Operators
and
Matrices
Volume 6, Number 1 (2012), 37–48

EXACTNESS OF UNIVERSAL FREE PRODUCTS OF FINITE DIMENSIONAL C^*–ALGEBRAS WITH AMALGAMATION

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(*Communicated by L. Rodman*)

Abstract. We investigate free products of finite dimensional C^*-algebras with amalgamation over diagonal subalgebras. We look to determine under what circumstances such a free product is exact and/or nuclear. We completely characterize exactness of $M_n *_D M_k$ where D is a unital subalgebra of both M_n and M_k. Our characterization depends both on the dimension of D and the embeddings of D into M_j and M_k. We also show that for free products of three finite dimensional algebras exactness fails. Lastly we look at some nonunital embeddings of a diagonal subalgebra into finite dimensional algebras.

Recall that for C^*-algebras A and B there are many possible norms on $A \otimes B$ for which the completion is a C^*-algebra. In particular there are two standard completions $A \otimes_{\min} B$ and $A \otimes_{\max} B$ corresponding to the 'smallest' and 'largest' possible tensor norms. We say that A is nuclear if these two tensor products correspond for all C^*-algebras B. We say that a C^*-algebra D is exact if given any short exact sequence of C^*-algebras

$$0 \to C \to B \to B/C \to 0$$

the associated sequence

$$0 \to C \otimes_{\min} D \to B \otimes_{\min} D \to B/C \otimes_{\min} D \to 0$$

is a short exact sequence. Both of these properties represent some appreciable level of 'niceness' for C^*-algebras. For more information about nuclear and exact C^*-algebras we refer the reader generally to [3].

In this paper we are interested in the question of whether the universal free product of matrix algebras, with and without amalgamation over diagonal subalgebras, are exact and/or nuclear. This question is solved in the case of the reduced free product, see [7] where it is shown that the reduced amalgamated free product of exact C^*-algebras is exact. Of course the universal free products are often not 'nice' in any reasonable sense; this is borne out in this paper by the fact that even simple finite dimensional C^*-algebras (matrix algebras over $\mathbb{C}$) quickly lose exactness and/or nuclearity when dealing with free products. However there are cases in which nuclearity is preserved under universal free products. This work was motivated by [6] where the question of nuclearity/exactness was discussed for free products of directed graph C^*-algebras.

Mathematics subject classification (2010): 46L09.
Keywords and phrases: Universal free products, exactness, nuclearity.

Since directed graph C^*-algebras are often free products of finite dimensional C^*-algebras this paper was the natural outgrowth of that investigation.

This work is related although different from [1, 8] where a related notion of $*$-wildness for finite dimensional free products was discussed. There is an important distinction between the present investigation and the aforementioned work: the free products in [1, 8] are all assumed to be unital. We look at a broader class of possible free products, allowing amalgamations over different diagonal subalgebras.

Unless specifically stated otherwise an algebra in this paper will be a C^*-algebra, and an isomorphism of algebras will mean a $*$-isomorphism. By M_j we mean the $j \times j$ matrices over $\mathbb{C}$; for the purposes of this paper we will always assume that $1 < j < \infty$. By a unital diagonal subalgebra of M_j we mean a subalgebra of the $j \times j$ diagonal matrices which contains the unit of M_j. The notation $A * B$ will denote the universal free product of the algebras A and B with no amalgamation. When A and B contain a common subalgebra D we will write $A *_D B$ to denote the universal free product of A and B with amalgamation over D.

We will use throughout some important results concerning nuclearity and exactness. More specifically we will need to recall that nuclearity and exactness are both preserved by quotients and that exactness passes to subalgebras (for these three facts see [3, IV.3.1.13, IV.3.4.19 and IV.3.4.3])

1. Amalgamation diagrams

Given M_j and M_k we intend to look at algebras of the form $M_j *_D M_k$ where D is a copy of $\mathbb{C}^n$ embedded into the two algebras as a diagonal subalgebra of the matrix algebras. Of course there are many different ways to do this embedding. We introduce some notation to describe how $\mathbb{C}^n$ embeds into M_j.

We use the diagram

$$M_j : \boxed{j_1}\boxed{j_2}\boxed{j_3}\cdots\boxed{j_n}\boxed{0}$$

to describe the embedding

$$\begin{bmatrix}\lambda_1\\ \lambda_2\\ \vdots\\ \lambda_n\end{bmatrix} \mapsto \begin{bmatrix}\lambda_1 I_{j_1} & 0 & 0 & \cdots & 0\\ 0 & \lambda_2 I_{j_2} & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & \lambda_n I_{j_n} & 0\\ 0 & 0 & \cdots & 0 & 0\end{bmatrix} \in \begin{bmatrix}M_{j_1} & 0 & 0 & \cdots & 0\\ 0 & M_{j_2} & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & 0 & M_{j_n} & 0\\ 0 & \cdots & 0 & 0 & 0\end{bmatrix} \subseteq M_j.$$

Here I_α is the $\alpha \times \alpha$ identity matrix in M_α. If there is a box containing a zero then we call such a box a *zero-box*. Further, in our notation there is at most one zero box.

Notice that the embedding diagram tells us:

1. the value of n,
2. and whether the embedding is non-unital, indicated by the presence of a zero-box.

For the purposes of our results it is safe to assume that through the use of elementary row operations that any zero-box is listed last.

Now when looking at the free product of two matrix algebras with amalgamation over $\mathbb{C}^n$ it is clear that just writing $M_j *_{\mathbb{C}^n} M_k$ will be unsuitable because it is not clear how we are embedding $\mathbb{C}^n$ into the two matrix algebras. To see the amalgamation we will use pairs of embedding diagrams. We will call a pair of embedding diagrams an *amalgamation diagram* since they represent the amalgamating subalgebra in a free product. We will present two examples to illustrate how this will work.

EXAMPLE 1. We start with an example from W. Paschke [4, Example 3.3]. There it is noted that with suitable amalgamation $M_{j+1} *_{\mathbb{C}^2} M_2$ is isomorphic to $M_{j+1} \otimes \mathscr{O}_j$, where $\mathscr{O}_j$ is the classical Cuntz algebra (see [5]). The amalgamation can be described using the amalgamation diagram

$$\begin{aligned} M_j &: \boxed{1}\boxed{j-1} \\ M_2 &: \boxed{1}\boxed{1}. \end{aligned}$$

Here the M_j-row represents $\mathbb{C}^2$ as the subalgebra of M_j given by

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_2 \end{bmatrix}.$$

The M_2-row represents the usual embedding of $\mathbb{C}^2$ as the diagonal subalgebra of M_2. Notice that for both embeddings $\mathbb{C}^2$ is a unital subalgebra of the associated algebra.

EXAMPLE 2. The next example is from [9, Chapter 6]. There it is shown that for unital A and appropriate choice of embedding we have that $M_j *_{\mathbb{C}} A$ is isomorphic to $M_j(A)$. For our notation we will look at the specific case of $A = M_k$ and amalgamation diagram

$$\begin{aligned} M_j &: \boxed{1}\boxed{0} \\ M_k &: \boxed{k}. \end{aligned}$$

Here the scalar multiples of the identity in M_k are matched up with the 1×1 entry in M_j.

EXAMPLE 3. Finally we have the following example which, while not of the form described above will allow us to make some computations later. The algebra

$$A := \begin{bmatrix} M_k & 0 \\ 0 & \mathbb{C} \end{bmatrix} *_{\mathbb{C}^{k+1}} \begin{bmatrix} \mathbb{C}^{k-1} & 0 \\ 0 & M_2 \end{bmatrix}$$

is isomorphic to M_{k+1}, where $\mathbb{C}^{k+1}$ is the canonical inclusion as diagonal matrices. Certainly there is an onto $*$-representation $\pi : A \to M_{k+1}$ induced by the inclusions

$$\iota_{k,1} : \begin{bmatrix} M_k & 0 \\ 0 & \mathbb{C} \end{bmatrix} \subseteq M_{k+1}$$
$$\iota_{k-1,2} \begin{bmatrix} \mathbb{C}^{k-1} & 0 \\ 0 & M_2 \end{bmatrix} \subseteq M_{k+1}.$$

We need only show that M_{k+1} satisfies the requisite universal property. So let B be a C^*-algebra and assume that we have $*$-representations $\pi_1 : \begin{bmatrix} M_k & 0 \\ 0 & \mathbb{C} \end{bmatrix} \to B$ and $\pi_2 : \begin{bmatrix} \mathbb{C}^{k-1} & 0 \\ 0 & M_2 \end{bmatrix} \to B$ with $\pi_1|_D = \pi_2|_D$ for the subalgebra of diagonal matrices D. Then for the elementary matrices $e_{i,j} \in M_{k+1}$ define

$$\pi(e_{i,j}) = \begin{cases} \pi_1(e_{i,j}) & 1 \leqslant i,j < k+1 \\ \pi_2(e_{k+1,k+1}) = \pi_1(e_{k+1,k+1}) & i = j = k+1 \\ \pi_1(e_{i,k})\pi_2(e_{k,k+1}) & 1 \leqslant i < k+1, j = k+1 \\ \pi_2(e_{k+1,k})\pi_1(e_{k,j}) & 1 \leqslant j < k+1, i = k+1 \end{cases}.$$

Notice that in the second case since $\pi_1|_D = \pi_2|_D$ which tells us that the second case is well defined. For general matrices we extend using linearity. We need only show that π induces a $*$-representation on M_{k+1}. To verify this we notice first that π is linear by construction. Next, to show that $\pi(A^*) = \pi(A)^*$ we only need show, by linearity, that $\pi(e_{i,j}{}^*) = \pi(e_{i,j})^*$ for all i, j. This is trivial if $1 \leqslant i, j \leqslant k$ or $i = j = k+1$ since π_1 and π_2 are $*$-representations. So assume that $i < k+1$ and consider

$$\begin{aligned} \pi(e_{i,k+1})^* &= (\pi_1(e_{i,k})\pi_2(e_{k,k+1}))^* \\ &= \pi_2(e_{k,k+1})^*\pi_1(e_{i,k+1})^* \\ &= \pi_2(e_{k+1,k})\pi_1(e_{k+1,i}) \\ &= \pi(e_{k+1,i}) = \pi(e_{i,k+1}{}^*). \end{aligned}$$

The third equality follows since π_1 and π_2 are $*$-representations. The case where $j < k+1$ and $i = k+1$ is similar.

We next need to show that π is multiplicative. We will consider products of the form $e_{i,m} = e_{i,j}e_{j,m}$. Again this follows using cases. If $1 \leqslant i, j, m < k+1$, or $i = j = m = k+1$ then $\pi(e_{i,m}) = \pi_1(e_{i,m}) = \pi_1(e_{i,j}e_{j,m}) = \pi_1(e_{i,j})\pi_1(e_{j,m}) = \pi(e_{i,j})\pi(e_{j,m})$. There are six remaining cases, we will do two of them, the remainder will follow in a similar fashion.

Assume that $j = k+1$ and $1 \leqslant i, m < k+1$, then

$$\begin{aligned} \pi(e_{i,j})\pi(e_{j,m}) &= \pi_1(e_{i,k})\pi_2(e_{k,k+1})\pi_2(e_{k+1,k})\pi_1(e_{k,m}) \\ &= \pi_1(e_{i,k})\pi_2(e_{k,k})\pi_1(e_{k,m}) \\ &= \pi_1(e_{i,k})\pi_1(e_{k,k})\pi_1(e_{k,m}) \\ &= \pi_1(e_{i,k}e_{k,k}e_{k,m}) = \pi_1(e_{i,m}) = \pi(e_{i,m}). \end{aligned}$$

Notice that in the third equality we used that π_2 is a homomorphism, in the next line we used that $\pi_1|_D = \pi_2|_D$, and then in the line after we use the fact that π_1 is a homomorphism.

Next consider the case that $i = k+1, m = k+1$ and $1 \leqslant j < k+1$ and compute

$$\begin{aligned}
\pi(e_{i,j})\pi(e_{j,m}) &= \pi_2(e_{k+1,k})\pi_1(e_{k,j})\pi_1(e_{j,k})\pi_2(e_{k,k+1}) \\
&= \pi_2(e_{k+1,k}\pi_1(e_{k,k})\pi_2(e_{k,k+1}) \\
&= \pi_2(e_{k+1,k}\pi_2(e_{k,k})\pi_2(e_{k,k+1}) \\
&= \pi_2(e_{k+1,k}e_{k,k}e_{k,k+1}) \\
&= \pi_2(e_{k+1,k+1}) = \pi(e_{i,j}e_{j,m}).
\end{aligned}$$

Similar calculations finish the remaining cases and then applying linearity completes the proof that M_{k+1} has the requisite universal property and hence is isomorphic to A.

The following will be useful in analyzing exactness and nuclearity for free products.

THEOREM 1. *If D is a C^*-subalgebra of A_1 and A_2 then there exists a canonical onto $*$-representation $\pi : A_1 * A_2 \to A_1 *_D A_2$. If, in addition, C is a C^*-algebra with $D \subseteq C \subseteq A_i$ for each $i = 1,2$ then there is a canonical onto $*$-representation $\sigma : A_1 *_D A_2 \to A_1 *_C A_2$.*

Proof. Let $\iota_i : A_i \to A_1 *_D A_2$ be the canonical inclusion (i.e. $A_i \subseteq A_1 *_D A_2$). Then by the universal property of $A_1 * A_2$ there exists a canonical $*$-representation $\iota_1 * \iota_2 : A_1 * A_2 \to A_1 *_D A_2$. This map is onto since a generating set for $A_1 *_D A_2$ is contained in the image of $\iota_1 * \iota_2$.

Next let $\beta_i : A_i \to A_1 *_C A_2$ be the canonical inclusion. Notice that $\beta_1(d) = \beta_2(d)$ for all $d \in D$ since $D \subseteq C$ and hence there is an induced $*$-representation $\beta_1 *_D \beta_2 : A_1 *_D A_2 \to A_1 *_C A_2$ which is onto for the same reason as the previous map. $\square$

The following is immediate since both nuclearity and exactness pass to quotients.

COROLLARY 1. *If $A * B$ is nuclear so is $A *_D B$ for any C^*-algebra D with $D \subseteq A$ and $D \subseteq B$. If $A *_D B$ is not exact then neither $A * B$ nor $A *_C B$ is exact for any subalgebra C with $D \subseteq C \subseteq A$ and $D \subseteq C \subseteq B$.*

Finally we have one more well-known example which will provide a standard non-exact C^*-algebra for our results.

EXAMPLE 4. The algebra $C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T})$ is isomorphic to the non-exact C^*-algebra $C^*(\mathbb{Z}) *_{\mathbb{C}} C^*(\mathbb{Z}) = C^*(\mathbb{Z} * \mathbb{Z}) = C^*(F_2)$ (see [11] for a proof that the latter is not exact).

2. Algebras of the form $M_j *_D M_k$

We have already seen two examples of these type of algebras, both of which were nuclear. The general case will be more complicated and will depend on the nature of D, and on the embedding diagrams for $D \subseteq M_j$ and $D \subseteq M_k$.

PROPOSITION 1. *The algebra $M_3 *_{\mathbb{C}^3} M_3$ is isomorphic to $M_3 \otimes (C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T}))$ and hence is not exact.*

Proof. We know from Example 3 that $M_3 = (M_2 \oplus \mathbb{C}) *_{\mathbb{C}^3} (\mathbb{C} \oplus M_2)$ and hence $M_3 *_{\mathbb{C}^3} M_3$ can be rewritten as

$$((M_2 \oplus \mathbb{C}) *_{\mathbb{C}^3} (\mathbb{C} \oplus M_2)) *_{\mathbb{C}^3} ((M_2 \oplus \mathbb{C}) *_{\mathbb{C}^3} (\mathbb{C} \oplus M_2)).$$

Of course by rearranging we can rewrite this as

$$((M_2 \oplus \mathbb{C}) *_{\mathbb{C}^3} (M_2 \oplus \mathbb{C})) *_{\mathbb{C}^3} ((\mathbb{C} \oplus M_2) *_{\mathbb{C}^3} (\mathbb{C} \oplus M_2))$$

which by Example 1 is isomorphic to $(M_2(C(\mathbb{T})) \oplus \mathbb{C}) *_{\mathbb{C}^3} (\mathbb{C} \oplus M_2(C(\mathbb{T})))$. The latter algebra has a canonical representation onto a generating set for the algebra $M_3 \otimes (C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T}))$ via the inclusion maps. It is a simple matter to see that the algebra $M_3 \otimes (C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T}))$ satisfies the universal property for

$$(M_2(C(\mathbb{T})) \oplus \mathbb{C}) *_{\mathbb{C}^3} (\mathbb{C} \oplus M_2(C(\mathbb{T}))).$$

Lack of exactness now follows using Example 4. $\square$

PROPOSITION 2. *If D is a unital diagonal subalgebra of M_j and M_k such that* $\dim D \geqslant 3$, *then $M_j *_D M_k$ is not exact.*

Proof. Consider the amalgamation diagram for $M_j *_D M_k$ given by:

$$\begin{array}{l} M_j : \boxed{j_1}\boxed{j_2}\cdots\boxed{j_l} \\ M_k : \boxed{k_1}\boxed{k_2}\cdots\boxed{k_l}. \end{array}$$

Let E_1 be the diagonal matrix in M_j given by $e_{1,1} + e_{j_1+1,j_1+1} + e_{j_1+j_2+1,j_1+j_2+1}$ and E_2 is the matrix in M_k given by $e_{1,1} + e_{k_1+1,k_1+1} + e_{k_1+k_2+1,k_1+k_2+1}$, then set $A = \{x \in M_j : xE_1 = E_1x = x\}$ and $B = \{x \in M_k : xE_2 = E_2x = x\}$. It is routine to verify that $A \cong M_3$ and $B \cong M_3$. Then applying [2, Proposition 2.4] with the canonical conditional expectations given by projections onto the appropriate subalgebras M_3 we have that $A *_E B \subset M_j *_D M_k$ and hence $M_j *_D M_k$ is not exact since a subalgebra of an exact C^*-algebra is exact. $\square$

We let m_i denote the minimum value in the ith column of the amalgamation diagram for $M_j *_{\mathbb{C}^k} M_k$. For example for the amalgamation diagram:

$$\begin{array}{l} M_{12} : \boxed{2}\boxed{5}\boxed{5} \\ M_{11} : \boxed{3}\boxed{1}\boxed{7}. \end{array}$$

$m_1 = 2$ and $m_2 = 1$ and $m_3 = 5$. We now define the *minimum value of the diagram* to be the sum of the m_i as i ranges over the columns of the amalgamation diagram. In the previous example this would be 8.

PROPOSITION 3. *Let D be a unital diagonal subalgebra of M_j and M_k. If the minimum value of the amalgamation diagram for $M_j *_D M_k$ is greater than or equal to 3, then $M_j *_D M_k$ is not exact.*

Proof. Again consider the amalgamation diagram for $M_j *_D M_k$ given by:

$$\begin{array}{l} M_j : \boxed{j_1}\boxed{j_2}\cdots\boxed{j_l} \\ M_k : \boxed{k_1}\boxed{k_2}\cdots\boxed{k_l}. \end{array}$$

Here we let E_1 be the matrix in M_j given by

$$\sum_{s=1}^{m_1} e_{s,s} + \sum_{s=J_2}^{J_2+m_2} e_{s,s} + \cdots + \sum_{s=J_l}^{J_l+m_l} e_{s,s}$$

where $J_r = \sum_{i \leqslant r-1} j_i + 1$ for $r > 2$ and similarly E_2 is the matrix in M_k given by

$$\sum_{s=1}^{m_1} e_{s,s} + \sum_{s=K_2}^{K_2+m_2} e_{s,s} + \cdots + \sum_{s=K_l}^{K_l+m_l} e_{s,s}$$

with similarly defined K_l. Now considering $A = \{x \in M_j : xE_1 = E_1 x = x\}$ and $B = \{x \in M_k : xE_2 = E_2 x = x\}$ we get by hypothesis that A and B are both isomorphic to M_t where t is the minimum value of the amalgamation diagram. Notice by the preceding proposition we know that $A *_{\mathbb{C}^t} B$ is not exact. Now applying [2, Proposition 2.4] we have that the non-exact algebra $A *_{\mathbb{C}^t} B \subseteq M_j *_D M_k$ and hence $M_j *_D M_k$ is not exact. $\square$

THEOREM 2. *Let D be a unital diagonal subalgebra of M_j and M_k such that the minimum value of the amalgamation diagram for $M_j *_D M_k$ is 2. If $\dim D = 2$ then the algebra is nuclear.*

Proof. We will show that such an algebra is a directed graph C^*-algebra and hence is nuclear. Let G be the directed graph with 2-vertices $\{v_1, v_2\}$ and $(j-1)+(k-1)$ edges $\{e_1, e_2, \cdots, e_{j-1}, f_1, f_2, \cdots, f_{k-1}\}$ with $r(e_i) = v_1$, $s(e_i) = v_2$ and $r(f_i) = v_2, s(f_i) = v_1$. We claim that $C^*(G)$ is isomorphic to $M_j *_D M_k$. Notice that $e_{i+1,1} \in M_j$ and $e_{j,k} \in M_k$ form a collection of partial isometries which form a Cuntz-Krieger family for the graph G. Further notice that this Cuntz-Krieger family generates the algebra $M_j *_D M_k$. By [10, Proposition 1.21] there is a $*$-representation $\pi : C^*(G) \to M_j *_D M_k$. Notice further that the directed graph thus constructed is cofinal and every cycle has an entry hence $C^*(G)$ is simple by [10, Proposition 4.2]. It follows that π is one-to-one and hence the free product algebra is isomorphic to $C^*(G)$ and is nuclear. $\square$

Notice that the minimum value of an amalgamation diagram for $M_j *_D M_k$ is never equal to 1. For unital amalgamations of finite dimensional algebras we have one case remaining.

PROPOSITION 4. *The algebra $M_2 *_{\mathbb{C}} M_2$ is not exact.*

Proof. Define $\pi_1 : M_2 \to M_2(C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T}))$ by

$$\pi_1\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & bz_1 \\ c\overline{z_1} & d \end{bmatrix}$$

where z_1 is the usual generator for $C(\mathbb{T})$ in the first copy of $C(\mathbb{T}) \subseteq C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T})$. A routine calculation shows that π_1 is a $*$-representation.

Next define $\pi_2 : M_2 \to M_2(C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T}))$ by

$$\pi_2\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \frac{a+d-c\overline{z_2}+bz_2}{2} & \frac{a-d-c\overline{z_2}+bz_2}{2} \\ \frac{a-d+c\overline{z_2}-bz_2}{2} & \frac{a+d+c\overline{z_2}+bz_2}{2} \end{bmatrix}$$

where z_2 is the usual generator for $C(\mathbb{T})$ in the second copy of $C(\mathbb{T}) \subseteq C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T})$. Again, a routine calculation shows that π_2 is a $*$-representation.

Now $\pi_1\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \pi_2\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right)$ and hence there is a $*$-representation $\pi_1 * \pi_2 : M_2 *_{\mathbb{C}} M_2 \to M_2(C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T}))$. Notice that

$$\begin{bmatrix} z_1 & 0 \\ 0 & 0 \end{bmatrix} = \pi_1\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\pi_2\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)$$

and

$$\begin{bmatrix} z_2 & 0 \\ 0 & 0 \end{bmatrix} = \pi_1\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\pi_2\left(\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\right)\pi_1\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

and hence the non-exact subalgebra $\begin{bmatrix} C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T}) & 0 \\ 0 & 0 \end{bmatrix}$ is contained as a subalgebra in the image of $\pi_1 * \pi_2$. It follows that $M_2 *_{\mathbb{C}} M_2$ can not be exact. $\square$

It is not hard to see that, in the previous proof, the mapping $\pi_1 * \pi_2$ is not one-to-one. This follows since $\mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^2$ is a subalgebra of $M_2 *_{\mathbb{C}} M_2$, but the image of $\mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^2$ under the mapping $\pi_1 * \pi_2$ is finite dimensional. However it is well known, see [3, Example IV.1.4.2] that $\mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^2$ is isomorphic to

$$\left\{\begin{bmatrix} f_{1,1}(t) & f_{1,2}(t) \\ f_{2,1}(t) & f_{2,2}(t) \end{bmatrix} : f_{i,j} \in C([0,1]), f_{1,2}(0) = f_{2,1}(0) = f_{1,2}(1) = f_{2,1}(1) = 0\right\}.$$

3. Algebras of the form $M_j *_D M_k *_D M_l$

We first point out that in this case the amalgamation diagram has some ambiguity since the inclusion of the two copies of D into each matrix algebras may not be the same. We will assume for the purposes of what follows that the two inclusions are the same (we do not deal here with the case where they differ although we expect similar results). We first note that $M_j *_D M_k *_D M_l = M_j *_D M_l *_D M_k$ and hence if any two of j,k, or l give rise to amalgamation diagrams with minimum value greater than or equal to 3, then the algebra $M_j *_D M_k *_D M_l$ is not exact.

THEOREM 3. *The algebra $M_2 *_{\mathbb{C}^2} M_2 *_{\mathbb{C}^2} M_2$ is isomorphic to $M_2(C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T}))$ and hence is not exact.*

Proof. By Example 1, the algebra $M_2 *_{\mathbb{C}^2} M_2$ is isomorphic to $M_2 \otimes C(\mathbb{T})$. Further there is a canonical $*$-isomorphism

$$\pi : (M_2 *_{\mathbb{C}^2} M_2) *_{M_2} (M_2 *_{\mathbb{C}^2} M_2) \to M_2 *_{\mathbb{C}^2} M_2 *_{\mathbb{C}^2} M_2.$$

Now assume that $\pi_i : C(\mathbb{T}) \otimes M_2 \to A$ are unital $*$-representations satisfying $\pi_1(1 \otimes d) = \pi_2(1 \otimes d)$ for all $d \in M_2$. For $a \in C(\mathbb{T})$ we define $\sigma_i(a) = \pi_i(a \otimes 1)$. Then there exists $\sigma_1 * \sigma_2 : C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T}) \to A$. Further, if we set $\sigma : M_2 \to A$ by $\sigma(d) = \sigma_1(1 \otimes d)$ then we know that $\sigma(d)\sigma_1(a) = \sigma_1(a)\sigma(d)$ and $\sigma(d)\sigma_2(a) = \sigma_2(a)\sigma(d)$ for all $a \in C(\mathbb{T})$ and $d \in M_2$ and hence $\sigma_1 * \sigma_2(x)\sigma(d) = \sigma(d)\sigma_1 * \sigma_2(x)$ for all $x \in C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T})$ and $d \in D$. It follows by the universal property of the tensor product that there exists $\tau : C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T}) \otimes M_2 \to A$ extending the canonical inclusions of $C(\mathbb{T}) \otimes M_2$ into $(C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T})) \otimes M_2$. Hence $(M_2 *_{\mathbb{C}^2} M_2) *_{M_2} (M_2 *_{\mathbb{C}^2} M_2)$ is isomorphic to $(C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T})) \otimes M_2$ which is not exact since it contains a copy of $C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T})$. $\square$

COROLLARY 2. *Let D be a diagonal subalgebra of M_2, then $M_2 *_D M_2 *_D M_2$ is not exact.*

PROPOSITION 5. *Let D be a unital diagonal subalgebra of M_j, M_k and M_l. The algebra $M_j *_D M_k *_D M_l$ is not exact.*

Proof. If the dimension of D is greater than or equal to 3 then by Proposition 2 the algebra can not be exact, so we look only at the case that $\dim D \leqslant 2$.

If $\dim D = 2$, then again we can assume without loss of generality that in the threefold amalgamation diagram for the free product that in any given row at most one box is not equal to 1. Thus at least one of $j,k,$ or l must equal 2. So without loss of generality assume that we have $l = 2$ and we are in the case of $M_j *_{\mathbb{C}^2} M_k *_{\mathbb{C}^2} M_2$. Now, as in Proposition 3 we can see that there is a copy of $M_2 *_{\mathbb{C}^2} M_2$ inside $M_j *_{\mathbb{C}^2} M_k$, and applying [2] again we have that $M_2 *_{\mathbb{C}^2} M_2 *_{\mathbb{C}^2} M_2$ is a subalgebra of $M_j *_{\mathbb{C}^2} M_k *_{\mathbb{C}^2} M_l$ and hence the latter is not exact.

The case of $\dim D = 1$ now follows by Corollary 1. $\square$

4. Free products with no amalgamation and some nonunital amalgamations

We know by applying Proposition 2 and Proposition 1 that the following is true.

PROPOSITION 6. *The algebra $M_j * M_k$ is not exact for any $k, j \geqslant 2$.*

We now focus on the case in which the diagonal subalgebra D contains the identity of M_j but not that of M_k. In this case the amalgamation diagram is of the form

$$\begin{array}{l} M_j : \boxed{j_1}\boxed{j_2}\cdots\boxed{j_m} \\ M_k : \boxed{k_1}\boxed{k_2}\cdots\boxed{k_m}\boxed{0} \end{array}$$

where m is the dimension of D. We will write $k(D)$ for the integer given by $k - \sum_{i=1}^{m} k_i$

THEOREM 4. *Let D be a unital diagonal subalgebra of M_j, where D is a diagonal subalgebra of M_k which does not contain the unit of M_k. Then the algebra $M_j *_D M_k$ is exact if and only if $M_j *_D M_{k-k(D)}$ is exact in which case $M_j *_D M_k$ is nuclear.*

Proof. Clearly, since $M_j *_D M_{k-k(D)}$ is a subalgebra of $M_j *_D M_k$ if the former is not exact neither is the latter, since exactness is preserved by subalgebras. We will focus on the case in which $M_j *_D M_{k-k(D)}$ is exact. This breaks down into two cases.

Case 1 ($\dim D = 1$): In this case, either $j = 1$ which is trivial, or $k - k(D) = 1$ which puts us in the context of Example 2.

Case 2 ($\dim D = 2$): In this case the subalgebra $M_j *_D M_{k-k(D)}$ is a directed graph algebra, see Proposition 2. The corresponding directed graph has two vertices $\{v_1, v_2\}$ and $j-1$ edges from v_1 to v_2 and $k - k(D) - 1$ edges from v_2 to v_1. Create a new graph G by adding a vertex v_3 and $k - k(D)$ edges from v_2 to v_3. We claim that the algebra $C^*(G)$ is isomorphic to $M_j *_D M_k$ and hence the algebra is nuclear.

Let $\{E, P\}$ be the Cuntz-Krieger system given by the generators for the graph C^*-algebra $M_j *_D M_{k-k(D)}$. Now look at the associated Cuntz-Krieger system

$$\left\{ E \cup \{e_{j,k} : 1 \leqslant j \leqslant k-1\}, P \cup \left\{ \sum_{m=k-k(D)+1}^{k} e_{m,m} \right\} \right\},$$

where $e_{i,j} \in M_k \subset M_j *_D M_k$. Notice that this new Cuntz-Krieger system generates $M_j *_D M_k$ as a C^*-algebra and hence a standard result for graph algebras, [10, Proposition 1.21], gives an onto representation : $\pi : C^*(G) \to M_j *_D M_k$. Now since graph algebras are nuclear the algebra $M_j *_D M_k$ is nuclear. $\square$

Finally we can make some progress on the general case. We know that there is a canonical onto $*$-representation $\pi : M_3 *_D M_3 \to M_3 *_{\mathbb{C}^3} M_3$ for any diagonal subalgebra D and hence $M_3 *_D M_3$ is not exact for any diagonal subalgebra D. We have also seen that $M_2 * M_2$ and the free product with unital amalgamation, $M_2 *_{\mathbb{C}} M_2$, are not exact.

Now write the amalgamation diagram for $M_j *_D M_k$ as

$$\begin{array}{r l} M_j : & \boxed{j_1}\boxed{j_2}\cdots\boxed{j_m}\boxed{0} \\ M_k : & \boxed{k_1}\boxed{k_2}\cdots\boxed{k_m}\boxed{0}. \end{array}$$

PROPOSITION 7. *Let D be a non-unital diagonal subalgebra of M_j and M_k. If $\dim D \geqslant 2$ then $M_j *_D M_k$ is not exact. If either $j - \sum j_i$ or $k - \sum k_i$ is greater than or equal to 2 then $M_j *_D M_k$ is not exact.*

Proof. We deal first with $\dim D \geqslant 2$. Notice that there will be an embedding of M_3 into M_j and M_k so that the subalgebra will have amalgamation diagram

$$\begin{array}{r l} M_3 : & \boxed{1}\boxed{1}\boxed{0} \\ M_3 : & \boxed{1}\boxed{1}\boxed{0} \end{array}$$

which will have as a quotient the non-exact algebra $M_3 *_{\mathbb{C}^3} M_3$ and hence $M_j *_D M_k$ will not be exact.

For the other situation we notice that there will be a subalgebra of the form $\mathbb{C} * (\mathbb{C} \oplus \mathbb{C})$. This non-unital C^*-algebra satisfies

$$(\mathbb{C} * (\mathbb{C} \oplus \mathbb{C}))^1 \cong (\mathbb{C} \oplus \mathbb{C}) *_{\mathbb{C}} (\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}).$$

The latter algebra is isomorphic to $C^*(\mathbb{Z}_2) *_{\mathbb{C}} C^*(\mathbb{Z}_3) \cong C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ which contains a copy of $C^*(\mathbb{Z} * \mathbb{Z})$ which is not exact. It follows that since the unitization of $\mathbb{C} * (\mathbb{C} \oplus \mathbb{C})$ is not exact the algebra is not either and hence $M_j *_D M_k$ is not exact. $\square$

The only case that remains is the free product $M_2 *_{\mathbb{C}} M_k$ with amalgamation diagram

$$\begin{array}{r l} M_2 : & \boxed{1}\boxed{0} \\ M_k : & \boxed{k-1}\boxed{0}. \end{array}$$

We do not, as yet, have a satisfactory answer for this situation.

REFERENCES

[1] S. ALBEVERIO, K. JUSHENKO, D. PROSKURIN, AND Y. SAMIOLENKO, *$*$-wildness of some classes of C^*-algebras*, Meth. Funct. Anal. Top., **12** (2006), 315–325.

[2] S. ARMSTRONG, K. DYKEMA, R. EXEL, AND H. LI, *On embeddings of full amalgamated free product C^*-algebras*, Proc. Amer. Math. Soc., **132** (2004), 2019–2030.

[3] B. BLACKADAR, *Operator Algebras: Theory of C^*-algebras and von Neumann Algebras*, Springer, New York, 2006.

[4] L. BROWN, *Ext of certain free product C^*-algebras*, J. Operator Theory, **6** (1981), 135–141.

[5] J. CUNTZ, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys., **57** (1977), 173–185.

[6] B. DUNCAN, *Certain free products of graph operator algebras*, J. Math. Anal. Appl., **364** (2010), 534–543.

[7] K. DYKEMA, *Exactness of reduced amalgamated free product C^*-algebras*, Forum Math., **16** (2004), 161–180.

[8] E. JUSHENKO AND K. SUKRETNIY, *On $*$-wildness of a free product of finite-dimensional C^*-algebras*, Meth. Funct. Anal. Top., **12** (2006), 151–156.

[9] T. LORING, *Lifting Solutions to Perturbing Problems in C^*-algebras*, Amer. Math. Soc., Providence, 1997.

[10] I. RAEBURN, *Graph Algebras*, Amer. Math. Soc., Providence, 2005.

[11] S. WASSERMANN, *On tensor products of certain group C^*-algebras*, J. Funct. Anal., **23** (1976), 239–254.

(Received June 9, 2009)

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