$
$$\tau(f,g) = \sum_{e\in E_1} \int_{a_e}^{b_e} (\iota_e f_e)'(x)\overline{(\iota_e g_e)'(x)}\,dx + \sum_{v\in V} \big(L_v \operatorname{tr}_v(\iota f) \,\big|\, \operatorname{tr}_v(\iota g)\big).$$

With

$$X_{v,0} := \begin{cases} X_v & \text{if } v \in V_0, \\ \{\xi \in X_v;\, \xi(v) = 0\} & \text{if } v \in V_1, \end{cases}$$

the condition (2.1) for $D(\tau)$ to be dense then decomposes into

$$X_{v,0} \neq X_v \qquad (v \in V_1),$$

or expressed differently, for all $v \in V_1$ there exists $\xi^v \in X_v$ such that $\xi^v(v) = 1$.

It is an easy task to translate the description of the associated operator H, given in Theorem 3.1, to the present case of local boundary conditions, as follows.

COROLLARY 5.1. *The operator H associated with τ is given by*

$$\begin{aligned} D(H) = \big\{ f \in \mathscr{H}_\Gamma;\, (f_e)_{e\in E_1} \in D(\hat{H}),\ \mathrm{tr}_v(\iota f) \in X_v, \\ Q_{v,0}\,\mathrm{str}_v(\iota f)' = Q_{v,0} L_v \,\mathrm{tr}_v(\iota f)\ (v \in V) \big\}, \end{aligned}$$

$$((Hf)_e)_{e\in E_1} = \hat{H}(f_e)_{e\in E_1},$$

$$(Hf)_v = \frac{1}{\mu_v}\big(L_v \,\mathrm{tr}_v(\iota f) - \mathrm{str}_v(\iota f)' \,\big|\, \xi^v\big) \qquad (v \in V_1).$$

Here, for $v \in V$ the mapping $\mathrm{str}_v \colon \prod_{e\in E_1} BV(a_e, b_e) \to \mathbb{K}^{E_{1,v}}$ is defined by $\mathrm{str}_v f := (\mathrm{str} f)|_{E_{1,v}}$, and $Q_{v,0}$ is the orthogonal projection onto $X_{v,0}$ in $\mathbb{K}^{E_{1,v}}$, for $v \in V_0$, or in $\mathbb{K}^{E_{1,v}\cup\{v\}}$, for $v \in V_1$. We will not put down further details here. Similarly, the conditions for $(e^{-tH})_{t\geqslant 0}$ to be positive and submarkovian, Theorem 4.1, can be spelled out in terms of the spaces X_v and the operators L_v. The statements are then analogous to [2, Theorem 3.5], where the case that $E = E_1$ and $V = V_0$ is treated.

COROLLARY 5.2. (a) *Assume that X_v is a sublattice of $\mathbb{K}^{E_{1,v}}$ ($v \in V_0$) or $\mathbb{K}^{E_{1,v}\cup\{v\}}$ ($v \in V_1$) and that $(e^{-tL_v})_{t\geqslant 0}$ positivity preserving, for all $v \in V$. Then $(e^{-tH})_{t\geqslant 0}$ is a positivitiy preserving C_0-semigroup on $\mathscr{H}_\Gamma$.*

(b) *Assume that X_v is a Stonean sublattice of $\mathbb{K}^{E_{1,v}}$ ($v \in V_0$) or $\mathbb{K}^{E_{1,v}\cup\{v\}}$ ($v \in V_1$) and that $(e^{-tL_v})_{t\geqslant 0}$ is a submarkovian C_0-semigroup on X_v, for all $v \in V$. Then $(e^{-tH})_{t\geqslant 0}$ is a submarkovian C_0-semigroup on $\mathscr{H}_\Gamma$.*

EXAMPLE 5.3. (local boundary conditions of continuity) This special case of local boundary conditions was studied in [2, Section 4]. In our framework, this example reads as follows. Let $X_v = \mathrm{lin}\{\mathbf{1}\}$, $L_v \in \mathbb{R}$, $l_v := (L_v\mathbf{1}\,|\,\mathbf{1})$ for $v \in V$. Then $X_{v,0} = \{0\}$ for $v \in V_1$ (which makes it clear that condition (2.1) is satisfied) and hence

$$Q_{v,0} = \begin{cases} (\cdot\,|\,\mathbf{1})\mathbf{1} & \text{if } v \in V_0, \\ 0 & \text{if } v \in V_1. \end{cases}$$

Functions $f \in D(\tau)$ are continuous on Γ, i.e., for $v \in V_1$ we have $f_v = (\mathrm{tr}_v f)(e,j)$ for all $(e,j) \in E_{1,v}$, and for $v \in V_0$ there exists $a_v(f) \in \mathbb{K}$ such that $a_v(f) = (\mathrm{tr}_v f)(e,j)$

for all $(e,j) \in E_{1,v}$ (note that we cannot write f_v in this case since f is not defined on V_0). The second part of the boundary conditions for $f \in D(H)$ translates to

$$\sum_{e\in E_{1,v,0}} f_e'(a_e+) - \sum_{e\in E_{1,v,1}} f_e'(b_e-) = l_v a_v(f) \quad (v \in V_0);$$

see also [2, Theorem 4.3]. In the setup considered in [4], these boundary conditions are called δ-type conditions; cf. [4, Section 3.2.1].

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