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NEAREST SOUTHEAST SUBMATRIX THAT MAKES MULTIPLE AN EIGENVALUE OF THE NORMAL NORTHWEST SUBMATRIX

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Abstract. Let A,B,C,D be four complex matrices, where $D \in \mathbb{C}^{m\times m}$ and $A \in \mathbb{C}^{n\times n}$ is a normal matrix. Let z_0 be an fixed eigenvalue of A. We find the distance (with respect to the 2-norm) from D to the set of matrices $X \in \mathbb{C}^{m\times m}$ such that z_0 is a multiple eigenvalue of the matrix

$$\begin{pmatrix} A & B \\ C & X \end{pmatrix}.$$

We also give an expression for one of the closest matrices.

1. Introduction

This paper is highly inspired by Malyshev [12] and Wei [14]. The Malyshev's paper is concerning to the distance from a matrix to the nearest matrix with a multiple eigenvalue (Wilkinson's problem). Wei solved the problem of finding the nearest matrix D' to D which reduces the rank of $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ to a specific integer.

We denote by $\|\cdot\|$ the matrix spectral norm or 2-norm. The spectrum of a square complex matrix M is denoted by $\Lambda(M)$. An important problem that has been studied for some decades is the description of the possible eigenvalues and Jordan canonical forms of square complex matrices partitioned in the shape

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

when some of the blocks A,B,C,D are fixed and the remaining blocks vary. Relevant results are due to Oliveira, Sá, Silva, Thompson, Wimmer and Zaballa, among others; see the survey paper by Cravo [4]. In [1], Beitia et al. studied the problem of analyzing the possible Jordan forms of the matrix $\left(\begin{smallmatrix} A & B \\ C' & D' \end{smallmatrix}\right)$ when A and B are fixed and C' and D' are close to C and D, respectively.

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The problem of the description of the possible eigenvalues and Jordan forms of the matrices of the form

$$\begin{pmatrix} A & B \\ C & X \end{pmatrix},$$

where $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{n\times m}, C \in \mathbb{C}^{m\times n}$ are fixed and X varies in $\mathbb{C}^{m\times m}$, has been particularly difficult. There are few results about it; see Cravo [4], problem (P_7) in pages 2520 and 2527. Moreover, we know no results on the Jordan forms of matrices $\left(\begin{smallmatrix} A & B \\ C & D' \end{smallmatrix}\right)$ when D' is close to $D \in \mathbb{C}^{m\times m}$. When all the eigenvalues of the matrix $G := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ are simple, the problem of finding the distance, $d(G)$, from D to the set of matrices $X \in \mathbb{C}^{m\times m}$ such that $\left(\begin{smallmatrix} A & B \\ C & X \end{smallmatrix}\right)$ has a multiple eigenvalue, is a kind of structured Wilkinson's problem. This problem has been addressed by means of the structured ε-pseudospectrum, defined as

$$\bigcup_{\substack{X \in \mathbb{C}^{m\times m} \\ \|X-D\| \leqslant \varepsilon}} \Lambda \begin{pmatrix} A & B \\ C & X \end{pmatrix};$$

see Du and Wei [5], where a characterization of the structured ε-pseudospectrum is given. Other characterizations can be seen in Hinrichsen and Kelb [9] and [6].

For $z_0 \in \mathbb{C}$, if we could know the minimum, $f(z_0)$, of the set

$$\{\|X - D\| : X \in \mathbb{C}^{m\times m} \text{ and } z_0 \text{ is a multiple eigenvalue of } \left(\begin{smallmatrix} A & B \\ C & X \end{smallmatrix}\right)\},$$

we would have

$$\min_{z_0 \in \mathbb{C}} f(z_0) = d(G).$$

In [8] the authors found an expression for $f(z_0)$ in terms of a singular value maximization, when $z_0 \notin \Lambda(A)$, A being any matrix of $\mathbb{C}^{n\times n}$. In the current paper we address this problem when A is a normal matrix and $z_0 \in \Lambda(A)$. The solution obtained can be easily extended to the case when z_0 is a semisimple (or nondefective) eigenvalue of A (normal or not). When z_0 is not an eigenvalue of A the solution of the problem involves matrices of polynomials in a real variable t and the inverse of square nonsingular matrices; the case when z_0 is an eigenvalue of A requires matrices of rational functions in t with a pole at $t = 0$ and the Moore-Penrose inverse instead.

If $\lambda_0 \in \Lambda(M)$, the algebraic multiplicity of λ_0 is denoted by $\mathrm{m}(\lambda_0, M)$. For a matrix $N \in \mathbb{C}^{p\times q}$ we denote by $\sigma_1(N) \geqslant \sigma_2(N) \geqslant \cdots$ its singular values, and by $N^\dagger$ its Moore–Penrose inverse. For a matrix X, we denote by $\mathrm{Im}(X)$ and $\mathrm{Ker}(X)$ its image and kernel subspaces. By O we denote the zero matrix of adequate size.

Moreover, as in [8], we can assume without loss of generality that $z_0 = 0$. Thus the problem we are going to solve, can be set as follows: Find the minimum

$$\min_{\substack{X \in \mathbb{C}^{m\times m} \\ \mathrm{m}\left(0, \left(\begin{smallmatrix} A & B \\ C & X \end{smallmatrix}\right)\right) \geqslant 2}} \|X - D\|. \tag{1}$$

where $A \in \mathbb{C}^{n\times n}$ is a singular normal matrix $B \in \mathbb{C}^{n\times m}, C \in \mathbb{C}^{m\times n}$ and $D \in \mathbb{C}^{m\times m}$.

If for all $X \in \mathbb{C}^{m\times m}$ it happens that $\mathrm{m}\left(0, \left(\begin{smallmatrix} A & B \\ C & X \end{smallmatrix}\right)\right) \leqslant 1$, we agree to say that the minimum distance (1) is infinite. Note that this case is possible considering $A = O \in \mathbb{C}^{n\times n}$ and $B = C = I_n$ for example, since for each $X \in \mathbb{C}^{n\times n}$ the matrix

$$\begin{pmatrix} O & I_n \\ I_n & X \end{pmatrix}$$

is nonsingular. In Section 4, the cases in which this distance is infinite will be determined.

To simplify we denote by $L_{n,m}$ the Cartesian product $\mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m} \times \mathbb{C}^{m\times n}$. For a triple of matrices $\alpha := (A,B,C) \in L_{n,m}$, and for $X \in \mathbb{C}^{m\times m}$ we denote

$$M(\alpha,X) := \begin{pmatrix} A & B \\ C & X \end{pmatrix}.$$

A lower bound of the minimum (1) was given in [8]. We will remember the notations that appear in [8, (11) and (12)] to recall this bound, and for their use in this paper: Given a triple $\alpha := (A,B,C) \in L_{n,m}$ and a matrix $D \in \mathbb{C}^{m\times m}$, we define for $t \in \mathbb{R}$,

$$\rho_\alpha(t) := \operatorname{rank}\left(\begin{array}{cc|cc} A & tI_n & B & O \\ O & A & O & B \end{array}\right) + \operatorname{rank}\left(\begin{array}{cc} A & tI_n \\ O & A \\ \hline C & O \\ O & C \end{array}\right) - \operatorname{rank}\begin{pmatrix} A & tI_n \\ O & A \end{pmatrix},$$

$$p_\alpha(t) := 2n + 2m - 2 - \rho_\alpha(t), \tag{2}$$

$$M_\alpha(t) := \left(I_{2n} - \begin{pmatrix} A & tI_n \\ O & A \end{pmatrix}\begin{pmatrix} A & tI_n \\ O & A \end{pmatrix}^\dagger\right)\begin{pmatrix} B & O \\ O & B \end{pmatrix} \tag{3}$$

$$N_\alpha(t) := \begin{pmatrix} C & O \\ O & C \end{pmatrix}\left(I_{2n} - \begin{pmatrix} A & tI_n \\ O & A \end{pmatrix}^\dagger\begin{pmatrix} A & tI_n \\ O & A \end{pmatrix}\right) \tag{4}$$

$$\begin{aligned} S_2^\alpha(t,D) := {} & \left(I_{2m} - N_\alpha(t)N_\alpha(t)^\dagger\right) \\ & \times \left(\begin{pmatrix} D & tI_m \\ O & D \end{pmatrix} - \begin{pmatrix} C & O \\ O & C \end{pmatrix}\begin{pmatrix} A & tI_n \\ O & A \end{pmatrix}^\dagger\begin{pmatrix} B & O \\ O & B \end{pmatrix}\right) \\ & \times \left(I_{2m} - M_\alpha(t)^\dagger M_\alpha(t)\right). \end{aligned} \tag{5}$$

We agree to write $\sup_{t\geqslant 0} f(t) = \infty$ if the function $f\colon [0,\infty) \to \mathbb{R}$ is not bounded above. Then the announced lower bound of (1) is given below.

PROPOSITION 1. ([8], Proposition 23)

$$\sup_{t\geqslant 0} \sigma_{p_\alpha(t)+1}\left(S_2^\alpha(t,D)\right) \leqslant \min_{\substack{X\in\mathbb{C}^{m\times m} \\ \mathrm{m}(0,M(\alpha,X))\geqslant 2}} \|X - D\|. \tag{6}$$

where

$$\sigma_j\big(S_2^{\alpha}(t,D)\big) := \begin{cases} \infty & \text{if } j < 1, \\ 0 & \text{if } j > 2m. \end{cases}$$

The aim of this paper is to prove that when A is normal and singular, the inequality (6) becomes an equality. Specifically, we prove the following result.

THEOREM 2. *Let $\alpha := (A,B,C) \in L_{n,m}$ be a triple of matrices, where A is normal and singular. Let $D \in \mathbb{C}^{m\times m}$. With the preceding notations, we have*

$$\sup_{t>0} \sigma_{p_\alpha(t)+1}\big(S_2^{\alpha}(t,D)\big) = \min_{\substack{X\in\mathbb{C}^{m\times m} \\ \mathrm{m}(0,M(\alpha,X))\geqslant 2}} \|X - D\|. \tag{7}$$

REMARK 1. Let us note that in this theorem we put $t > 0$ instead of $t \geqslant 0$. In fact, once (7) is proved then by (6) we have

$$\sigma_{p_\alpha(0)+1}\big(S_2^{\alpha}(0,D)\big) \leqslant \sup_{t>0} \sigma_{p_\alpha(t)+1}\big(S_2^{\alpha}(t,D)\big).$$

Hence,

$$\sup_{t\geqslant 0} \sigma_{p_\alpha(t)+1}\big(S_2^{\alpha}(t,D)\big) = \sup_{t>0} \sigma_{p_\alpha(t)+1}\big(S_2^{\alpha}(t,D)\big).$$

This work is organized as follows. In Section 2, we give a simplified expression for $S_2^{\alpha}(t,D)$, and we reformulate Theorem 2 in Theorem 5. In Section 3, we introduce the auxiliary results we are going to use in this work. We analyze the asymptotic behavior of the singular values of $S_2^{\alpha}(t,D)$, both for $t \to 0^+$ and $t \to \infty$, and we establish the existence of the limits

$$\lim_{t\to 0^+} \sigma_{p_\alpha(t)+1}\big(S_2^{\alpha}(t,D)\big) \text{ and } \lim_{t\to\infty} \sigma_{p_\alpha(t)+1}\big(S_2^{\alpha}(t,D)\big),$$

in Section 4. We prove Theorem 5 in the following sections until the end of Section 8. Namely, in Section 5, we calculate the minimum (1) when the supremum

$$\sup_{t>0} \sigma_{p_\alpha(t)+1}\big(S_2^{\alpha}(t,D)\big)$$

is attained at a point t_0 such that $0 < t_0 < \infty$ and we prove equality (7). In Section 6, we study the case when

$$\sup_{t>0} \sigma_{p_\alpha(t)+1}\big(S_2^{\alpha}(t,D)\big) = \lim_{t\to\infty} \sigma_{p_\alpha(t)+1}\big(S_2^{\alpha}(t,D)\big),$$

and, in Sections 7 and 8, we consider the case when

$$\sup_{t>0} \sigma_{p_\alpha(t)+1}\big(S_2^{\alpha}(t,D)\big) = \lim_{t\to 0^+} \sigma_{p_\alpha(t)+1}\big(S_2^{\alpha}(t,D)\big),$$

finishing the proof of Theorem 5. In Section 9, we give a more general result that falls within the scope of this article. This is the case in which z_0 is a semisimple eigenvalue of a not necessarily normal matrix A.

2. Reformulation of the main result

We denote by M^* the conjugate transpose of each complex matrix M. In this section we are going to reformulate Theorem 2, simplifying the expression of $S_2^{\alpha}(t,D)$ for $t>0$ when the triple α undergoes a transformation of unitary similarity given by the unitary matrix U that diagonalizes A. For this purpose we need some properties of the Moore-Penrose inverse, which can be seen in [2, Proposition 6.1.6, p. 225].

LEMMA 3. *Given a matrix $A\in\mathbb{C}^{p\times q}$, then we have*
(1) *$I_p - AA^{\dagger}$ and $I_q - A^{\dagger}A$ are orthogonal projectors.*
(2) *If $S_1\in\mathbb{C}^{p\times p}$ and $S_2\in\mathbb{C}^{q\times q}$ are unitary, then $(S_1AS_2)^{\dagger} = S_2^*A^{\dagger}S_1^*$.*

LEMMA 4. *Let $U\in\mathbb{C}^{n\times n}$ be a unitary matrix and $D\in\mathbb{C}^{m\times m}$. Then for the triple of matrices $\beta := (U^*AU, U^*B, CU)\in L_{n,m}$ and each $t>0$ we have $S_2^{\alpha}(t,D) = S_2^{\beta}(t,D)$.*

Proof. To simplify this demonstration, we introduce the following notations:

$$L(t)=\begin{pmatrix} D & tI_m\\ O & D\end{pmatrix},\quad V=\begin{pmatrix} U & O\\ O & U\end{pmatrix},\quad F(t)=\begin{pmatrix} A & tI_n\\ O & A\end{pmatrix},\quad F_1(t)=V^*F(t)V,$$

$$G=\begin{pmatrix} B & O\\ O & B\end{pmatrix},\quad G_1=V^*G,\quad H=\begin{pmatrix} C & O\\ O & C\end{pmatrix},\quad H_1=HV.$$

First, as the matrix V is unitary, by Lemma 3, we deduce that $(V^*F(t)V)^{\dagger} = V^*F(t)^{\dagger}V$. Hence, from (3) and (4), we obtain

$$\begin{aligned} M_{\beta}(t) &= (I_{2n}-F_1(t)F_1(t)^{\dagger})G_1 = (I_{2n}-V^*F(t)VV^*F(t)^{\dagger}V)V^*G = V^*M_{\alpha}(t),\\ N_{\beta}(t) &= H_1(I_{2n}-F_1(t)^{\dagger}F_1(t)) = HV(I_{2n}-V^*F(t)^{\dagger}VV^*F(t)V) = N_{\alpha}(t)V.\end{aligned}$$

Similarly, as the matrix V is unitary, we see that $(V^*M_{\alpha}(t))^{\dagger} = M_{\alpha}(t)^{\dagger}V$ and $(N_{\alpha}(t)V)^{\dagger} = V^*N_{\alpha}(t)^{\dagger}$. Therefore,

$$\begin{aligned} I_{2m}-N_{\beta}(t)N_{\beta}(t)^{\dagger} &= I_{2m}-N_{\alpha}(t)N_{\alpha}(t)^{\dagger},\\ I_{2m}-M_{\beta}(t)^{\dagger}M_{\beta}(t) &= I_{2m}-M_{\alpha}(t)^{\dagger}M_{\alpha}(t).\end{aligned}$$

Finally, from $H_1F_1(t)G_1 = HF(t)G$, by (5), we infer that

$$\begin{aligned} S_2^{\beta}(t,D) &= \left(I_{2m}-N_{\alpha}(t)N_{\alpha}(t)^{\dagger}\right)\left(L(t)-HF(t)G\right)\left(I_{2m}-M_{\alpha}(t)^{\dagger}M_{\alpha}(t)\right)\\ &= S_2^{\alpha}(t,D). \qquad \square\end{aligned}$$

REMARK 2. Let us note that if $\alpha=(A,B,C)$ and $\beta=(U^*AU,U^*B,CU)$ are two triples of matrices of $L_{n,m}$ with U unitary, then 0 is a multiple eigenvalue of $M(\alpha,X)$ if and only if it is a multiple eigenvalue of $M(\beta,X)$. Hence, by the previous lemma, in the proof of Theorem 2 there is no loss of generality if we consider the triple of matrices β.

Now, we are going to apply Lemma 4 to compute $S_2^{\alpha}(t,D)$. As the matrix A is normal, let $U \in \mathbb{C}^{n\times n}$ be a unitary matrix such that

$$U^*AU = \begin{pmatrix} O & O \\ O & \Sigma \end{pmatrix},$$

where $\Sigma \in \mathbb{C}^{n_2\times n_2}, 1 \leqslant n_2 < n$, is a invertible diagonal matrix. So, it is understood that $A \neq O$; the case when $A = O$ will be considered later in Remark 4. Let us consider the partition $n = n_1 + n_2$ in block matrices:

$$\left(\begin{array}{c|c} U^*AU & U^*B \\ \hline CU & D \end{array}\right) = \left(\begin{array}{cc|c} O & O & B_1 \\ O & \Sigma & B_2 \\ \hline C_1 & C_2 & D \end{array}\right), \quad B_1 \in \mathbb{C}^{n_1\times m}, C_1 \in \mathbb{C}^{m\times n_1}. \tag{8}$$

By Lemma 4, $S_2^{\alpha}(t,D) = S_2^{\beta}(t,D)$, where $\beta := (U^*AU, U^*B, CU)$. We will compute $S_2^{\beta}(t,D)$ for $t > 0$.

First, let us call

$$F(t) := \begin{pmatrix} O & O & tI_{n_1} & O \\ O & \Sigma & O & tI_{n_2} \\ O & O & O & O \\ O & O & O & \Sigma \end{pmatrix};$$

therefore,

$$F(t)^\dagger = \begin{pmatrix} O & O & O & O \\ O & \Sigma^{-1} & O & -t\Sigma^{-2} \\ t^{-1}I_{n_1} & O & O & O \\ O & O & O & \Sigma^{-1} \end{pmatrix},$$

and

$$F(t)F(t)^\dagger = \begin{pmatrix} I_{n_1} & O & O & O \\ O & I_{n_2} & O & O \\ O & O & O & O \\ O & O & O & I_{n_2} \end{pmatrix}, \quad F(t)^\dagger F(t) = \begin{pmatrix} O & O & O & O \\ O & I_{n_2} & O & O \\ O & O & I_{n_1} & O \\ O & O & O & I_{n_2} \end{pmatrix}.$$

Hence, from (3) and (4),

$$M_\beta(t) = \begin{pmatrix} O & O \\ O & O \\ O & B_1 \\ O & O \end{pmatrix}, \quad N_\beta(t) = \begin{pmatrix} C_1 & O & O & O \\ O & O & O & O \end{pmatrix}.$$

Consequently,

$$I_{2m} - N_\beta(t)N_\beta(t)^\dagger = \begin{pmatrix} I_m - C_1C_1^\dagger & O \\ O & I_m \end{pmatrix}, \quad I_{2m} - M_\beta(t)^\dagger M_\beta(t) = \begin{pmatrix} I_m & O \\ O & I_m - B_1^\dagger B_1 \end{pmatrix}.$$

Last,

$$\begin{pmatrix} C_1 & C_2 & O & O \\ O & O & C_1 & C_2 \end{pmatrix} F(t)^\dagger \begin{pmatrix} B_1 & O \\ B_2 & O \\ O & B_1 \\ O & B_2 \end{pmatrix} = \begin{pmatrix} C_2\Sigma^{-1}B_2 & -tC_2\Sigma^{-2}B_2 \\ t^{-1}C_1B_1 & C_2\Sigma^{-1}B_2 \end{pmatrix},$$

From the three last equalities and (5) we deduce that for $t>0$,

$$S_2^{\beta}(t,D)=\begin{pmatrix}(I_m-C_1C_1^{\dagger})(D-C_2\Sigma^{-1}B_2) & t(I_m-C_1C_1^{\dagger})(I_m+C_2\Sigma^{-2}B_2)(I_m-B_1^{\dagger}B_1)\\ -t^{-1}C_1B_1 & (D-C_2\Sigma^{-1}B_2)(I_m-B_1^{\dagger}B_1)\end{pmatrix}.$$

Thus, by Lemma 4, it follows that for $t>0$

$$S_2^{\alpha}(t,D)=\begin{pmatrix}P_CL_1 & tP_CL_2P_B\\ -t^{-1}C_1B_1 & L_1P_B\end{pmatrix}, \tag{9}$$

where $P_C:=I_m-C_1C_1^{\dagger}$, $P_B:=I_m-B_1^{\dagger}B_1$, $L_1:=D-C_2\Sigma^{-1}B_2$ and $L_2:=I_m+C_2\Sigma^{-2}B_2$. However, from this point on, in order to simplify the demonstration, we only consider the expression of $S_2^{\alpha}(t,D)$ given in (9). Moreover, by Remark 2, we can assume the triple $\alpha=(A,B,C)$ is in the form (U^*AU,U^*B,CU) that was given in (8). From the definition of $p_{\alpha}(t)$ given in (2) we infer that

$$p_{\alpha}(t)=2m+n_1-2-\operatorname{rank}(B_1)-\operatorname{rank}(C_1)$$

for $0<t<\infty$.

From now on, we will abbreviate $S_2^{\alpha}(t,D)$ by $S_2(t)$. With these considerations, when $A\neq O$, Theorem 2 can be reformulated in the following way.

THEOREM 5. *Let* $\alpha=(A,B,C)\in L_{n,m}$ *be a triple of matrices*

$$A:=\begin{pmatrix}O & O\\ O & \Sigma\end{pmatrix},\quad B:=\begin{pmatrix}B_1\\ B_2\end{pmatrix},\quad C:=(C_1,C_2),$$

with $B_1\in\mathbb{C}^{n_1\times m}$, $C_1\in\mathbb{C}^{m\times n_1}$ *and* $\Sigma\in\mathbb{C}^{n_2\times n_2}$ *an invertible diagonal matrix,* $n_1\geqslant 1$. *Let us define*

$$h:=2m+n_1-1-\operatorname{rank}(B_1)-\operatorname{rank}(C_1). \tag{10}$$

Given $D\in\mathbb{C}^{m\times m}$. *For* $t>0$ *let us also define*

$$S_2(t):=\begin{pmatrix}P_CL_1 & tP_CL_2P_B\\ -t^{-1}C_1B_1 & L_1P_B\end{pmatrix}, \tag{11}$$

where

$$P_C:=I_m-C_1C_1^{\dagger},\quad P_B:=I_m-B_1^{\dagger}B_1, \tag{12}$$

and

$$L_1:=D-C_2\Sigma^{-1}B_2,\quad L_2:=I_m+C_2\Sigma^{-2}B_2. \tag{13}$$

Then

$$\sup_{t>0}\sigma_h(S_2(t))=\min_{\substack{X\in\mathbb{C}^{m\times m}\\ \mathrm{m}(0,M(\alpha,X))\geqslant 2}}\|X-D\|.$$

REMARK 3. Suppose there exists a $t_1 > 0$ such that $\sigma_h\big(S_2(t_1)\big) = 0$, equivalently $\operatorname{rank}\big(S_2(t_1)\big) \leqslant h-1$. By Section 4 of [8] we see that

$$\operatorname{rank}\begin{pmatrix} A & t_1I_n & B & O \\ O & A & O & B \\ C & O & D & t_1I_m \\ O & C & O & D \end{pmatrix} = \operatorname{rank}\left(\begin{array}{cc|cc} A & B & t_1I_n & O \\ C & D & O & t_1I_m \\ \hline O & O & A & B \\ O & O & C & D \end{array}\right)$$

$$= \rho_\alpha(t_1) + \operatorname{rank}\big(S_2(t_1)\big) \leqslant \rho_\alpha(t_1) + h - 1.$$

But, as

$$h - 1 = p_\alpha(t_1) = 2m + 2n - 2 - \rho_\alpha(t_1),$$

we infer that

$$\operatorname{rank}\left(\begin{array}{cc|cc} A & B & t_1I_n & O \\ C & D & O & t_1I_m \\ \hline O & O & A & B \\ O & O & C & D \end{array}\right) \leqslant 2m + 2n - 2.$$

As it can be seen in [12, pages 444–445], this inequality implies that 0 is a multiple eigenvalue of $M(\alpha, D)$. Thus, by Proposition 1, Theorem 5 is already proved in this case. Therefore, *from now on we will assume that* $\sigma_h\big(S_2(t)\big) > 0$ *for* $t > 0$.

REMARK 4. When the normal matrix A is the $n \times n$ zero matrix, the *statement* of Theorem 5 is reduced to

$$\sup_{t>0} \sigma_k \begin{pmatrix} P_C D & tP_C P_B \\ -t^{-1}CB & DP_B \end{pmatrix} = \min_{\substack{X \in \mathbb{C}^{m\times m} \\ \mathrm{m}(0, M(\alpha, X)) \geqslant 2}} \|X - D\|, \tag{14}$$

where $k := 2m + n - 1 - \operatorname{rank}(B) - \operatorname{rank}(C)$. The proof of (14) might be done following similar reasoning to the $A \neq O$ case, replacing B_1 by B, C_1 by C, L_1 by D, L_2 by I_m, and removing Σ, B_2 and C_2.

3. Auxiliary results

In this section, we are going to introduce some results that will be used in this work. In the first one, we give some properties of the Moore-Penrose inverse, which can be seen in [2, Proposition 6.1.6, page 225; Fact 6.4.8, page 235] and [3].

LEMMA 6. *Let* $A \in \mathbb{C}^{p\times q}$ *be a matrix. Then*
(1) $\operatorname{Ker}(I_p - AA^\dagger) = \operatorname{Im}(A)$, $\operatorname{Im}(I_p - AA^\dagger) = \operatorname{Ker}(A^*) = \operatorname{Ker}(A^\dagger)$.
(2) $\operatorname{Ker}(I_q - A^\dagger A) = \operatorname{Im}(A^*) = \operatorname{Im}(A^\dagger)$, $\operatorname{Im}(I_q - A^\dagger A) = \operatorname{Ker}(A)$.
(3) $x \in \operatorname{Im}(A)$ *if and only if* $x = AA^\dagger x$; $x \in \operatorname{Im}(A^*)$ *if and only if* $x^* = x^*A^\dagger A$.
(4) *If* $\operatorname{rank}(A) = p$, *then* $AA^\dagger = I_p$; *if* $\operatorname{rank}(A) = q$, *then* $A^\dagger A = I_q$.
(5) *Let* $F \in \mathbb{C}^{q\times r}$. *If* $\operatorname{rank}(A) = \operatorname{rank}(F) = q$, *then* $(AF)^\dagger = F^\dagger A^\dagger$.

With this lemma and Lemma 3, we get the following properties for the matrices $P_C = (I_m - C_1C_1^\dagger)$ and $P_B = (I_m - B_1^\dagger B_1)$, defined in (12).

LEMMA 7. (1) *P_C and P_B are orthogonal projectors.*
(2) $\operatorname{Ker}(P_C) = \operatorname{Im}(C_1)$, $\operatorname{Im}(P_C) = \operatorname{Ker}(C_1^*) = \operatorname{Ker}(C_1^\dagger)$.
(3) $\operatorname{Ker}(P_B) = \operatorname{Im}(B_1^*) = \operatorname{Im}(B_1^\dagger)$, $\operatorname{Im}(P_B) = \operatorname{Ker}(B_1)$.
(4) *If* $\operatorname{rank}(C_1) = \operatorname{rank}(B_1) = n_1$, *then* $(C_1B_1)^\dagger = B_1^\dagger C_1^\dagger$.

We will need in Sections 6 and 8 the following lemma.

LEMMA 8. ([8], Lemma 33) *Let $\{t_k\}_{k=1}^\infty$ be a sequence of real numbers which tends to ∞ when $k \to \infty$. Let $G \in \mathbb{C}^{p\times p}$ be a matrix and let $x_k, y_k \in \mathbb{C}^{p\times 1}$, $k = 1,2,\dots$ be vector sequences such that*

(i) $\lim_{k\to\infty} Gy_k = 0$,

(ii) $\sup_{k=1,2,\dots} \|t_k(x_k)^*G\| \leqslant T < \infty$, *where T is a positive constant.*

Then

$$\lim_{k\to\infty} t_k(x_k)^*Gy_k = 0.$$

With respect to the asymptotic behavior of the eigenvalues of matrix functions, we have the following result.

LEMMA 9. ([11], Lemma 5) *Let $F(t) = G(t) + t^{-1}H \in \mathbb{C}^{p\times p}$ where $G(t)$ is a Hermitian matrix function analytic on an open interval $J \subset \mathbb{R}$ around 0, and H is a constant Hermitian matrix such that* $\operatorname{rank}(H) = r$. *Assume that H has a spectral decomposition*

$$H = (V_1,V_2)\begin{pmatrix}\Lambda_r & O\\ O & O\end{pmatrix}(V_1,V_2)^*,$$

with unitary $V = (V_1,V_2)$ and $\Lambda_r \in \mathbb{R}^{r\times r}$ is a diagonal matrix with nonzero diagonal entries. Then as t approaches 0, r eigenvalues of $F(t)$ tend in absolute value to ∞, and the rest to the eigenvalues of $V_2^*G(0)V_2$.

Hence we will deduce the following result for singular values, which will be used in Section 4.

LEMMA 10. *Let $K(t) = L(t) + t^{-1}M \in \mathbb{C}^{p\times p}$, where $L(t)$ is an analytic matrix function on an open interval $J \subset \mathbb{R}$ around 0 and* $\operatorname{rank}(M) = s$. *Consider the singular value decomposition of M*

$$M = (P_1,P_2)\begin{pmatrix}\Sigma_s & O\\ O & O\end{pmatrix}(Q_1,Q_2)^*,$$

with unitary (P_1,P_2) and (Q_1,Q_2) and $\Sigma_s \in \mathbb{R}^{s\times s}$. Then as t approaches 0, s singular values of $K(t)$ tend to ∞, and the rest to the singular values of the matrix $P_2^*L(0)Q_2$.

Proof. Observe that in the matrix function, valued in $\mathbb{C}^{2p\times 2p}$,

$$N(t) = \begin{pmatrix} O & K(t) \\ K^*(t) & O \end{pmatrix} = \begin{pmatrix} O & L(t) \\ L^*(t) & O \end{pmatrix} + t^{-1}\begin{pmatrix} O & M \\ M^* & O \end{pmatrix} = R(t) + t^{-1}S,$$

the matrices $R(t)$ and S are Hermitian, $R(t)$ is analytic around 0 and $\operatorname{rank}(S) = 2s$. Let us note that, by the Jordan-Wielandt lemma [13, Theorem 4.2], the eigenvalues of $N(t)$ are

$$\pm\sigma_1(K(t)),\ldots,\pm\sigma_p(K(t)).$$

Consider the unitary matrix $(V_1,V_2)\in\mathbb{C}^{2p\times 2p}$, with

$$V_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & P_1 \\ Q_1 & -Q_1 \end{pmatrix} \in \mathbb{C}^{2p\times 2s}, \quad V_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} P_2 & P_2 \\ Q_2 & -Q_2 \end{pmatrix} \in \mathbb{C}^{2p\times 2(p-s)}.$$

Then

$$S = (V_1,V_2)\begin{pmatrix} \Sigma_s & O & O \\ O & -\Sigma_s & O \\ O & O & O \end{pmatrix}(V_1,V_2)^*,$$

is a spectral decomposition of S. Hence, by Lemma 9 as $t\to 0$ we deduce that $2s$ eigenvalues of $N(t)$ tend in absolute value to ∞; and the rest to the eigenvalues of the matrix

$$V_2^*R(0)V_2 = \frac{1}{2}\begin{pmatrix} Q_2^*L^*(0)P_2 + P_2^*L(0)Q_2 & Q_2^*L^*(0)P_2 - P_2^*L(0)Q_2 \\ -Q_2^*L^*(0)P_2 + P_2^*L(0)Q_2 & -Q_2^*L^*(0)P_2 - P_2^*L(0)Q_2 \end{pmatrix}.$$

Taking the unitary matrix

$$X = \frac{1}{\sqrt{2}}\begin{pmatrix} -I & -I \\ -I & I \end{pmatrix},$$

we deduce that the eigenvalues of $V_2^*R(0)V_2$ are the eigenvalues of

$$X^*V_2^*R(0)V_2X = \begin{pmatrix} O & P_2^*L(0)Q_2 \\ (P_2^*L(0)Q_2)^* & O \end{pmatrix},$$

that is, $\pm\sigma_1(P_2^*L(0)Q_2),\ldots,\pm\sigma_{p-s}(P_2^*L(0)Q_2)$. $\square$

To conclude this section, we give some results about the singular values of matrix functions of a real variable. The first one can be seen in [10, Theorem 4.3.17, page 442 and Corollary 4.3.20, page 443].

LEMMA 11. *Let $F(t)\in\mathbb{C}^{q\times q}$ be an analytic matrix function on an open set $\Omega\subset\mathbb{R}$. Then, there exist unitary matrix functions $U(t),V(t)$ and a diagonal matrix function $\Sigma(t) = \operatorname{diag}(\tilde\sigma_1(t),\tilde\sigma_2(t),\ldots,\tilde\sigma_p(t))\in\mathbb{R}^{q\times q}$, all of which are analytic on Ω, such that for $t\in\Omega$,*

$$U(t)^*F(t)V(t) = \Sigma(t).$$

Moreover

$$\tilde\sigma_i'(t) = \operatorname{Re}\left(u_i^*(t)F'(t)v_i(t)\right).$$

Another result, which will be used in Section 5, is the following one [10, Proposition 4.3.21, page 443].

LEMMA 12. *Let Ω be an open subset of $\mathbb{R}$ and $F:\Omega\to\mathbb{C}^{m\times n}$ be an analytic matrix function on Ω. If the function $\sigma_i\big(F(t)\big)$ has a positive local maximum (or minimum) at $t_0\in\Omega$, then there exist a pair of singular vectors $u\in\mathbb{C}^{m\times 1}$, $v\in\mathbb{C}^{n\times 1}$ of $F(t_0)$ corresponding to $\sigma_i\big(F(t_0)\big)$ such that*

$$\operatorname{Re}\big(u^*F'(t_0)v\big)=0.$$

4. Asymptotic behavior of the singular values

In this section, we analyze the asymptotic behavior of the singular values of the matrix function $S_2(t)$ defined in (11), both when $t\to 0^+$ and $t\to\infty$. We start with the $t\to 0^+$ case.

LEMMA 13. *Let $S_2(t)$ be the matrix function in (11), and assume that $s=\operatorname{rank}(C_1B_1)$. Then as $t\to 0^+$, the first s singular values of $S_2(t)$ tend to ∞ and the remaining $2m-s$ ones satisfy*

$$\lim_{t\to 0^+}\sigma_{s+k}\left(S_2(t)\right)=\sigma_k\begin{pmatrix} P_CL_1(I_m-(C_1B_1)^{\dagger}C_1B_1) & O\\ O & (I_m-C_1B_1(C_1B_1)^{\dagger})L_1P_B\end{pmatrix}.$$

for $k=1,\dots,2m-s$. If $\operatorname{rank}(B_1)=\operatorname{rank}(C_1)=n_1$, then as $t\to 0^+$, the first n_1 singular values of $S_2(t)$ tend to ∞, and the remaining $2m-n_1$ ones satisfy

$$\lim_{t\to 0^+}\sigma_{n_1+k}\left(S_2(t)\right)=\sigma_k\begin{pmatrix} P_CL_1P_B & O\\ O & P_CL_1P_B\end{pmatrix}\ \textit{for } k=1,\dots,2m-n_1.$$

REMARK 5. Note that the block $P_CL_1P_B$ in the last matrix is repeated.

Proof. First, by (11), we have

$$S_2(t)=\begin{pmatrix} P_CL_1 & tP_CL_2P_B\\ O & L_1P_B\end{pmatrix}+t^{-1}\begin{pmatrix} O & O\\ -C_1B_1 & O\end{pmatrix}=L(t)+t^{-1}M,$$

with $L(t)$ analytic in a neighborhood of 0 and $\operatorname{rank}(M)=\operatorname{rank}(C_1B_1)=s$. Let (U_1,U_2), (V_1,V_2) be unitary matrices of $\mathbb{C}^{m\times m}$, with $U_2,V_2\in\mathbb{C}^{m\times(m-s)}$, such that

$$(U_1,U_2)^*(-C_1B_1)(V_1,V_2)=\begin{pmatrix}\Sigma_s & O\\ O & O\end{pmatrix}, \tag{15}$$

with $\Sigma_s\in\mathbb{R}^{s\times s}$, gives us the singular value decomposition of $-C_1B_1$. Therefore, considering the unitary matrices

$$P:=\begin{pmatrix} O & O & I_m\\ U_1 & U_2 & O\end{pmatrix},\quad Q:=\begin{pmatrix} V_1 & V_2 & O\\ O & O & I_m\end{pmatrix},$$

we deduce that

$$P^*MQ = \begin{pmatrix} \Sigma_s & O \\ O & O \end{pmatrix}.$$

Calling

$$P_2 := \begin{pmatrix} O & I_m \\ U_2 & O \end{pmatrix}, \quad Q_2 := \begin{pmatrix} V_2 & O \\ O & I_m \end{pmatrix},$$

by Lemma 10 we see that when $t \to 0^+$, the first s singular values of $S_2(t)$ tend to ∞, and the rest to the singular values of

$$P_2^* L(0) Q_2 = \begin{pmatrix} O & U_2^* L_1 P_B \\ P_C L_1 V_2 & O \end{pmatrix}.$$

Hence, for $k = 1, 2, \ldots, 2m - s$,

$$\lim_{t \to 0^+} \sigma_{s+k}\left(S_2(t)\right) = \sigma_k \begin{pmatrix} P_C L_1 V_2 & O \\ O & U_2^* L_1 P_B \end{pmatrix}. \tag{16}$$

By (15), $-C_1 B_1 V_1 = U_1 \Sigma_s$ and $-(C_1 B_1)^* U_1 = V_1 \Sigma_s$, from Lemma 6(1)(2) we get first

$$\begin{aligned} U_1 = -C_1 B_1 V_1 \Sigma_s^{-1} &\Rightarrow \operatorname{Im}(U_1) \subset \operatorname{Im}(C_1 B_1) = \operatorname{Ker}(I_m - C_1 B_1 (C_1 B_1)^\dagger), \\ V_1 = -(C_1 B_1)^* U_1 \Sigma_s^{-1} &\Rightarrow \operatorname{Im}(V_1) \subset \operatorname{Im}((C_1 B_1)^*) = \operatorname{Ker}(I_m - (C_1 B_1)^\dagger C_1 B_1). \end{aligned}$$

But, given that $I_m - C_1 B_1 (C_1 B_1)^\dagger$ and $I_m - (C_1 B_1)^\dagger C_1 B_1$ are orthogonal projectors in virtue of Lemma 3(1), we infer that

$$U_1^* (I_m - C_1 B_1 (C_1 B_1)^\dagger) = O, \quad (I_m - (C_1 B_1)^\dagger C_1 B_1) V_1 = O. \tag{17}$$

Similarly, from (15) and Lemma 6(1)(2), we see that

$$\begin{aligned} (C_1 B_1)^* U_2 = O &\Rightarrow \operatorname{Im}(U_2) \subset \operatorname{Ker}((C_1 B_1)^*) = \operatorname{Im}(I_m - C_1 B_1 (C_1 B_1)^\dagger), \\ C_1 B_1 V_2 = O &\Rightarrow \operatorname{Im}(V_2) \subset \operatorname{Ker}(C_1 B_1) = \operatorname{Im}(I_m - (C_1 B_1)^\dagger C_1 B_1). \end{aligned}$$

Thus

$$U_2^* (I_m - C_1 B_1 (C_1 B_1)^\dagger) = U_2^*, \quad (I_m - (C_1 B_1)^\dagger C_1 B_1) V_2 = V_2.$$

Consequently, with (17) and these two last equations, we deduce that

$$\begin{aligned} (O, P_C L_1 V_2) &= P_C L_1 (I_m - (C_1 B_1)^\dagger C_1 B_1)(V_1, V_2), \\ \begin{pmatrix} O \\ U_2^* L_1 P_B \end{pmatrix} &= \begin{pmatrix} U_1^* \\ U_2^* \end{pmatrix} (I_m - C_1 B_1 (C_1 B_1)^\dagger) L_1 P_B. \end{aligned}$$

Substituting these equations in (16) we prove the lemma in the first case.

To prove the lemma in the $\operatorname{rank}(B_1) = \operatorname{rank}(C_1) = n_1$ case, as $s = \operatorname{rank}(C_1 B_1) = n_1$, it is sufficient to see that $I_m - C_1 B_1 (C_1 B_1)^\dagger = P_C$, $I_m - (C_1 B_1)^\dagger C_1 B_1 = P_B$. In fact, given that $\operatorname{rank}(B_1) = \operatorname{rank}(C_1) = n_1$, then $C_1^\dagger C_1 = B_1 B_1^\dagger = I_{n_1}$ by Lemma 6(4). And $(C_1B_1)^\dagger = B_1^\dagger C_1^\dagger$ by Lemma 7(4). Hence

$$I_m - C_1B_1(C_1B_1)^\dagger = I_m - C_1B_1B_1^\dagger C_1^\dagger = I_m - C_1C_1^\dagger = P_C,$$
$$I_m - (C_1B_1)^\dagger C_1B_1 = I_m - B_1^\dagger C_1^\dagger C_1 B_1 = I_m - B_1^\dagger B_1 = P_B. \qquad \square$$

For the $t \to \infty$ case, we have the following result.

LEMMA 14. *Let $S_2(t)$ be the matrix function in* (11)*. Let us call $L := P_C L_2 P_B$, and assume that $\ell = \operatorname{rank}(P_C L_2 P_B)$. Then as $t \to \infty$, the first ℓ singular values of $S_2(t)$ tend to ∞, and the remaining $2m - \ell$ ones satisfy*

$$\lim_{t\to\infty} \sigma_{\ell+k}(S_2(t)) = \sigma_k \begin{pmatrix} P_C(I_m - LL^\dagger)L_1 & O \\ O & L_1(I_m - L^\dagger L)P_B \end{pmatrix},$$

for $k = 1, \ldots, 2m - \ell$.

REMARK 6. Let us note that the matrix in the right hand side is $2m \times 2m$.

Proof. Let $(U_1, U_2), (V_1, V_2)$ be unitary matrices of $\mathbb{C}^{m\times m}$, with $U_2, V_2 \in \mathbb{C}^{m\times(m-\ell)}$, that perform the singular value decomposition of L

$$(U_1,U_2)^* L (V_1,V_2) = \begin{pmatrix} \Sigma_\ell & O \\ O & O \end{pmatrix}, \tag{18}$$

with $\Sigma_\ell \in \mathbb{R}^{\ell\times\ell}$. Applying a similar reasoning to the one of the previous lemma for the matrix function

$$\hat{S}_2(t) = \begin{pmatrix} P_C L_1 & t^{-1}L \\ -tC_1B_1 & L_1 P_B \end{pmatrix},$$

we find that as $t \to \infty$, the first ℓ singular values of $S_2(t)$ tend to ∞, and the remaining $2m - \ell$ ones satisfy

$$\lim_{t\to\infty} \sigma_{\ell+k}(S_2(t)) = \sigma_k \begin{pmatrix} U_2^* P_C L_1 & O \\ O & L_1 P_B V_2 \end{pmatrix} \text{ for } k = 1, \ldots, 2m-\ell. \tag{19}$$

Let us note that as P_C and P_B are orthogonal projectors, then $P_C L = L$ and $P_B L^* = L^*$. Hence, from (18) and by Lemma 6(1)(2), we obtain first

$$P_C U_1 \Sigma_\ell = P_C L V_1 = L V_1 \Rightarrow \operatorname{Im}(P_C U_1) \subset \operatorname{Im} L = \operatorname{Ker}(I_m - LL^\dagger),$$
$$P_B V_1 \Sigma_\ell = P_B L^* U_1 = L^* U_1 \Rightarrow \operatorname{Im}(P_B V_1) \subset \operatorname{Im} L^* = \operatorname{Ker}(I_m - L^\dagger L).$$

Therefore by Lemma 3(1) $I_m - LL^\dagger$ and $I_m - L^\dagger L$ are orthogonal projectors, then

$$U_1^* P_C (I_m - LL^\dagger) = O, \quad (I_m - L^\dagger L) P_B V_1 = O. \tag{20}$$

Similarly, from (18) and Lemma 6(1)(2), we have

$$O = U_2^* L = U_2^* P_C L \text{ and } O = L V_2 = L P_B V_2.$$

Thus

$$U_2^* P_C (I_m - LL^\dagger) = U_2^* P_C, \quad (I_m - L^\dagger L) P_B V_2 = P_B V_2.$$

Substituting these two last equalities and (20) in (19) we have proved the lemma. $\square$

REMARK 7. Taking into account the expression for h given in (10), Lemmas 13 and 14, and Proposition 1, we conclude that if

$$2m + n_1 - 1 - \operatorname{rank}(B_1) - \operatorname{rank}(C_1) \leqslant \max\{\operatorname{rank}(C_1 B_1), \operatorname{rank}(P_C L_2 P_B)\}, \tag{21}$$

then $\sup_{t>0} \sigma_h (S_2(t)) = \infty$; that is, there is no matrix $X \in \mathbb{C}^{m\times m}$ such that 0 is a multiple eigenvalue of $M(\alpha, X)$. Consequently, Theorem 5 is proved in this case. It can be demonstrated that inequality (21) is equivalent to

$$\operatorname{rank}(B_1) = \operatorname{rank}(C_1) = m \text{ and } (m = n_1 \text{ or } m = n_1 - 1).$$

Therefore, *from here on we will assume that*

$$2m + n_1 - 1 - \operatorname{rank}(B_1) - \operatorname{rank}(C_1) > \max\{\operatorname{rank}(C_1 B_1), \operatorname{rank}(P_C L_2 P_B)\}.$$

REMARK 8. Given Theorem 5, we can assert that

$$\min_{\substack{X \in \mathbb{C}^{m\times m} \\ \mathrm{m}(0, M(\alpha, X)) \geqslant 2}} \|X - D\| = \infty$$

if and only if inequality (21) is satisfied.

In the next section the proof of Theorem 5 *starts and continues until the end of Section* 8.

5. When the supremum is a maximum

Given $t_0 \neq 0$, in agreement with the notations (10) and (11), let us call

$$\sigma_0 := \sigma_h (S_2(t_0)),$$

where we assume $\sigma_0 > 0$. Let

$$u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad v := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{22}$$

be a pair of singular vectors of $S_2(t_0)$ associated with σ_0, where $u_1, u_2, v_1, v_2 \in \mathbb{C}^{m\times 1}$.

Using [12, Section 4] and [7, Section 4] we will establish some properties of u, v. First, as $S_2(t_0)v = \sigma_0 u$ and $S_2(t_0)^* u = \sigma_0 v$, from (11) and (22) we get

$$P_C L_1 v_1 + t_0 P_C L_2 P_B v_2 = \sigma_0 u_1, \text{ and } t_0 P_B L_2^* P_C u_1 + P_B L_1^* u_2 = \sigma_0 v_2.$$

Hence, as $\sigma_0 > 0$, from the two previous equalities we deduce that $u_1 \in \mathrm{Im}(P_C)$ and $v_2 \in \mathrm{Im}(P_B)$. Thus, by Lemma 7(2)(3) we have $C_1^* u_1 = 0, P_C u_1 = u_1$ and $B_1 v_2 = 0, P_B v_2 = v_2$. Theses equalities jointly with $S_2(t_0)v = \sigma_0 u$ and $S_2(t_0)^* u = \sigma_0 v$, imply the following equations.

$$P_C L_1 v_1 + t_0 P_C L_2 v_2 = \sigma_0 u_1, \tag{23}$$
$$-t_0^{-1} C_1 B_1 v_1 + L_1 v_2 = \sigma_0 u_2, \tag{24}$$
$$L_1^* u_1 - t_0^{-1} B_1^* C_1^* u_2 = \sigma_0 v_1, \tag{25}$$
$$t_0 P_B L_2^* u_1 + P_B L_1^* u_2 = \sigma_0 v_2, \tag{26}$$
$$C_1^* u_1 = C_1^\dagger u_1 = 0, \tag{27}$$
$$B_1 v_2 = 0, \tag{28}$$
$$P_C u_1 = u_1, \tag{29}$$
$$P_B v_2 = v_2. \tag{30}$$

Substituting (29) in (23) we see that $P_C(L_1 v_1 + t_0 L_2 v_2 - \sigma_0 u_1) = 0$. Therefore from Lemma 7(2) we have $L_1 v_1 + t_0 L_2 v_2 - \sigma_0 u_1 \in \mathrm{Im}(C_1)$. Consequently by Lemma 6(3),

$$C_1 C_1^\dagger (L_1 v_1 + t_0 L_2 v_2 - \sigma_0 u_1) = L_1 v_1 + t_0 L_2 v_2 - \sigma_0 u_1. \tag{31}$$

Multiplying to the right equations (23)–(26) by $u_1^*, u_2^*, v_1^*, v_2^*$ respectively, conjugating (29) and (30), *i.e.*, $u_1^* P_C = u_1^*$, $v_2^* P_B = v_2^*$, we conclude that

$$\begin{aligned}
&u_1^* L_1 v_1 + t_0 u_1^* L_2 v_2 = \sigma_0 u_1^* u_1,\\
&-t_0^{-1} u_2^* C_1 B_1 v_1 + u_2^* L_1 v_2 = \sigma_0 u_2^* u_2,\\
&v_1^* L_1^* u_1 - t_0^{-1} v_1^* B_1^* C_1^* u_2 = \sigma_0 v_1^* v_1,\\
&t_0 v_2^* L_2^* u_1 + v_2^* L_1^* u_2 = \sigma_0 v_2^* v_2.
\end{aligned}$$

Subtracting the conjugate of the third equation from the first one and the conjugate of the fourth equation from the second one, we conclude that

$$\sigma_0(u_1^* u_1 - v_1^* v_1) = t_0 u_1^* L_2 v_2 + t_0^{-1} u_2^* C_1 B_1 v_1 = -\sigma_0(u_2^* u_2 - v_2^* v_2). \tag{32}$$

Multiplying (24) and (25) by u_1^* and v_2^* from the right-hand side, respectively and using $u_1^* C_1 = 0$ (27) and $B_1 v_2 = 0$ (28), we obtain

$$u_1^* L_1 v_2 = \sigma_0 u_1^* u_2, \quad v_2^* L_1^* u_1 = \sigma_0 v_2^* v_1.$$

Hence, subtracting the conjugate of the second equation from the first one, we see that $\sigma_0(u_1^* u_2 - v_1^* v_2) = 0$. As $\sigma_0 \neq 0$, we infer that

$$u_1^* u_2 = v_1^* v_2. \tag{33}$$

REMARK 9. Note that equations (23)–(33) remain valid for each pair of singular vectors associated with a nonzero singular value of $S_2(t)$ for $t \neq 0$. This remark will be important in Sections 6 and 8.

Now assume that $\sigma_h(S_2(t))$ attains a relative extremum $\sigma_0 := \sigma_h(S_2(t_0)) > 0$ at $t_0 \neq 0$. Then, by Lemma 12, there exists a pair of singular vectors u, v of $S_2(t_0)$ corresponding to $\sigma_h(S_2(t_0))$ such that

$$\operatorname{Re}\left(u^* S_2'(t_0) v\right) = \operatorname{Re}\left(u^* \begin{pmatrix} O & P_C L_2 P_B \\ t_0^{-2} C_1 B_1 & O \end{pmatrix} v\right) = 0.$$

Partitioning the vectors u, v according (22), we have

$$\operatorname{Re}(t_0^{-2} u_2^* C_1 B_1 v_1 + u_1^* P_C L_2 P_B v_2) = 0.$$

Since $t_0 \neq 0$ and $u_1^* P_C L_2 P_B v_2 = u_1^* L_2 v_2$ (by (29) and (30)), we deduce that $\operatorname{Re}(t_0^{-1} u_2^* C_1 B_1 v_1 + t_0 u_1^* L_2 v_2) = 0$. Hence, from (32), we see that

$$u_1^* u_1 = v_1^* v_1, \quad u_2^* u_2 = v_2^* v_2. \tag{34}$$

Now let us define the matrices

$$V := [v_1, v_2] \in \mathbb{C}^{m\times 2}, \quad U := [u_1, u_2] \in \mathbb{C}^{m\times 2}.$$

By (33) and (34), we have $V^*V = U^*U$. Hence, the matrix

$$D_0 := D - \sigma_0 U V^\dagger,$$

satisfies $\|D - D_0\| = \sigma_0$ and

$$D_0 V = DV - \sigma_0 U, \quad U^* D_0 = U^* D - \sigma_0 V^*. \tag{35}$$

(see [8], page 1208, (35)) Consequently, to prove Theorem 5 in this case, it suffices to prove that 0 is a multiple eigenvalue of the matrix $M(\alpha, D_0)$.

Since $\operatorname{rank}(V^*V) \geqslant 1$, we have two possibilities: $\operatorname{rank} V = 1$ or $\operatorname{rank} V = 2$. In the $\operatorname{rank} V = 1$ case, we will analyze the subcases when $v_2 \neq 0$ and when $v_2 = 0$.

5.1. $\operatorname{rank} V = 2$

Note that $\operatorname{rank} V = 2$ implies that v_1 and v_2 are linearly independent. Hence, to prove that 0 is a multiple eigenvalue of $M(\alpha, D_0)$ it suffices to see that

$$\begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} \begin{pmatrix} z_2 & z_1 \\ w_2 & w_1 \\ v_2 & v_1 \end{pmatrix} = \begin{pmatrix} z_2 & z_1 \\ w_2 & w_1 \\ v_2 & v_1 \end{pmatrix} \begin{pmatrix} 0 & -t_0 \\ 0 & 0 \end{pmatrix},$$

with

$$\begin{aligned} z_2 &= -t_0^{-1} B_1 v_1, \quad z_1 = -C_1^\dagger (L_1 v_1 + t_0 L_2 v_2 - \sigma_0 u_1), \\ w_2 &= -\Sigma^{-1} B_2 v_2, \; w_1 = t_0 \Sigma^{-2} B_2 v_2 - \Sigma^{-1} B_2 v_1. \end{aligned}$$

By $B_1 v_2 = 0$ (28) and $D_0 v_i = D v_i - \sigma_0 u_i$ for $i = 1, 2$ (35), the problem reduces to verifying the equalities

$$-t_0^{-1} C_1 B_1 v_1 - C_2 \Sigma^{-1} B_2 v_2 + D v_2 = \sigma_0 u_2,$$
$$-C_1 C_1^\dagger (L_1 v_1 + t_0 L_2 v_2 - \sigma_0 u_1) + t_0 C_2 \Sigma^{-2} B_2 v_2 - C_2 \Sigma^{-1} B_2 v_1 + D v_1 - \sigma_0 u_1 = -t_0 v_2.$$

By (13) we have $L_1 = D - C_2\Sigma^{-1}B_2$ and $L_2 = I_m + C_2\Sigma^{-2}B_2$, the two previous equalities are reduced to

$$-t_0^{-1}C_1B_1v_1 + L_2v_2 = \sigma_0 u_2,$$
$$-C_1C_1^{\dagger}(L_1v_1 + t_0L_2v_2 - \sigma_0u_1) + L_1v_1 + t_0L_2v_2 - \sigma_0u_1 = 0,$$

which are true by (24) and (31), respectively.

5.2. $\operatorname{rank} V = 1$ **and** $v_2 \neq 0$

Observe that in this case $v_1 = \lambda v_2$ and $u_1 = \lambda u_2$, for some $\lambda \in \mathbb{C}$. Hence, as $v_2 \neq 0$, to prove that 0 is a multiple eigenvalue of $M(\alpha, D_0)$ it suffices to find a vector $w \in \mathbb{C}^{n_1\times 1}$ such that

$$\begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix}\begin{pmatrix} 0 & w \\ -\Sigma^{-1}B_2v_2 & -\Sigma^{-2}B_2v_2 \\ v_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & w \\ -\Sigma^{-1}B_2v_2 & -\Sigma^{-2}B_2v_2 \\ v_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{36}$$

because this means that the columns of the matrix

$$\begin{pmatrix} 0 & w \\ -\Sigma^{-1}B_2v_2 & -\Sigma^{-2}B_2v_2 \\ v_2 & 0 \end{pmatrix}$$

form a Jordan chain of 0 as eigenvalue of $M(\alpha, D_0)$.

Multiplying the matrices in (36), we have

$$\begin{pmatrix} B_1v_2 & 0 \\ -B_2v_2 + B_2v_2 & -\Sigma^{-1}B_2v_2 \\ -C_2\Sigma^{-1}B_2v_2 + D_0v_2 & C_1w - C_2\Sigma^{-2}B_2v_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\Sigma^{-1}B_2v_2 \\ 0 & v_2 \end{pmatrix}. \tag{37}$$

By (28) $B_1v_2 = 0$, so the $(1,1)$-entries in (37) are equal. By (35) $D_0v_2 = Dv_2 - \sigma_0u_2$, hence, by the definition of L_1,

$$-C_2\Sigma^{-1}B_2v_2 + D_0v_2 = Dv_2 - C_2\Sigma^{-1}B_2v_2 - \sigma_0u_2 = L_1v_2 - \sigma_0u_2.$$

As $v_1 = \lambda v_2$ and $B_1v_2 = 0$, then $B_1v_1 = 0$. From (24), $L_1v_2 = \sigma_0u_2$, thus the $(3,1)$-entries in (37) are equal. Equating the $(3,2)$-entries, and by the definition of L_2, we have

$$C_1w - C_2\Sigma^{-2}B_2v_2 = v_2, \quad C_1w = v_2 + C_2\Sigma^{-2}B_2v_2, \quad C_1w = L_2v_2.$$

Thus the vector w must satisfy $C_1w = L_2v_2$. This vector exists if and only if $L_2v_2 \in \operatorname{Im} C_1 = \operatorname{Ker} P_C$ by Lemma 7(2).

As $B_1v_1 = 0$, $u_1 = \lambda u_2$, $v_1 = \lambda v_2$, from (23) and (29) we have

$$\lambda P_CL_1v_2 + t_0P_CL_2v_2 = \lambda\sigma_0u_2;$$

since $L_1v_2 = \sigma_0u_2$, we obtain $t_0P_CL_2v_2 = 0$. But $t_0 \neq 0$, so $P_CL_2v_2 = 0$. Therefore, $L_2v_2 \in \operatorname{Ker} P_C$.

5.3. $\operatorname{rank} V = 1$ **and** $v_2 = 0$

As $v_2 = 0$, then $u_2 = 0$, $v_1 \neq 0$ and $u_1 \neq 0$. Hence, to prove that 0 is a multiple eigenvalue of $M(\alpha, D_0)$ it suffices to find a vector $w \in \mathbb{C}^{n_1 \times 1}$, such that

$$\begin{pmatrix} 0 & -u_1^* C_2 \Sigma^{-1} & u_1^* \\ w^* & -u_1^* C_2 \Sigma^{-2} & 0 \end{pmatrix} \begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -u_1^* C_2 \Sigma^{-1} & u_1^* \\ w^* & -u_1^* C_2 \Sigma^{-2} & 0 \end{pmatrix}. \tag{38}$$

This means that the vectors

$$\begin{pmatrix} w \\ -\Sigma^{-2} C_2^* u_1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -\Sigma^{-1} C_2^* u_1 \\ u_1 \end{pmatrix}$$

form a Jordan chain on the left of 0 as an eigenvalue of $M(\alpha, D_0)$. Multiplying the matrices in (38), we will have to prove the following equality.

$$\begin{pmatrix} u_1^* C_1 & -u_1^* C_2 + u_1^* C_2 & -u_1^* C_2 \Sigma^{-1} B_2 + u_1^* D_0 \\ 0 & -u_1^* C_2 \Sigma^{-1} & w^* B_1 - u_1^* C_2 \Sigma^{-2} B_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -u_1^* C_2 \Sigma^{-1} & u_1^* \end{pmatrix}.$$

The $(1,1)$-entries are equal, because $u_1^* C_1 = 0$ by (27). Let us see the reasons of the equality of the $(1,3)$-entries. By (35), $u_1^* D_0 = u_1^* D - \sigma_0 v_1^*$. So,

$$-u_1^* C_2 \Sigma^{-1} B_2 + u_1^* D_0 = 0 \Longleftrightarrow -u_1^* C_2 \Sigma^{-1} B_2 + u_1^* D = \sigma_0 v_1^*.$$

By the definition of L_1, the last equality is equivalent to $u_1^* L_1 = \sigma_0 v_1^*$. As $u_2 = 0$, (25) implies $L_1^* u_1 = \sigma_0 v_1^*$. Finally to prove the equality of the $(2,3)$-entries, we construct a vector w such that

$$w^* B_1 - u_1^* C_2 \Sigma^{-2} B_2 = u_1^*.$$

By the definition of L_2, the vector w must satisfy $w^* B_1 = u_1^* L_2$; that is $B_1^* w = L_2^* u_1$. Such a w exists if and only if $L_2^* u_1 \in \operatorname{Im} B_1^* = \operatorname{Ker} P_B$, by Lemma 7(3). Since $u_2 = v_2 = 0$, $t_0 \neq 0$, and (26), $P_B L_2^* u_1 = 0$.

REMARK 10. We have proved Theorem 5 when the function $t \mapsto \sigma_h(S_2(t))$ has a positive local extremum at a point $t_0 \neq 0$. Note that if for a positive integer q we have $\sigma_{h+q}(S_2(t)) \neq 0$, for $t \neq 0$, we can apply the same reasoning to the function $t \mapsto \sigma_{h+q}(S_2(t))$. Therefore, as in [8, Corollary 30], we deduce the following result.

THEOREM 15. *The function $t \mapsto \sigma_h(S_2(t))$ has no relative minimum in $(0, \infty)$. Moreover for each positive integer q, either $\sigma_{h+q}(S_2(t)) = 0$ for $t \neq 0$, or the function $t \mapsto \sigma_{h+q}(S_2(t))$ has no relative minimum in $(0, \infty)$.*

6. When the supremum is the limit at ∞

In this section, we suppose that the limit

$$\lim_{t\to\infty} \sigma_h\big(S_2(t)\big)$$

is finite and positive, let us call it σ_0.

Observe first that Lemma 14 requires $h > \operatorname{rank}(P_C L_2 P_B)$ because the limit above is finite. Consider now a sequence of real numbers $\{t_k\}_{k=1}^{\infty}$ which tends to ∞ when $k\to\infty$, and let $\hat{\sigma}_k := \sigma_h\big(S_2(t_k)\big)$. Then

$$\lim_{t\to\infty} \hat{\sigma}_k = \sigma_0.$$

For each k, let

$$u^k := \begin{pmatrix} u_1^k \\ u_2^k \end{pmatrix}, \quad v^k := \begin{pmatrix} v_1^k \\ v_2^k \end{pmatrix}, \quad u_i^k, v_i^k \in \mathbb{C}^{m\times 1}, \quad i = 1,2,$$

be pairs of singular vectors of $S_2(t_k)$, associated with $\hat{\sigma}_k$. As the vectors u^k and v^k are unitary, the sequence $\{(u^k, v^k)\}_{k=1}^{\infty}$ has a convergent subsequence, say to (u,v). In order to simplify we will denote the terms of this subsequence with the same index k. Then

$$\lim_{k\to\infty} u^k = u =: \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \lim_{k\to\infty} v^k = v =: \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

For each sufficiently large k, the equalities (23)–(30), (32), and (33) are satisfied for t_k, u^k, v^k and $\hat{\sigma}_k$ instead of t_0, u, v and σ_0. Hence, taking limits, we infer that

$$\lim_{k\to\infty} P_C L_2 v_2^k = P_C L_2 v_2 = 0, \tag{39}$$

$$L_1 v_2 = \sigma_0 u_2, \tag{40}$$

$$L_1^* u_1 = \sigma_0 v_1, \tag{41}$$

$$\lim_{k\to\infty} t_k P_B L_2^* u_1^k = \sigma_0 v_2 - P_B L_1^* u_2, \tag{42}$$

$$\lim_{k\to\infty} P_B L_2^* u_1^k = P_B L_2^* u_1 = 0, \tag{43}$$

$$C_1^* u_1 = C_1^\dagger u_1 = 0, \tag{44}$$

$$B_1 v_2 = 0, \tag{45}$$

$$\lim_{k\to\infty} t_k (u_1^k)^* L_2 v_2^k = \sigma_0(u_1^* u_1 - v_1^* v_1) = -\sigma_0(u_2^* u_2 - v_2^* v_2), \tag{46}$$

$$u_1^* u_2 = v_1^* v_2. \tag{47}$$

We are going to apply Lemma 8 to $t_k (u_1^k)^* L_2 v_2^k = t_k (u_1^k)^* P_C L_2 P_B v_2^k$, for each k, because $(u_1^k)^* = (u_1^k)^* P_C$ and $v_2^k = P_B v_2^k$, by (29) and (30), respectively. Let $x_k := u_1^k$, $y_k := v_2^k$ and $G := P_C L_2 P_B$. Then by (39) we have

$$\lim_{k\to\infty} G y_k = \lim_{k\to\infty} P_C L_2 P_B v_2^k = \lim_{k\to\infty} P_C L_2 v_2^k = 0.$$

On the other hand, $\|t_k x_k^* G\| = \|t_k (u_1^k)^* P_C L_2 P_B\| = \|t_k P_B L_2^* u_1^k\|$ is bounded in virtue of (42). Thus, applying Lemma 8, we see that

$$\lim_{k\to\infty} t_k (u_1^k)^* L_2 v_2^k = \lim_{k\to\infty} t_k (u_1^k)^* P_C L_2 P_B v_2^k = 0.$$

Substituting this equality in (46), we conclude that $u_1^* u_1 = v_1^* v_1$ and $u_2^* u_2 = v_2^* v_2$. Hence, if we consider the matrices $V := [v_1, v_2], U := [u_1, u_2]$, from the two preceding equalities and (47), we have $U^*U = V^*V$. Therefore, as in Section 5, the matrix

$$D_0 := D - \sigma_0 U V^\dagger,$$

satisfies $\|D - D_0\| = \sigma_0$ and

$$D_0 v_2 = D v_2 - \sigma_0 u_2, \quad u_1^* D_0 = u_1^* D - \sigma_0 v_1^*.$$

By the definition of L_1, given in (13), from (40) and (41), we see that

$$D_0 v_2 = C_2 \Sigma^{-1} B_2 v_2, \quad u_1^* D_0 = u_1^* C_2 \Sigma^{-1} B_2. \tag{48}$$

Hence, to prove Theorem 5 in this case, it suffices to prove that 0 is a multiple eigenvalue of the matrix $M(\alpha, D_0)$. Once here, we are going to consider two cases: $v_2 \neq 0$ and $v_2 = 0$.

6.1. $v_2 \neq 0$

As v_2 is nonzero, to prove that 0 is a multiple eigenvalue of the matrix $M(\alpha, D_0)$, it suffices to find a vector $w \in \mathbb{C}^{n_1 \times 1}$ such that

$$\begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} \begin{pmatrix} 0 & w \\ -\Sigma^{-1} B_2 v_2 & -\Sigma^{-2} B_2 v_2 \\ v_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & w \\ -\Sigma^{-1} B_2 v_2 & -\Sigma^{-2} B_2 v_2 \\ v_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Multiplying these matrices, as $B_1 v_2 = 0$ by (45), and $D_0 v_2 = C_2 \Sigma^{-1} B_2 v_2$ by (48), the problem is reduced to find a vector w that satisfies $C_1 w - C_2 \Sigma^{-2} B v_2 = v_2$. That is, using the definition of L_2, given in (13), it suffices to find w such that

$$C_1 w = L_2 v_2.$$

Hence, there exists w if and only if $L_2 v_2 \in \operatorname{Im} C_1$, or which is equivalent by Lemma 7(2), if and only if $L_2 v_2 \in \operatorname{Ker} P_C$, which is true by (39).

6.2. $v_2 = 0$

In this case $u_2 = 0$ and $u_1 \neq 0$. Thus, it suffices to find a vector $w \in \mathbb{C}^{n_1 \times 1}$ such that

$$\begin{pmatrix} 0 & -u_1^* C_2 \Sigma^{-1} & u_1^* \\ w^* & -u_1^* C_2 \Sigma^{-2} & 0 \end{pmatrix} \begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -u_1^* C_2 \Sigma^{-1} & u_1^* \\ w^* & -u_1^* C_2 \Sigma^{-2} & 0 \end{pmatrix}.$$

Multiplying these matrices, as $u_1^*C_1 = 0$ by (44) and $u_1^*D_0 = u_1^*C_2\Sigma^{-1}B_2$ by (48), it suffices to find a vector w such that

$$w_1^*B_1 - u_1^*C_2\Sigma^{-2}B_2 = u_1^* \Leftrightarrow B_1^*w_1 = L_2^*u_1,$$

having used the definition of L_2, given in (13). Consequently, there exists w if and only if $L_2^*u_1 \in \operatorname{Im} B_1$, or which is equivalent by Lemma 7(3), if and only if $L_2^*u_1 \in \operatorname{Ker} P_B$; which is true by (43).

Two final remarks on Section 6

REMARK 11. Let us observe that in the part of the proof of Theorem 5, given in this section, we have not used the equality

$$\sup_{t>0} \sigma_h\big(S_2(t)\big) = \lim_{t\to\infty} \sigma_h\big(S_2(t)\big). \tag{49}$$

Actually, all we have used is the fact that

$$\lim_{t\to\infty} \sigma_h\big(S_2(t)\big)$$

is finite and positive. This assumption implies (49), since by Proposition 1 and Lemma 14, we have the following result.

PROPOSITION 16. *Let $L := P_C L_2 P_B$. If $h > \operatorname{rank}(L)$ and*

$$\sigma_0 := \sigma_{h-\operatorname{rank}(L)} \begin{pmatrix} P_C(I_m - LL^\dagger)L_1 & O \\ O & L_1(I_m - L^\dagger L)P_B \end{pmatrix}$$

is positive, then

$$\min_{\substack{X\in\mathbb{C}^{m\times m} \\ \mathrm{m}(0,M(\alpha,X))\geqslant 2}} \|X - D\| = \sigma_0.$$

Moreover,

$$\sup_{t>0} \sigma_h\left(S_2(t)\right) = \lim_{t\to\infty} \sigma_h\left(S_2(t)\right).$$

REMARK 12. Let $p > \operatorname{rank}(P_C L_2 P_B)$. By Lemma 14 the limit

$$\lim_{t\to\infty} \sigma_p\big(S_2(t)\big)$$

is finite. Let us assume it is positive. Following again all the reasoning of this section, we can prove that there exists a matrix $Y \in \mathbb{C}^{m\times m}$ such that

$$\|Y - D\| = \lim_{t\to\infty} \sigma_p\big(S_2(t)\big), \text{ and } \mathrm{m}\left(0, M(\alpha, Y)\right) \geqslant 2.$$

Besides, by Proposition 1 and Lemma 14, we have the following result.

PROPOSITION 17.

(1) *Let* $L := P_C L_2 P_B$. *Assume* $p > \operatorname{rank}(L)$. *Then*

$$\min_{\substack{X\in\mathbb{C}^{m\times m}\\ \mathrm{m}(0,M(\alpha,X))\geqslant 2}} \|X-D\| \leqslant \sigma_{p-\operatorname{rank}(L)} \begin{pmatrix} P_C(I_m - LL^\dagger)L_1 & O \\ O & L_1(I_m - L^\dagger L)P_B \end{pmatrix},$$

if this singular value is positive.

(2) *For each positive integer* q *the limit*

$$\lim_{t\to\infty} \sigma_{h+q}\left(S_2(t)\right)$$

is equal to σ_0 *or to* 0, *where* σ_0 *is defined in Proposition* 16.

7. When the supremum is the limit at 0, and $\operatorname{rank}(B_1) < n_1$ or $\operatorname{rank}(C_1) < n_1$

In this section, we assume that $\operatorname{rank}(B_1) < n_1$ or $\operatorname{rank}(C_1) < n_1$, and we suppose that the limit

$$\lim_{t\to 0^+} \sigma_h\big(S_2(t)\big)$$

is finite and positive, let us call it σ_0.

To shorten notation, we write s instead of $\operatorname{rank}(C_1B_1)$. First, let us observe that Lemma 13 warrants the existence of the limit. Moreover, by the same lemma and denoting

$$T_1 := I_m - (C_1B_1)^\dagger C_1B_1, \quad T_2 := I_m - C_1B_1(C_1B_1)^\dagger,$$

as $h > s$, we have

$$\lim_{t\to 0^+} \sigma_h\left(S_2(t)\right) = \sigma_{h-s} \begin{pmatrix} P_C L_1 T_1 & O \\ O & T_2 L_1 P_B \end{pmatrix} = \sigma_0 > 0.$$

We are going to prove some properties of the singular vectors of $P_C L_1 T_1$ and $T_2 L_1 P_B$. Assume that σ_0 is a singular value of $P_C L_1 T_1$ and let (u,v) be a pair of singular vectors corresponding to it. As $P_C L_1 T_1 v = \sigma_0 u$, by Lemma 7(2), we have $u \in \operatorname{Im}(P_C) = \operatorname{Ker}(C_1^*)$, that is $P_C u = u$, $u^* C_1 = 0$. On the other hand, as $T_1 L_1^* P_C u = T_1 L_1^* u = \sigma_0 v$,

$$L_1^* u - \sigma_0 v = (C_1B_1)^\dagger C_1 B_1 L_1^* u \in \operatorname{Im}(C_1B_1)^\dagger = \operatorname{Im}(C_1B_1)^* \subset \operatorname{Im}(B_1^*).$$

Hence, by Lemma 6(3), we see that $u^* L_1 - \sigma_0 v^* = (u^* L_1 - \sigma_0 v^*) B_1^\dagger B_1$. Thus, as (u,v) is a pair singular vectors of $P_C L_1 T_1$ associated with σ_0, we infer that

$$P_C L_1 T_1 v = \sigma_0 u, \tag{50}$$

$$T_1 L_1^* u = \sigma_0 v, \tag{51}$$

$$P_C u = u,\ u^* C_1 = 0, \tag{52}$$

$$u^* L_1 - \sigma_0 v^* = (u^* L_1 - \sigma_0 v^*) B_1^\dagger B_1. \tag{53}$$

Similarly, using Lemmas 7(3) and 6(3), if (x,y) is a pair of singular vectors of $T_2L_1P_B$ associated with σ_0, we conclude that

$$T_2L_1y = \sigma_0 x, \tag{54}$$
$$P_BL_1^*T_2x = \sigma_0 y, \tag{55}$$
$$P_By = y, B_1y = 0, \tag{56}$$
$$L_1y - \sigma_0 x = C_1C_1^\dagger(L_1y - \sigma_0 x). \tag{57}$$

To conclude the proof of Theorem 5 in this case, we are going to consider two cases: (1) σ_0 is a singular value of $P_CL_1T_1$; (2) σ_0 is a singular value of $T_2L_1P_B$.

7.1. σ_0 is a singular value of $P_CL_1T_1$

Let (u,v) be a pair of singular vectors of $P_CL_1T_1$ associated with σ_0. For the entire subsection let

$$D_0 := D - \sigma_0 uv^*.$$

It is clear that $\|D - D_0\| = \sigma_0$. Besides, by (53), we have

$$(-(u^*L_1 - \sigma_0 v^*)B_1^\dagger, -u^*C_2\Sigma^{-1}, u^*)\begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} = 0. \tag{58}$$

At this point, we consider two subcases: (1) $\operatorname{rank}(B_1) < n_1$ and (2) $\operatorname{rank}(C_1) < n_1 = \operatorname{rank}(B_1)$.

7.1.1. $\operatorname{rank}(B_1) < n_1$

In this case, there exists a nonzero vector $z \in \mathbb{C}^{n_1\times 1}$ such that $z^*B_1 = 0$. Thus,

$$(z^*, 0, 0)\begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} = 0.$$

This, together with (58), proves that 0 is a multiple eigenvalue of $M(\alpha, D_0)$.

7.1.2. $\operatorname{rank}(C_1) < n_1 = \operatorname{rank}(B_1)$

As $\operatorname{rank}(C_1) < n_1$ there exists a nonzero vector $z \in \mathbb{C}^{n_1\times 1}$ such that $C_1z = 0$. Therefore

$$\begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix}\begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix} = 0.$$

Thus, by (58), to prove that 0 is a multiple eigenvalue of $M(\alpha, D_0)$ it suffices to see that

$$\left(-(u^*L_1-\sigma_0 v^*)B_1^\dagger, -u^*C_2\Sigma^{-1}, u^*\right)\begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix} = -(u^*L_1-\sigma_0 v^*)B_1^\dagger z = 0.$$

Since $\operatorname{rank}(B_1)=n_1$, (51) implies $L_1^*u-\sigma_0 v=(C_1B_1)^\dagger C_1B_1L_1^*u$ and $B_1B_1^\dagger=I_{n_1}$. Thus

$$(u^*L_1-\sigma_0 v^*)B_1^\dagger z = u^*L_1(C_1B_1)^\dagger C_1B_1B_1^\dagger z = u^*L_1(C_1B_1)^\dagger C_1 z = 0,$$

because $C_1 z = 0$.

7.2. σ_0 is a singular value of $T_2L_1P_B$

Let (x,y) be a pair of singular vectors of $T_2L_1P_B$ associated with σ_0. In this subsection we define

$$D_0 := D - \sigma_0 x y^*.$$

Again we have $\|D-D_0\| = \sigma_0$. From (57), we see that

$$\begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix}\begin{pmatrix} -C_1^\dagger(L_1y-\sigma_0 x) \\ -\Sigma^{-1}B_2 y \\ y \end{pmatrix} = 0. \tag{59}$$

Now we will consider two subcases: (1) $\operatorname{rank}(C_1) < n_1$ and (2) $\operatorname{rank}(B_1) < n_1 = \operatorname{rank}(C_1)$.

7.2.1. $\operatorname{rank}(C_1) < n_1$

In this case there exists a nonzero vector $z \in \mathbb{C}^{n_1\times 1}$ such that $C_1 z = 0$. Hence,

$$\begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix}\begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix} = 0.$$

This, together with (59) proves that 0 is a multiple eigenvalue of $M(\alpha, D_0)$.

7.2.2. $\operatorname{rank}(B_1) < n_1 = \operatorname{rank}(C_1)$

As $\operatorname{rank}(B_1) < n_1$ there is a nonzero vector $z \in \mathbb{C}^{n\times 1}$ such that $z^*B_1 = 0$. So

$$(z^*, 0, 0)\begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} = 0.$$

Therefore, to demonstrate that 0 is a multiple eigenvalue of $M(\alpha, D_0)$, it suffices to see that

$$(z^*, 0, 0)\begin{pmatrix} -C_1^\dagger(L_1y - \sigma_0 x) \\ -\Sigma^{-1}B_2y \\ y \end{pmatrix} = -z^*C_1^\dagger(L_1y - \sigma_0 x) = 0.$$

Since $\operatorname{rank}(C_1) = n_1$, (54) implies $L_1y - \sigma_0 x = C_1B_1(C_1B_1)^\dagger L_1y$ and $C_1^\dagger C_1 = I_{n_1}$. Consequently

$$z^*C_1^\dagger(L_1y - \sigma_0 x) = z^*C_1^\dagger C_1B_1(C_1B_1)^\dagger L_1y = z^*B_1(C_1B_1)^\dagger L_1y = 0,$$

because $z^*B_1 = 0$.

Two final remarks on Section 7

REMARK 13. Let us observe that in the part of the proof of Theorem 5, given in this section, we have not used the equality

$$\sup_{t>0} \sigma_h\big(S_2(t)\big) = \lim_{t\to 0^+} \sigma_h\big(S_2(t)\big). \tag{60}$$

Actually, all we have used is the fact that

$$\lim_{t\to 0^+} \sigma_h\big(S_2(t)\big)$$

is finite and positive. This assumption implies (60), since by Proposition 1 and Lemma 13, we have the following result.

PROPOSITION 18. *Let* $M := C_1B_1$. *Assume that* $\operatorname{rank}(B_1) < n_1$ *or* $\operatorname{rank}(C_1) < n_1$. *If* $h > \operatorname{rank}(C_1B_1)$ *and*

$$\sigma_0 := \sigma_{h-\operatorname{rank}(M)}\begin{pmatrix} P_CL_1(I_m - M^\dagger M) & O \\ O & (I_m - MM^\dagger)L_1P_B \end{pmatrix}$$

is positive, then

$$\min_{\substack{X\in\mathbb{C}^{m\times m} \\ \mathrm{m}(0,M(\alpha,X))\geqslant 2}} \|X - D\| = \sigma_0.$$

Moreover,

$$\sup_{t>0} \sigma_h\left(S_2(t)\right) = \lim_{t\to 0^+} \sigma_h\left(S_2(t)\right).$$

REMARK 14. Let $p > \operatorname{rank}(C_1B_1)$. By Lemma 13 the limit

$$\lim_{t\to 0^+} \sigma_p\big(S_2(t)\big)$$

is finite. Let us assume it is positive. Following again all the reasoning of this section, we can prove that there exists a matrix $Y \in \mathbb{C}^{m\times m}$ such that

$$\|Y - D\| = \lim_{t\to 0^+} \sigma_p\big(S_2(t)\big), \text{ and } \mathrm{m}\left(0, M(\alpha, Y)\right) \geqslant 2.$$

Besides, by Proposition 1 and Lemma 13, we have the following result.

PROPOSITION 19. *Assume that* $\operatorname{rank}(B_1) < n_1$ *or* $\operatorname{rank}(C_1) < n_1$.
(1) *Let* $M := C_1B_1$. *Suppose that* $p > \operatorname{rank}(C_1B_1)$. *Then*

$$\min_{\substack{X\in\mathbb{C}^{m\times m}\\ \mathrm{m}(0,M(\alpha,X))\geqslant 2}} \|X-D\| \leqslant \sigma_{p-\operatorname{rank}(M)}\begin{pmatrix} P_CL_1(I_m-M^{\dagger}M) & O \\ O & (I_m-MM^{\dagger})L_1P_B \end{pmatrix},$$

if this singular value is positive.
(2) *For each positive integer* q *it follows that the limit*

$$\lim_{t\to 0^+}\sigma_{h+q}(S_2(t))$$

is equal to σ_0 *or to* 0, *where* σ_0 *is defined in Proposition* 18.

8. When the supremum is the limit at 0, and $\operatorname{rank}(B_1)=\operatorname{rank}(C_1)=n_1$

In this section, we assume that $\operatorname{rank}(B_1)=\operatorname{rank}(C_1)=n_1$, and we consider the case when

$$\sup_{t>0}\sigma_h\big(S_2(t)\big) = \lim_{t\to 0^+}\sigma_h\big(S_2(t)\big).$$

As $\operatorname{rank}(B_1)=\operatorname{rank}(C_1)=n_1$, by Lemma 6(5), we have $C_1^{\dagger}C_1 = B_1B_1^{\dagger} = I_{n_1}$; this fact will be used frequently along the section. Besides from (10) it follows that $h = 2m-n_1-1$. Since $\operatorname{rank}(C_1B_1)=n_1$, by Lemma 13 we have

$$\lim_{t\to 0^+}\sigma_h\big(S_2(t)\big) = \lim_{t\to 0^+}\sigma_{h+1}\big(S_2(t)\big) = \sigma_{m-n_1}(P_CL_1P_B) =: \sigma_0 > 0.$$

Thus there exists an $\varepsilon>0$ such that the functions $t\mapsto\sigma_h(S_2(t))$ and $t\mapsto\sigma_{h+1}(S_2(t))$ are nonincreasing on the interval $(0,\varepsilon)$.

Let us suppose that σ_0 *is a multiple singular value of* $P_CL_1P_B$. Then there are pairs of singular vectors $(u_1,v_1),(u_2,v_2)$ of $P_CL_1P_B$ associated with σ_0 so that $U^*U = I_2 = V^*V$, where $U := [u_1,u_2]$ and $V := [v_1,v_2]$. Define now the matrix

$$D_0 := D - \sigma_0UV^*.$$

Since $\|UV^*\| = 1$, it follows that $\|D-D_0\| = \sigma_0$ and $U^*D_0 = U^*D-\sigma_0V^*$. Given that $L_1 = D - C_2\Sigma^{-1}B_2$, by (52) and (53) we have

$$(-(u_i^*L_1-\sigma_0v_i^*)B_1^{\dagger}, -u_i^*C_2\Sigma^{-1}, u_i^*)\begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} = 0, \quad i=1,2;$$

that is, 0 is a multiple eigenvalue of $M(\alpha,D_0)$.

From now let us assume that σ_0 *is a simple singular value of* $P_CL_1P_B$. We will consider the matrix function $t\mapsto tS_2(t)$, which is analytic on $\mathbb{R}$. Then, by Lemma 11, there must be some $2m\times 2m$ unitary matrix functions

$$U(t) := (U_1(t),U_2(t),\ldots,U_{2m}(t)), V(t) := (V_1(t),V_2(t),\ldots,V_{2m}(t))$$

and a diagonal matrix function $\Sigma(t) = \mathrm{diag}\,(\tilde{\sigma}_1(t), \tilde{\sigma}_2(t), \ldots, \tilde{\sigma}_{2m}(t)) \in \mathbb{R}^{2m\times 2m}$, all analytic on $\mathbb{R}$, so that for each $t \neq 0$ we have

$$U(t)^* t S_2(t) V(t) = \Sigma(t) \Leftrightarrow U(t)^* S_2(t) V(t) = \mathrm{diag}(\tilde{\sigma}_i(t)/t).$$

Observe that for some interval $(0,a)$, with $a > 0$, we can assume without loss of generality that all the functions $\tilde{\sigma}_i(t)$ are nonnegative on it. Let j,k be now the unique subscripts such that

$$\lim_{t\to 0^+} \frac{\tilde{\sigma}_j(t)}{t} = \lim_{t\to 0^+} \frac{\tilde{\sigma}_k(t)}{t} = \sigma_0.$$

Thus, it is correct to assume that for each positive t sufficiently close to 0 we have $\tilde{\sigma}_j(t) \geqslant \tilde{\sigma}_k(t)$. Define the functions

$$f(t) := \frac{\tilde{\sigma}_j(t)}{t}, \quad g(t) := \frac{\tilde{\sigma}_k(t)}{t}.$$

Then, we see that

$$\lim_{t\to 0^+} f(t) = \lim_{t\to 0^+} g(t) = \sigma_0,$$

and there exists a $b > 0$ such that $f(t), g(t)$ are analytic on $(0,b)$, and for $t \in (0,b)$ we have the inequality $f(t) \geqslant g(t)$.

Let us denote

$$u(t) := \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = U_j(t), \quad v(t) := \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = V_j(t),$$

and

$$x(t) := \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = U_k(t), \quad y(t) := \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = V_k(t),$$

where $U_j(t), U_k(t)$ and $V_j(t), V_k(t)$ are the j-th and k-th columns of $U(t)$ and $V(t)$, respectively. Since they are analytic functions, we infer that the following limits exist

$$\lim_{t\to 0^+} u(t) = u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \lim_{t\to 0^+} v(t) = v := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

and

$$\lim_{t\to 0^+} x(t) = x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \lim_{t\to 0^+} y(t) = y := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Moreover, $(u(t), v(t))$ and $(x(t), y(t))$ are pairs of singular vectors of $S_2(t)$ associated with the singular values $f(t)$ and $g(t)$, respectively. Therefore, for each $t \in (0,b)$ the equalities (23)–(33) but for $(u(t), v(t))$ (instead of u, v) and $f(t)$ (instead of σ_0), and for $(x(t), y(t))$ (instead of u, v) and $g(t)$ (instead of σ_0), respectively, are satisfied.

First note that from (24) we deduce that

$$\lim_{t\to 0^+} C_1 B_1 v_1(t) = C_1 B_1 v_1 = 0.$$

Thus, as $\operatorname{rank}(C_1) = n_1$, we have $B_1 v_1 = 0$, which is equivalent to $P_B v_1 = v_1$ by Lemma 7(3). Similarly, as $\operatorname{rank}(B_1) = n_1$, being aware of Remark 9 and taking limits in (25) when $t \to 0^+$, we conclude that $C_1^* u_2 = 0$, which is equivalent to $P_C u_2 = u_2$ by Lemma 7(2).

Now, being aware of Remark 9 and considering equations (23)–(30), changing t_0 by t in them and as $\operatorname{Im}(C_1) = \operatorname{Ker}(P_C)$, $\operatorname{Im}(B_1^*) = \operatorname{Ker}(P_B)$, by Lemma 7(2)(3), when $t \to 0^+$ we infer that

$$P_C L_1 v_1 = \sigma_0 u_1, \tag{61}$$
$$\lim_{t\to 0^+} t^{-1} C_1 B_1 v_1(t) = L_1 v_2 - \sigma_0 u_2, \tag{62}$$
$$L_1 v_2 - \sigma_0 u_2 \in \operatorname{Im}(C_1) = \operatorname{Ker}(P_C), \tag{63}$$
$$\lim_{t\to 0^+} t^{-1} B_1^* C_1^* u_2(t) = L_1^* u_1 - \sigma_0 v_1, \tag{64}$$
$$L_1^* u_1 - \sigma_0 v_1 \in \operatorname{Im}(B_1^*) = \operatorname{Ker}(P_B), \tag{65}$$
$$P_B L_1^* u_2 = \sigma_0 v_2, \tag{66}$$
$$C_1^* u_1 = C_1^* u_2 = 0, \tag{67}$$
$$B_1 v_1 = B_1 v_2 = 0, \tag{68}$$
$$P_C u_1 = u_1,\, P_C u_2 = u_2, \tag{69}$$
$$P_B v_1 = v_1,\, P_B v_2 = v_2. \tag{70}$$

Remark that all the above properties are true also for (x,y).

Now, let (z,w) be a pair of singular vectors of $P_C L_1 P_B$ associated with the simple singular value σ_0. Let us see that there exist vectors $a := (a_1, a_2)$ and $b := (b_1, b_2)$ of $\mathbb{C}^{1\times 2}$ such that

$$(u_1, u_2) = za,\ (v_1, v_2) = wa,\ (x_1, x_2) = zb,\ (y_1, y_2) = wb, \tag{71}$$

where $ab^* = 0$ and $\|a\|^2 = \|b\|^2 = 1$.

First note that, as $P_B v_i = v_i$ and $P_C u_i = u_i, i = 1,2$, by (61) and (66) equation (71) is equivalent to

$$\begin{cases} P_C L_1 v_2 = \sigma_0 P_C u_2, \\ P_B L_1^* u_1 = \sigma_0 P_B v_1. \end{cases}$$

These last equalities are true by (63) and (65), respectively.

Hence, if we consider the matrices $V := [v_1, v_2], U := [u_1, u_2] \in \mathbb{C}^{m\times 2}$, from (71) we find that

$$U^* U = V^* V. \tag{72}$$

Thus, as in Section 5, the matrix

$$D_0 := D - \sigma_0 U V^\dagger,$$

satisfy $\|D - D_0\| = \sigma_0$ and $D_0 V = DV - \sigma_0 U$. Remark that all the above properties are true also for $X := [x_1, x_2], Y := [y_1, y_2]$.

So, to prove Theorem 5 in this case, it suffices to prove that 0 is a multiple eigenvalue of the matrix $M(\alpha, D_0)$, where $D_0 := D - \sigma_0 U V^\dagger$ or $D_0 := D - \sigma_0 X Y^\dagger$, respectively. The following lemma allows us to reduce the possible cases.

LEMMA 20. *With the preceding notations, we have*
(1) $\operatorname{rank}(U) = \operatorname{rank}(V) = \operatorname{rank}(X) = \operatorname{rank}(Y) = 1$,
(2) *if* $v_1 = 0$ *then* $y_2 = 0$,
(3) *if* $v_2 = 0$ *then* $y_1 = 0$.

Proof. (1) is immediate by (71). For demonstrating (2), let us assume now that $v_1 = 0$, hence $v_2 \neq 0$. Since u, y are orthogonal, we have $v_2^* y_2 = 0$, i.e. by (71) $\bar{a}_2 b_2 = 0$. Then $b_2 = 0$, consequently $y_2 = 0$. In a similar way (3) is proved. □

At this moment, by the preceding lemma, the possible cases to analyze are two: (1) $v_1 = 0$ or $v_2 = 0$; (2) $u_1 = \alpha u_2$, $v_1 = \alpha v_2$, $x_1 = \beta x_2$ and $y_1 = \beta y_2$, with scalar nonzero α, β.

8.1. $v_1 = 0$ or $v_2 = 0$

First let us suppose that $v_1 = 0$ and let $D_0 := D - \sigma_0 U V^\dagger$. Note that $u_1 = 0$. Hence v_2 and u_2 are nonzero vectors. To prove Theorem 5 in this case, we will search a pair of eigenvectors of $M(\alpha, D_0)$ associated with the eigenvalue 0, one on the left and other on the right, so that they are orthogonal.

We are going to prove that

$$\begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} \begin{pmatrix} -C_1^\dagger (L_1 v_2 - \sigma_0 u_2) \\ -\Sigma^{-1} B_2 v_2 \\ v_2 \end{pmatrix} = 0.$$

Since $B_1 v_2 = 0$ by property (68) and $D_0 v_2 = D v_2 - \sigma_0 u_2$, we just need to check

$$-C_1 C_1^\dagger (L_1 v_2 - \sigma_0 u_2) - C_2 \Sigma^{-1} B_2 v_2 + D v_2 - \sigma_0 u_2 = 0.$$

Or which is the same,

$$C_1 C_1^\dagger (L_1 v_2 - \sigma_0 u_2) = L_1 v_2 - \sigma_0 u_2,$$

because by (13), $L_1 = D - C_2 \Sigma^{-1} B_2$. That is, by Lemma 6(3), it suffices to prove that $L_1 v_2 - \sigma_0 u_2 \in \operatorname{Im} C_1$. Which is true by (63).

On the other hand, since $P_B v_2 = v_2$, from (66) we conclude that $L_1^* u_2 - \sigma_0 v_2 \in \operatorname{Ker}(P_B) = \operatorname{Im}(B_1^\dagger)$. Hence, reasoning in a similar manner and using $u_2^* D_0 = u_2^* D - \sigma_0 v_2^*$, it follows that

$$(-(u_2^* L_1 - \sigma_0 v_2^*) B_1^\dagger, -u_2^* C_2 \Sigma^{-1}, u_2^*) \begin{pmatrix} O & O & B_1 \\ O & \Sigma & B_2 \\ C_1 & C_2 & D_0 \end{pmatrix} = 0.$$

By the definition (13), $L_2 = I_m + C_2 \Sigma^{-2} B_2$. Moreover $v_2^* B_1^\dagger = 0$, by (68). Let us denote by ϕ the following scalar:

$$\begin{aligned} \phi &:= (-(u_2^* L_1 - \sigma_0 v_2^*) B_1^\dagger, -u_2^* C_2 \Sigma^{-1}, u_2^*) \begin{pmatrix} -C_1^\dagger (L_1 v_2 - \sigma_0 u_2) \\ -\Sigma^{-1} B_2 v_2 \\ v_2 \end{pmatrix} \\ &= u_2^* L_1 B_1^\dagger C_1^\dagger (L_1 v_2 - \sigma_0 u_2) + u_2^* L_2 v_2. \end{aligned}$$

In order to prove Theorem 5 in this case we are going to see that $\phi = 0$.

From (62),

$$L_1 v_2 - \sigma_0 u_2 = \lim_{t\to 0^+} t^{-1} C_1 B_1 v_1(t) \Rightarrow C_1^\dagger (L_1 v_2 - \sigma_0 u_2) = \lim_{t\to 0^+} t^{-1} C_1^\dagger C_1 B_1 v_1(t).$$

But, since $C_1^\dagger C_1 = I_{n_1}$, we have

$$C_1^\dagger (L_1 v_2 - \sigma_0 u_2) = \lim_{t\to 0^+} t^{-1} B_1 v_1(t).$$

Thus

$$\phi = u_2^* L_1 B_1^\dagger \lim_{t\to 0^+} t^{-1} B_1 v_1(t) + u_2^* L_2 v_2 = \lim_{t\to 0^+} u_2(t)^* L_1 B_1^\dagger \lim_{t\to 0^+} t^{-1} B_1 v_1(t) + u_2^* L_2 v_2;$$

that is,

$$\phi = \lim_{t\to 0^+} \frac{u_2(t)^* L_1 B_1^\dagger B_1 v_1(t)}{t} + u_2^* L_2 v_2.$$

By (26) we find that

$$t u_1^*(t) L_2 P_B + u_2^*(t) L_1 P_B = f(t) v_2(t)^* \Rightarrow t u_1^*(t) L_2 P_B + u_2^*(t) L_1 - u_2^*(t) L_1 B_1^\dagger B_1 = f(t) v_2(t)^*.$$

Therefore

$$u_2^*(t) L_1 B_1^\dagger B_1 = t u_1^*(t) L_2 P_B + u_2^*(t) L_1 - f(t) v_2(t)^*.$$

Consequently

$$\phi = \lim_{t\to 0^+} \frac{t u_1^*(t) L_2 P_B v_1(t) + u_2^*(t) L_1 v_1(t) - f(t) v_2(t)^* v_1(t)}{t} + u_2^* L_2 v_2,$$

and, as $P_B v_1 = 0$ by (70),

$$\phi = \lim_{t\to 0^+} \frac{u_2^*(t) L_1 v_1(t) - f(t) v_2(t)^* v_1(t)}{t} + u_2^* L_2 v_2.$$

By (23), $P_C L_1 v_1(t) + t P_C L_2 v_2(t) = f(t) u_1(t)$. Hence we know that $L_1 v_1(t) = f(t) u_1(t) - t P_C L_2 v_2(t) + C_1 C_1^\dagger L_1 v_1(t)$. Since $u_2^* P_C = u_2^*$, it follows that

$$\phi = \lim_{t\to 0^+} \frac{f(t) u_2(t)^* u_1(t) + u_2(t)^* C_1 C_1^\dagger L_1 v_1(t) - f(t) v_2(t)^* v_1(t)}{t}.$$

But, by (33), we have $u_2(t)^* u_1(t) = v_2(t)^* v_1(t)$. Therefore

$$\phi = \lim_{t\to 0^+} \frac{u_2(t)^* C_1 C_1^\dagger L_1 v_1(t)}{t} = \lim_{t\to 0^+} \frac{u_2(t)^* C_1 B_1 B_1^\dagger C_1^\dagger L_1 v_1(t)}{t},$$

because $B_1 B_1^\dagger = I_{n_1}$. Finally, we will apply Lemma 8. Taking $x(t) := u_2(t)$, $y(t) := B_1^\dagger C_1^\dagger L_1 v_1(t)$ and $G = C_1 B_1$, we obtain

$$\lim_{t\to 0^+} G y(t) = \lim_{t\to 0^+} C_1 B_1 B_1^\dagger C_1^\dagger L_1 v_1(t) = 0,$$

$$\lim_{t\to 0^+} \frac{x(t)^* G}{t} = \lim_{t\to 0^+} \frac{u_2(t)^* C_1 B_1}{t} = L_1^* u_1 - \sigma_0 v_1 = 0,$$

using (64) and that $u_1(t), v_1(t) \to 0$. Thus, by Lemma 8 we have $\phi = 0$.

If $v_2 = 0$, since by Lemma 20(3), $y_1 = 0$, it suffices to repeat the preceding reasoning for the pair (x,y), with the matrix $D_0 := D - \sigma_0 XY^\dagger$.

8.2. $u_1 = \alpha u_2$**,** $v_1 = \alpha v_2$**,** $x_1 = \beta x_2$**, and** $y_1 = \beta y_2$**, with** $\alpha\beta \neq 0$**.**

From (71) we infer that there exist two nonzero complex numbers δ, η such that

$$u = \begin{pmatrix} \delta\alpha z \\ \delta z \end{pmatrix}, v = \begin{pmatrix} \delta\alpha w \\ \delta w \end{pmatrix}, \quad x = \begin{pmatrix} \eta\beta z \\ \eta z \end{pmatrix}, y = \begin{pmatrix} \eta\beta w \\ \eta w \end{pmatrix}.$$

Since v, y are orthogonal, $\bar{\delta}\eta(\bar{\alpha}\beta + 1)w^* w = 0$. Consequently

$$\bar{\alpha}\beta + 1 = 0. \tag{73}$$

On the other hand, applying Lemma 11, for $t \in (0, \varepsilon)$, one has

$$f'(t) = \operatorname{Re}\left(u^*(t) S_2'(t) v(t)\right) = \operatorname{Re}\left(\begin{pmatrix} u_1(t)^* & u_2(t)^* \end{pmatrix} \begin{pmatrix} O & P_C L_2 P_B \\ t^{-2} C_1 B_1 & O \end{pmatrix} \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}\right).$$

Since $u_1(t)^* P_C L_2 P_B v_2(t) = u_1(t)^* L_2 v_2(t)$, we get

$$f'(t) = \operatorname{Re}\left(t^{-2} u_2(t)^* C_1 B_1 v_1(t) + u_1(t)^* L_2 v_2(t)\right) = t^{-2} u_2(t)^* C_1 B_1 v_1(t) + u_1(t)^* L_2 v_2(t),$$

because of (32). As $C_1 B_1 = C_1 B_1 (C_1 B_1)^\dagger C_1 B_1$ and $(C_1 B_1)^\dagger = B_1^\dagger C_1^\dagger$, by Lemma 7-4, we obtain

$$t^{-2} u_2(t)^* C_1 B_1 v_1(t) = \frac{u_2(t)^* C_1 B_1}{t} B_1^\dagger C_1^\dagger \frac{C_1 B_1 v_1(t)}{t}.$$

Thus, from (64) and (62), we see that

$$\lim_{t \to 0^+} t^{-2} u_2(t)^* C B v_1(t) = (u_1^* L_1 - \sigma_0 v_1^*) B_1^\dagger C_1^\dagger (L_1 v_2 - \sigma_0 u_2).$$

Therefore, as $v_1^* B_1^\dagger = 0$ and $C_1^\dagger u_2 = 0$, we infer that

$$\lim_{t \to 0^+} f'(t) = u_1^* (L_1 B_1^\dagger C_1^\dagger L_1 + L_2) v_2. \tag{74}$$

Similarly, for $g(t)$ we obtain

$$\lim_{t \to 0^+} g'(t) = x_1^* (L_1 B_1^\dagger C_1^\dagger L_1 + L_2) y_2. \tag{75}$$

Now, since the functions $f(t), g(t)$ are strictly nonincreasing and f', g' are continuous functions, we see that $f'(t), g'(t)$ are nonpositive. As there exist the limits of $f'(t), g'(t)$ when $t \to 0^+$, given in (74) and (75), we deduce that

$$u_1^* (L_1 B_1^\dagger C_1^\dagger L_1 + L_2) v_2 \leqslant 0 \text{ and } x_1^* (L_1 B_1^\dagger C_1^\dagger L_1 + L_2) y_2 \leqslant 0. \tag{76}$$

Using the expressions obtained at the beginning of this subsection for u,v,x,y, we get

$$u_1^*(L_1B_1^\dagger C_1^\dagger L_1+L_2)v_2=|\delta|^2\bar{\alpha}z^*(L_1B_1^\dagger C_1^\dagger L_1+L_2)w,$$
$$x_1^*(L_1B_1^\dagger C_1^\dagger L_1+L_2)y_2=|\eta|^2\bar{\beta}z^*(L_1B_1^\dagger C_1^\dagger L_1+L_2)w.$$

Thus, from (76) we obtain

$$\bar{\alpha}z^*(L_1B_1^\dagger C_1^\dagger L_1+L_2)w\leqslant 0 \text{ and } \bar{\beta}z^*(L_1B_1^\dagger C_1^\dagger L_1+L_2)w\leqslant 0.$$

Denote in a short while $\chi:=z^*(L_1B_1^\dagger C_1^\dagger L_1+L_2)w\in\mathbb{C}$. Hence, as $\bar{\alpha}\beta+1=0$ by (73), from the preceding inequalities, we find that

$$-\beta^{-1}\chi\leqslant 0 \text{ and } \bar{\beta}\chi\leqslant 0.$$

Consequently, since $\beta\neq 0$, these two inequalities are only possible if $\chi=0$. That is, we have proved that if (z,w) is a pair of singular vectors of $P_CL_1P_B$ associated with the singular value σ_0, then $z^*(L_1B_1^\dagger C_1^\dagger L_1+L_2)w=0$. Therefore, for the pair (u,v) one has

$$u_2^*(L_1B_1^\dagger C_1^\dagger L_1+L_2)v_2=0. \tag{77}$$

Next, defining the matrix $D_0:=D-\sigma_0UV^\dagger$ we are going to prove that 0 is a multiple eigenvalue of $M(\alpha,D_0)$. In a similar way to that of Subsection 8.1, given that

$$\begin{pmatrix} O & O & B_1\\ O & \Sigma & B_2\\ C_1 & C_2 & D_0\end{pmatrix}\begin{pmatrix}-C_1^\dagger(L_1v_2-\sigma_0u_2)\\ -\Sigma^{-1}B_2v_2\\ v_2\end{pmatrix}=0,$$

and

$$(-(u_2^*L_1-\sigma_0v_2^*)B_1^\dagger,-u_2^*C_2\Sigma^{-1},u_2^*)\begin{pmatrix} O & O & B_1\\ O & \Sigma & B_2\\ C_1 & C_2 & D_0\end{pmatrix}=0,$$

to prove that 0 is a multiple eigenvalue of $M(\alpha,D_0)$, it suffices to see that

$$\phi=(u_2^*L_1-\sigma_0v_2^*)B_1^\dagger C_1^\dagger(L_1v_2-\sigma_0u_2)+u_2^*L_2v_2=0.$$

That is as $v_2^*B_1^\dagger=0$ and $C_1^\dagger u_2=0$, it suffices to see that

$$u_2^*(L_1B_1^\dagger C_1^\dagger L_1+L_2)v_2=0,$$

which is true by (77). *This completes the proof of Theorem* 5.

Final remark on Section 8

REMARK 15. In Section 7, Proposition 18, we have proved that if $\operatorname{rank}(B_1)<n_1$ or $\operatorname{rank}(C_1)<n_1$, then

$$\sup_{t>0}\sigma_h(S_2(t))=\lim_{t\to 0^+}\sigma_h(S_2(t))$$

whenever this limit is >0. Let us assume that $\operatorname{rank}(B_1)=\operatorname{rank}(C_1)=n_1$. If the limit

$$\lim_{t\to 0^+}\sigma_h\left(S_2(t)\right)$$

is finite and positive, the following question arises: does the equality

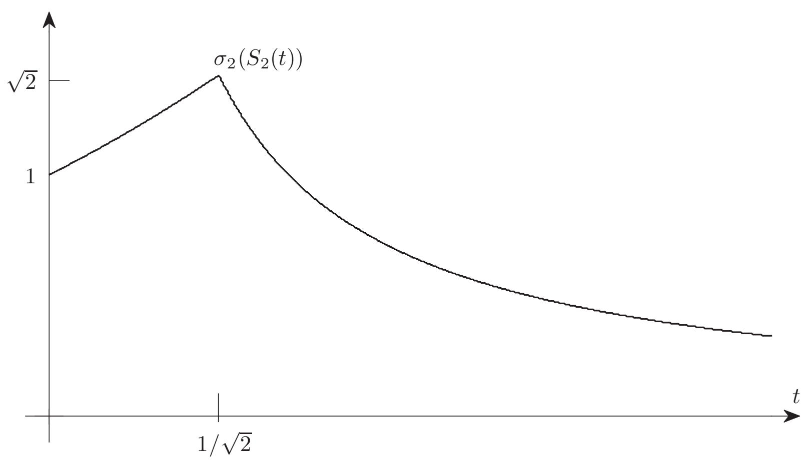
$$\sup_{t>0}\sigma_h\left(S_2(t)\right)=\lim_{t\to 0^+}\sigma_h\left(S_2(t)\right)$$

always hold? The answer is negative, as it can be seen in the following example. Let us consider the matrix of $\mathbb{C}^{3\times 3}$

$$\begin{pmatrix} 0 & B \\ C & D \end{pmatrix} := \left(\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right) \Rightarrow S_2(t)=\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & t \\ \hline -1/t & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

Then, $h=2$ and

$$\sigma_2\left(S_2(t)\right)=\begin{cases} \sqrt{\dfrac{t^2+2+t\sqrt{t^2+4}}{2}} & \text{if } t\in(0,1/\sqrt{2}], \\ 1/t & \text{if } t\in[1/\sqrt{2},\infty). \end{cases}$$



We have

$$\lim_{t\to 0^+}\sigma_2\left(S_2(t)\right)=1>0,$$

but the supremum is attained at $t_0=1/\sqrt{2}$ and its value is $\sqrt{2}$.

9. Scope of the results

Let $\alpha := (A,B,C) \in L_{n,m}$. Let $T \in \mathbb{C}^{n\times n}$ an invertible matrix and consider the triple $\alpha_T := (TAT^{-1}, TB, CT^{-1})$. It is easy to see that $M(\alpha,X)$ has a double 0 eigenvalue if and only if $M(\alpha_T,X)$ has a double 0 eigenvalue, for $X \in \mathbb{C}^{m\times m}$. Hence

$$\min_{\substack{X\in\mathbb{C}^{m\times m}\\ \mathrm{m}(0,M(\alpha,X))\geqslant 2}} \|X-D\| = \min_{\substack{X\in\mathbb{C}^{m\times m}\\ \mathrm{m}(0,M(\alpha_T,X))\geqslant 2}} \|X-D\|.$$

Moreover, it is clear that $p_\alpha(t) = p_{\alpha_T}(t)$.

Finally, we wish to note that applying the same reasoning of this work, we can obtain the following result, more general than Theorem 2.

THEOREM 21. *Let $\alpha := (A,B,C) \in L_{n,m}$ be any triple of matrices, where 0 is a semisimple eigenvalue of A. Let $D \in \mathbb{C}^{m\times m}$. Let Q be an invertible matrix such that*

$$QAQ^{-1} = \begin{pmatrix} O & O \\ O & A_1 \end{pmatrix},$$

where A_1 is an invertible matrix. Let $\beta := (QAQ^{-1}, QB, CQ^{-1})$. Then,

$$\min_{\substack{X\in\mathbb{C}^{m\times m}\\ \mathrm{m}(0,M(\alpha,X))\geqslant 2}} \|X-D\| = \sup_{t>0} \sigma_{p_\beta(t)+1}\left(S_2^\beta(t,D)\right).$$

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REFERENCES

[1] M. A. BEITIA, I. DE HOYOS, I. ZABALLA, *The change of similarity invariants under row perturbations*, Linear Algebra Appl. **429** (2008), 1302–1333.
[2] D. S. BERNSTEIN, *Matrix Mathematics*, Princeton University Press, 2005.
[3] S. L. CAMPBELL AND C. D. MEYER, *Generalized Inverses of Linear Transformations*, Pitman, London, 1979.
[4] G. CRAVO, *Matrix completion problems*, Linear Algebra Appl. **430** (2009), 2511–2540.
[5] K. DU, Y. WEI, *Structured pseudospectra and structured sensitivity of eigenvalues*, J. Comput. Appl. Math. **197** (2006), 502–519.
[6] J. M. GONZÁLEZ DE DURANA, J. M. GRACIA, *Geometric multiplicity margin for a submatrix*, Linear Algebra Appl. **349** (2002), 77–104.
[7] J. M. GRACIA, *Nearest matrix with two prescribed eigenvalues*, Linear Algebra Appl. **401** (2005), 277–294.
[8] J. M. GRACIA, F. E. VELASCO, *Nearest southeast submatrix that makes multiple a prescribed eigenvalue. Part 1*, Linear Algebra Appl. **430** (2009), 1196–1215.
[9] D. HINRICHSEN, B. KELB, *Spectral value sets: a graphical tool for robustness analysis*, Systems Control Lett. **21** (1993), 127–136.
[10] D. HINRICHSEN, A. J. PRITCHARD, *Mathematical systems theory I*, Springer, 2000.

[11] T. HU, L. QIU, *On structured perturbation of Hermitian matrices*, Linear Algebra Appl. **275-276** (1998), 287–314.
[12] A. N. MALYSHEV, *A formula for the 2-norm distance from a matrix to the set of matrices with multiple eigenvalues*, Numer. Math. **83** (3) (1999) 443–454.
[13] G. W. STEWART, J. G. SUN, *Matrix perturbation theory*, Academic Press, 1990.
[14] M. WEI, *Perturbation theory for the Eckart-Young-Mirsky theorem and the constrained total least squares problem*, Linear Algebra Appl. **280** (1998), 267–287.

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