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INVERSE NODAL PROBLEMS FOR THE STURM–LIOUVILLE OPERATOR WITH EIGENPARAMETER DEPENDENT BOUNDARY CONDITIONS

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Abstract. An inverse nodal problem consists in reconstructing this operator from the given zeros of their eigenfunctions. In this work, we are concerned with the inverse nodal problem of the Sturm-Liouville operator with eigenparameter dependent boundary conditions on a finite interval. We prove uniqueness theorems: a dense subset of nodal points uniquely determine the parameters of the boundary conditions and the potential function of the Sturm-Liouville equation; and provide a constructive procedure for the solution of the inverse nodal problems.

1. Introduction

Inverse spectral problems consist in recovering operators from their spectral characteristics. Such problems play an important role in mathematics and have many applications in natural sciences (see, for example, [9, 18, 20, 22, 28, 29]). In 1988, the inverse nodal problem was posed and solved for Sturm-Liouville problems by J. R. McLaughlin [21], who showed that knowledge of a dense subset of nodal points of the eigenfunctions alone can determine the potential function of the Sturm-Liouville problem up to a constant. Some numerical schemes were provided by O. H. Hald and J. R. McLaughlin [11] for the reconstruction of the potential. This is the so-called inverse nodal problem. From the physical point of view this corresponds to finding, e.g., the density of a string or a beam from the zero-amplitude positions of their eigenvibrations. Later, some remarkable results of inverse nodal problems of the Sturm-Liouville operators were obtained (for example, refer to [2, 3, 4, 5, 8, 11, 13, 14, 15, 16, 21, 23, 24, 25, 27, 30]).

For the Sturm-Liouville operator with eigenparameter dependent boundary conditions, C. T. Fulton [10] and J. Walter [26] gave extensive bibliographies of work in this area: Fulton also discussed various physical applications of the Sturm-Liouville operator of this class. In 1996, P. J. Browne and B. D. Sleeman [2] extended inverse nodal results of Hald and Mclaughlin concerning the inverse problem for the regular Sturm-Liouville problem on a finite interval to the case in which the boundary conditions are

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eigenparameter dependent. Nowadays there are only a number of papers devoted to inverse nodal problems for the Sturm-Liouville operator with eigenparameter dependent boundary conditions [2].

In this work, we consider the regular differential equation

$$-y''(x)+q(x)y(x)=\lambda y(x),\ x\in(0,1), \tag{1.1}$$

where $q(x)$ is a real-valued, absolutely continuous function on $(0,1)$ and a boundary-value problem

$$y(0)\cos\alpha-y'(0)\sin\alpha=0,\ \ y'(1)=\Theta(\lambda)y(1), \tag{1.2}$$

where $\alpha\in[0,\pi)$ and $\Theta(\lambda)=\frac{\Theta_1(\lambda)}{\Theta_2(\lambda)}$ is a rational function, $\Theta_1(\lambda)$ and $\Theta_2(\lambda)$ are relatively prime polynomials with real coefficients and no common zeros. In addition, if $\deg(\Theta_2(\lambda))\geqslant\deg(\Theta_1(\lambda))$ then we set $m=\deg(\Theta_2(\lambda))$ and assume that $\Theta_2(\lambda)$ is monic and $\Theta_1(\lambda)=A_m\lambda^m+\cdots+A_0$ where $A_m\in R$ (it may be zero), and if $\deg(\Theta_2(\lambda))<\deg(\Theta_1(\lambda))$ then we set $m=\deg(\Theta_1(\lambda))$ and assume that $\Theta_1(\lambda)$ is monic and $\Theta_2(\lambda)=A_{m-1}\lambda^{m-1}+\cdots+A_0$ where $A_{m-1}\in R$ (it may be zero).

If the boundary conditions (1.2) do not contain the eigenvalue parameter λ, then the solution of the inverse nodal problem is given [4, 5, 11, 14, 15, 16, 21, 27] and other works. P. J. Browne and B. D. Sleeman [2] considered the inverse nodal problem of the problem (1.1) with the parameter boundary conditions $y'(0)-hy(0)=0$ and $(a\lambda+b)y(1)=(c\lambda+d)y'(1)$, where $ad-bc>0$ and $c\neq 0$. Here there is an interesting problem: in the study of vibrating systems, the most natural experiment for finding the nodal positions is to excite the vibrating system at a natural frequency and take measurements of the positions which are the zeros (or nodes) of the eigenfunctions. Suppose we are given nodal information about the eigenfunctions arising from a problem such as (1.1) and (1.2). To what extent does this determine the potential $q(x)$ and the boundary conditions (1.2)? Inverse nodal problems for the problem (1.1) and (1.2) are not studied yet.

Asymptotics and oscillation results for the Sturm-Liouville problem (1.1) and (1.2) was given [1]. In this work, firstly, we derive a detailed asymptotic formula for the nodal points; secondly, we prove the uniqueness theorem and provide an algorithm for solving the inverese nodal problem; finally, we also show connections of these problems with inverse spectral problems of the Sturm-Liouville operator with an eigenparameter dependent boundary condition. We note that the obtained results are natural generalizations of the well-known results on inverse nodal problems for the Sturm-Liouville operators which were studied in [2, 21] and other works.

2. Main results

Let $\phi(x,\lambda)$ be the solution of (1.1) satisfying the following initial conditions

$$\phi(0,\lambda)=\sin\alpha,\quad \phi'(0,\lambda)=\cos\alpha.$$

Then the spectra of the boundary-value problem (1.1) and (1.2) are the zero-sequences $\{\lambda_n^{(i)}\}_{n=0}^{\infty}$ of the entire function

$$\Phi(\lambda)=\Theta_1(\lambda)\phi(1,\lambda)-\Theta_2(\lambda)\phi'(1,\lambda).$$

The spectra $\{\lambda_n^{(i)}\}_{n\geqslant 0}$ of the problem (1.1) and (1.2) is bounded from below and discrete. In particular, for large n all eigenvalues are algebraically simple and real. The spectra $\{\lambda_n^{(i)}\}_{n\geqslant 0}$ of the problem (1.1) and (1.2) are given asymptotically for $n\to\infty$ by (see [1])

$$\lambda_n^{(i)} = (n-m_i)^2\pi^2 + c_i + o\left(\frac{1}{n}\right), \tag{2.1}$$

where

$$(m_i,\ \ c_i) = \begin{cases} \left(m,\ 2\cot\alpha - 2A_m + \int_0^1 q(t)dt\right) & (i=1); \\ \left(m-\frac{1}{2},\ -2A_m + \int_0^1 q(t)dt\right) & (i=2); \\ \left(m-\frac{1}{2},\ 2\cot\alpha + 2A_{m-1} + \int_0^1 q(t)dt\right) & (i=3); \\ \left(m-1,\ 2A_{m-1} + \int_0^1 q(t)dt\right) & (i=4), \end{cases}$$

and

$i=1$ stands for the case $\alpha\neq 0$, $\deg(\Theta_1(\lambda))\leqslant\deg(\Theta_2(\lambda))=m$;
$i=2$ stands for the case $\alpha=0$, $\deg(\Theta_1(\lambda))\leqslant\deg(\Theta_2(\lambda))=m$;
$i=3$ stands for the case $\alpha\neq 0$, $m=\deg(\Theta_1(\lambda))>\deg(\Theta_2(\lambda))$;
$i=4$ stands for the case $\alpha=0$, $m=\deg(\Theta_1(\lambda))>\deg(\Theta_2(\lambda))$.

Let $\phi(x,\lambda_n^{(i)})$ be the eigenfunction corresponding to the eigenvalue $\lambda_n^{(i)}$ of the Sturm-Liouville operator (1.1) and (1.2). For sufficiently large n we get $\phi(x,\lambda_n^{(i)})$ has exactly $n-m+1$ nodal points in $(0,1)$ for $\deg(\Theta_2(\lambda))<\deg(\Theta_1(\lambda))$ and $\lim_{\lambda\to\infty}\Theta(\lambda)$ $=+\infty$, or $n-m$ nodal points in $(0,1)$ for other cases [1]. Suppose $x_n^{(i)j}$ is the jth nodal point of the eigenfunction $\phi(x,\lambda_n^{(i)})$ in $(0,1)$. In other words, $\phi(x_n^{(i)j},\lambda_n^{(i)})=0$. Let $I_n^{(i)j}=(x_n^{(i)j},x_n^{(i)j+1})$, and the nodal length $l_n^{(i)j}$ by

$$l_n^{(i)j}\overset{def}{=}x_n^{(i)j+1}-x_n^{(i)j}.$$

We also define the function $j_n(x)$ to be the largest index j such that $0\leqslant x_n^{(i)j}\leqslant x$.

Denote $X^{(i)}\overset{def}{=}\{x_n^{(i)j}\}$. $X^{(i)}$ is called the set of nodal points of the Sturm-Liouville operator (1.1) and (1.2).

We consider the following inverse nodal problem.

PROBLEM. Fix $i\in\{1,2,3,4\}$. From given nodal points set $X^{(i)}$ or its subset $X_0^{(i)}$ which is dense in $(0,1)$, how to find the boundary condition parameter α, some quantities for $\Theta(\lambda)$ and the potential $q(x)$?

The main theorems are the following.

THEOREM 2.1. *Fix* $x\in[0,\pi]$ *and* $i\in\{1,2,3,4\}$. *Let* $\{x_n^{(i)j}\}\subset X^{(i)}$ *be chosen such that*

$$\lim_{n\to\infty} x_n^{(i)j}=x.$$

Then the following finite limits exist and the corresponding equalities hold.

(a) In the case $i=1,3$:

$$\begin{aligned} &\lim_{n\to\infty} n\left(x_n^{(i)j}-\frac{j-\frac{1}{2}}{n}\right)\stackrel{def}{=} f_i(x),\\ &\lim_{n\to\infty}(n-m_i)^2\left(x_n^{(i)j}-\frac{j-\frac{1}{2}}{n-m_i}\right)\stackrel{def}{=} g_i(x), \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} f_i(x) &= m_i x,\\ g_i(x) &= -\frac{c_i}{2\pi^2}x+\frac{1}{\pi^2}\left[\cot\alpha+\frac{1}{2}\int_0^x q(t)dt\right]. \end{aligned} \tag{2.3}$$

(b) In the case $i=2,4$:

$$\begin{aligned} &\lim_{n\to\infty} n\left(x_n^{(i)j}-\frac{j}{n}\right)\stackrel{def}{=} f_i(x),\\ &\lim_{n\to\infty}(n-m_i)^2\left(x_n^{(i)j}-\frac{j}{n-m_i}\right)\stackrel{def}{=} g_i(x), \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} f_i(x) &= m_i x,\\ g_i(x) &= -\frac{c_i}{2\pi^2}x+\frac{1}{2\pi^2}\int_0^x q(t)dt. \end{aligned} \tag{2.5}$$

Here

$$m_i=\begin{cases} m & (i=1);\\ m-\frac{1}{2} & (i=2,3);\\ m-1 & (i=4). \end{cases}$$

Let us now formulate a uniqueness theorem and provide a constructive procedure for the solution of the inverse nodal problem.

Define $\overline{q}\stackrel{def}{=}\int_0^1 q(x)dx$.

THEOREM 2.2. *Fix $i\in\{1,2,3,4\}$. Let $X_0^{(i)}\subset X^{(i)}$ be a subset of nodal points which is dense in $(0,1)$. Then, the specification of $X_0^{(i)}$ uniquely determines the potential $q(x)-\overline{q}$ in $(0,1)$, and the parameters m, α, A_{m-1} and A_m of the boundary conditions. The potential $q(x)-\overline{q}$, and the numbers m, α, A_{m-1} and A_m can be constructed via the following algorithm:*

(1) for each $x\in[0,1]$ choose a sequence $\{x_n^{(i)j}\}\subset X_0^{(i)}$ such that $x_n^{(i)j}\to x$ as $n\to\infty$;

(2) find the functions $f_i(x)$, $g_i(x)$ via (2.2)-(2.5) and in turn, calculate

(a) In the case $i=1,3$*:*

$$
\begin{gathered}
m=\begin{cases} f_1(1) & (i=1) \\ f_3(1)+\frac{1}{2} & (i=3); \end{cases} \\
\cot\alpha=\pi^2 g_i(0); \\
A_m=\pi^2 g_1(1),\ \ A_{m-1}=-\pi^2 g_3(1); \\
q(x)-\overline{q}=\begin{cases} 2\pi^2\frac{d}{dx}g_1(x)-2\cot\alpha-2A_m & (i=1); \\ 2\pi^2\frac{d}{dx}g_3(x)-2\cot\alpha+2A_{m-1} & (i=3). \end{cases}
\end{gathered} \tag{2.6}
$$

(b) In the case $i=2,4$*:*

$$
\begin{gathered}
m=\begin{cases} f_2(1)+\frac{1}{2} & (i=2) \\ f_4(1)-1 & (i=4); \end{cases} \\
A_m=\pi^2 g_2(1),\ \ A_{m-1}=-\pi^2 g_4(1); \\
q(x)-\overline{q}=\begin{cases} 2\pi^2\frac{d}{dx}g_2(x)-2A_m & (i=2); \\ 2\pi^2\frac{d}{dx}g_4(x)+2A_{m-1} & (i=4). \end{cases}
\end{gathered} \tag{2.7}
$$

Using only the nodal data, we can reconstruct the potential. Our reconstruction formulae are direct and automatically implies the uniqueness of this inverse problem (1.1) and (1.2).

Finally we give the following incomplete inverse spectral problem. Suppose that $q(x)$ is known on a part of the interval, namely, for $x\in(b,1)$. The inverse problem is to construct $q(x)$ for $x\in(0,b)$ from a part of the spectrum of the boundary value problem.

We denote the boundary-value problem (1.1) and (1.2) by $B{=}B(q,\alpha,\Theta(\lambda))$. Together with B we consider a boundary value problem $\widetilde{B}=B(\widetilde{q},\widetilde{\alpha},\Theta(\lambda))$ of the same form:

$$-y''(x)+\widetilde{q}(x)y(x)=\lambda y(x),\ x\in(0,1), \tag{2.8}$$

with the boundary conditions

$$y(0)\cos\tilde{\alpha}-y'(0)\sin\tilde{\alpha}=0,\ \ y'(1)=\Theta(\lambda)y(1), \tag{2.9}$$

where $\widetilde{q}(x)$ is a real-valued, absolutely continuous function on $(0,1)$ and $\widetilde{\alpha}\in[0,\pi)$. We agree that if a certain symbol δ denotes an object related to B, then $\widetilde{\delta}$ will denote an analogous object related to $\widetilde{B}$.

THEOREM 2.3. *Fix* $b\in(0,\frac{1}{2})$ *and* $i\in\{1,2,3,4\}$*. Let* $q(x)=\widetilde{q}(x)$ *a.e. on* $(b,1)$*. Let* $\Lambda\subset\mathbf{N}$ *be a subset of positive integer numbers, and let* $\Omega\overset{def}{=}\{\lambda_n^{(i)}\}_{n\in\Lambda}$ *be a part of the spectrum of the boundary-value problem* (1.1) *and* (1.2)*, and the system of the functions* $\left\{\cos\left(2\sqrt{\lambda_n^{(i)}}x\right)\right\}_{n\in\Lambda}$ *is complete in* $L^2(0,b)$*. If* $\Omega=\widetilde{\Omega}$*, then* $q(x)=\widetilde{q}(x)$ *a.e. on* $(0,1)$ *and* $\alpha=\widetilde{\alpha}$*.*

Let us go on to the investigation of an incomplete inverse nodal problem when nodal points are given only on a part of the interval $(0,1)$. For $X_0^{(i)} \subset X^{(i)}$, we denote $\Lambda_{X_0^{(i)}} \overset{def}{=} \{m(n) : \exists j, x_{m(n)}^{(i)j} \in X_0^{(i)}\}$, where $m(n)$ is a sequence of natural numbers. The set $X_0^{(i)}$ is called twin if together with each of its point $x_n^{(i)j}$, the set $X_0^{(i)}$ contains at least one of adjacent nodal $x_n^{(i)j-1}$ or/and $x_n^{(i)j+1}$.

THEOREM 2.4. *Fix $b \in (0, \frac{1}{2})$ and $i \in \{1,2,3,4\}$. Let $X_0^{(i)} \subset X^{(i)} \bigcap (b,1)$ be a dense on $(b,1)$ twin subset of nodal points, $m(n) \in \Lambda_{X_0^{(i)}}$, and the functions*

$$\left\{ \cos\left(2\sqrt{\lambda_{m(n)}^{(i)} - \overline{q}}x\right)\right\}$$

is complete in $L^2(0,b)$. If $X_0^{(i)} = \widetilde{X}_0^{(i)}$ and $\overline{q} = \overline{\widetilde{q}}$, then $q(x) = \widetilde{q}(x)$ a.e. on $(0,1)$ and $\alpha = \widetilde{\alpha}$.

THEOREM 2.5. *Fix $b \in (0, \frac{1}{2})$ and $i \in \{1,2,3,4\}$. Let $X_0^{(i)} \subset X^{(i)} \bigcap (b,1)$ be a dense on $(b,1)$ twin subset of nodal points and $m(n)$ be a sequence of natural numbers such that*

$$m(n) = \frac{n}{\sigma}(1+\varepsilon_n),\ 0 < \sigma \leqslant 1,\ \varepsilon_n \to 0. \tag{2.10}$$

If $X_0^{(i)} = \widetilde{X}_0^{(i)}$, and $\overline{q} = \overline{\widetilde{q}}$ and $\sigma > 2b$, then $q(x) = \widetilde{q}(x)$ a.e. on $(0,1)$ and $\alpha = \widetilde{\alpha}$.

3. Proofs

By [1], for large n eigenfunctions corresponding to the eigenvalue $\lambda_n^{(i)}$ have $n-m$ zeroes in the case of $\deg(\Theta_1(\lambda)) \leqslant \deg(\Theta_2(\lambda)) = m$ and for the case $m = \deg(\Theta_1(\lambda)) > \deg(\Theta_2(\lambda))$, the number of zeroes is $n-m+1$ if $\lim_{\lambda\to\infty}\Theta(\lambda) = +\infty$ and $n-m$ if $\lim_{\lambda\to\infty}\Theta(\lambda) = -\infty$. Here we give asymptotic formulas of the nodal points for the Sturm-Liouville operator (1.1) and (1.2).

LEMMA 3.1. *For sufficiently large n, the asymptotic formulas of the nodal points for the eigenfunction $\phi\left(x, \lambda_n^{(i)}\right)$ of the Sturm-Liouville operator (1.1) and (1.2) are the following:*

(a) In the case $i = 1,3$:

$$x_n^{(i)j} = \frac{j-\frac{1}{2}}{n-m_i} - \frac{(j-\frac{1}{2})c_i}{2(n-m_i)^3\pi^2} + \frac{\cot\alpha + \frac{1}{2}\int_0^{x_n^{(i)j}} q(t)dt}{(n-m_i)^2\pi^2} + O\left(\frac{1}{n^3}\right),\ n\to\infty \tag{3.1}$$

with

$$j = \begin{cases} 1,2,\cdots,n-m \textit{ for } i=1 \textit{ and } i=3 \textit{ with } \lim_{\lambda\to\infty}\Theta(\lambda) = -\infty, \\ 1,2,\cdots,n-m+1 \textit{ for } i=3 \textit{ with } \lim_{\lambda\to\infty}\Theta(\lambda) = +\infty; \end{cases}$$

(b) In the case $i=2,4$*:*

$$x_n^{(i)j}=\frac{j}{n-m_i}-\frac{jc_i}{2(n-m_i)^3\pi^2}+\frac{\int_0^{x_n^{(i)j}}q(t)dt}{2(n-m_i)^2\pi^2}+O\left(\frac{1}{n^3}\right),\ n\to\infty \tag{3.2}$$

with

$$j=\begin{cases}1,2,\cdots,n-m \ \textit{for}\ i=2\ \textit{and}\ i=4\ \textit{with}\ \lim_{\lambda\to\infty}\Theta(\lambda)=-\infty,\\ 1,2,\cdots,n-m+1\ \textit{for}\ i=4\ \textit{with}\ \lim_{\lambda\to\infty}\Theta(\lambda)=+\infty,\end{cases}$$

uniformly with respect to j.

Proof. The eigenfunction $\phi\left(x,\lambda_n^{(i)}\right)$ has the following asymptotic formula for $n\to\infty$ uniformly in x [1, 14]:

$$\phi\left(x,\lambda_n^{(i)}\right)=\sin\alpha\cos\left(\sqrt{\lambda_n^{(i)}}x\right)+\frac{\cos\alpha+\frac{1}{2}\sin\alpha\int_0^x q(t)dt}{\sqrt{\lambda_n^{(i)}}}\sin\left(\sqrt{\lambda_n^{(i)}}x\right)+O\left(\frac{1}{n^2}\right) \tag{3.3}$$

for $\alpha\neq 0$ (i.e. $i=1,3$), and for $\alpha=0$ (i.e. $i=2,4$):

$$\phi\left(x,\lambda_n^{(i)}\right)=\frac{\sin\left(\sqrt{\lambda_n^{(i)}}x\right)}{\sqrt{\lambda_n^{(i)}}}-\frac{\int_0^x q(t)dt}{2\lambda_n^{(i)}}\cos\left(\sqrt{\lambda_n^{(i)}}x\right)+O\left(\frac{1}{n^3}\right). \tag{3.4}$$

For $i=1$ and large n eigenfunctions corresponding to the eigenvalue $\lambda_n^{(1)}$ have $n-m$ zeroes in $(0,1)$ and for $i=3$, the number of zeroes is $n-m+1$ if $\lim_{\lambda\to\infty}\Theta(\lambda)=+\infty$ and $n-m$ if $\lim_{\lambda\to\infty}\Theta(\lambda)=-\infty$. Therefore, zeroes may be labeled in a natural order; denote $x_n^{(i)j}$ as the jth zero of the $\phi\left(x,\lambda_n^{(i)}\right)$.

Then, for $i=1,3$, from

$$\begin{aligned}0&=\phi\left(x_n^{(i)j},\lambda_n^{(i)}\right)\\ &=\sin\alpha\cos\left(\sqrt{\lambda_n^{(i)}}x_n^{(i)j}\right)+\frac{\cos\alpha+\frac{1}{2}\sin\alpha\int_0^{x_n^{(i)j}}q(t)dt}{\sqrt{\lambda_n^{(i)}}}\sin\left(\sqrt{\lambda_n^{(i)}}x_n^{(i)j}\right)+O\left(\frac{1}{n^2}\right),\end{aligned}$$

we obtain

$$\cot\left(\sqrt{\lambda_n^{(i)}}x_n^{(i)j}\right)=-\frac{\cot\alpha+\frac{1}{2}\int_0^{x_n^{(i)j}}q(t)dt}{\sqrt{\lambda_n^{(i)}}}+O\left(\frac{1}{n^2}\right). \tag{3.5}$$

Using Taylor's expansions for the arctangent, we obtain the following asymptotic formula for nodal points as $n\to\infty$ uniformly in j:

$$\sqrt{\lambda_n^{(i)}}x_n^{(i)j}=\left(j-\frac{1}{2}\right)\pi+\frac{\cot\alpha+\frac{1}{2}\int_0^{x_n^{(i)j}}q(t)dt}{\sqrt{\lambda_n^{(i)}}}+O\left(\frac{1}{n^2}\right),$$

which implies

$$x_n^{(i)j} = \frac{(j-\frac{1}{2})\pi}{\sqrt{\lambda_n^{(i)}}} + \frac{\cot\alpha + \frac{1}{2}\int_0^{x_n^{(i)j}} q(t)dt}{\lambda_n^{(i)}} + O\left(\frac{1}{n^3}\right). \tag{3.6}$$

From (2.1), we get the following asymptotic formulae

$$\begin{aligned} \frac{1}{\sqrt{\lambda_n^{(i)}}} &= \frac{1}{(n-m_i)\pi} - \frac{c_i}{2n^3\pi^3} + o\left(\frac{1}{n^4}\right), \\ \frac{1}{\lambda_n^{(i)}} &= \frac{1}{(n-m_i)^2\pi^2} - \frac{c_i}{n^4\pi^4} + o\left(\frac{1}{n^5}\right). \end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.6) we have,

$$x_n^{(i)j} = \frac{j-\frac{1}{2}}{n-m_i} - \frac{(j-\frac{1}{2})c_i}{2(n-m_i)^3\pi^2} + \frac{\cot\alpha + \frac{1}{2}\int_0^{x_n^{(i)j}} q(t)dt}{(n-m_i)^2\pi^2} + O\left(\frac{1}{n^3}\right),\ n\to\infty.$$

For $i=2$ and large n eigenfunctions corresponding to the eigenvalue $\lambda_n^{(2)}$ have $n-m$ zeroes in $(0,1)$ and for $i=4$, the number of zeroes is $n-m+1$ if $\lim_{\lambda\to\infty}\Theta(\lambda) = +\infty$ and $n-m$ if $\lim_{\lambda\to\infty}\Theta(\lambda) = -\infty$. Therefore, zeroes may be labeled in a natural order; denote $x_n^{(i)j}$ as the jth zero of the $\phi\left(x, \lambda_n^{(i)}\right)$.

For $i = 2,4$, from

$$0 = \phi\left(x_n^{(i)j}, \lambda_n^{(i)}\right) = \frac{\sin\left(\sqrt{\lambda_n^{(i)}}x\right)}{\sqrt{\lambda_n^{(i)}}} - \frac{\int_0^x q(t)dt}{2\lambda_n^{(i)}}\cos\left(\sqrt{\lambda_n^{(i)}}x\right) + O\left(\frac{1}{n^3}\right),$$

we obtain

$$\tan\left(\sqrt{\lambda_n^{(i)}}x_n^{(i)j}\right) = \frac{\int_0^{x_n^{(i)j}} q(t)dt}{2\sqrt{\lambda_n^{(i)}}} + O\left(\frac{1}{n^2}\right). \tag{3.8}$$

Using Taylor's expansions for the arctangent, we obtain the following asymptotic formula for nodal points as $n\to\infty$ uniformly in j:

$$\sqrt{\lambda_n^{(i)}}x_n^{(i)j} = j\pi + \frac{\int_0^{x_n^{(i)j}} q(t)dt}{2\sqrt{\lambda_n^{(i)}}} + O\left(\frac{1}{n^2}\right),$$

which implies

$$x_n^{(i)j} = \frac{j\pi}{\sqrt{\lambda_n^{(i)}}} + \frac{\int_0^{x_n^{(i)j}} q(t)dt}{2\lambda_n^{(i)}} + O\left(\frac{1}{n^3}\right). \tag{3.9}$$

From (3.7), we get

$$x_n^{(i)j} = \frac{j}{n-m_i} - \frac{jc_i}{2(n-m_i)^3\pi^2} + \frac{\int_0^{x_n^{(i)j}} q(t)dt}{2(n-m_i)^2\pi^2} + O\left(\frac{1}{n^3}\right), \; n \to \infty.$$

The proof of theorem is finished. $\square$

In the above results, the estimate is independent of j. As a result,

$$l_n^{(i)j} \stackrel{def}{=} x_n^{(i)j+1} - x_n^{(i)j} = \frac{1}{n} + o\left(\frac{1}{n}\right). \tag{3.10}$$

COROLLARY 3.2. *From Lemma* 3.1 *it follows that the sets* $X^{(i)} = \{x_n^{(i)j}\}$ *is dense in* $[0,1]$.

Now we can give the proofs of the theorems in this work.

Proof of Theorem 2.1. Using the asymptotic expansions (3.1) and (3.2) for nodal points and the fact that $\lim_{n\to\infty} x_n^{(i)j} = x$, we get

$$\lim_{n\to\infty} \frac{j-\frac{1}{2}}{n-m_i} = x, \quad \lim_{n\to\infty} \frac{j}{n-m_i} = x.$$

Moreover, we obtain

$$\begin{aligned}
\lim_{n\to\infty} n\left(x_n^{(i)j} - \frac{j-\frac{1}{2}}{n}\right) &= \lim_{n\to\infty} n\left[\frac{j-\frac{1}{2}}{n-m_i} - \frac{j-\frac{1}{2}}{n} + O\left(\frac{1}{n^2}\right)\right] \\
&= \lim_{n\to\infty}\left[\frac{(j-\frac{1}{2})m_i}{n-m_i} + O(\frac{1}{n})\right] \\
&= m_i x = f_i(x),
\end{aligned}$$

$$\begin{aligned}
&\lim_{n\to\infty}(n-m_i)^2\left[x_n^{(i)j} - \frac{j-\frac{1}{2}}{n-m_i}\right] \\
&= \lim_{n\to\infty}(n-m_i)^2\left[\frac{j-\frac{1}{2}}{n-m_i} - \frac{(j-\frac{1}{2})c_i}{2(n-m_i)^3\pi^2} + \frac{\cot\alpha + \frac{1}{2}\int_0^{x_n^{(i)j}} q(t)dt}{(n-m_i)^2\pi^2} - \frac{j-\frac{1}{2}}{n-m_i} + O\left(\frac{1}{n^3}\right)\right] \\
&= \lim_{n\to\infty}\left[-\frac{(j-\frac{1}{2})c_i}{2(n-m_i)\pi^2} + \frac{\cot\alpha + \frac{1}{2}\int_0^{x_n^{(i)j}} q(t)dt}{\pi^2}\right] + O\left(\frac{1}{n}\right) \\
&= -\frac{c_i x}{2\pi^2} + \frac{\cot\alpha + \frac{1}{2}\int_0^x q(t)dt}{\pi^2} = g_i(x)
\end{aligned}$$

for the case $i=1,3$, and

$$\begin{aligned}\lim_{n\to\infty} n\left(x_n^{(i)j}-\frac{j}{n}\right) &= \lim_{n\to\infty} n\left[\frac{j}{n-m_i}-\frac{j}{n}+O\left(\frac{1}{n^2}\right)\right]\\ &= \lim_{n\to\infty}\left[\frac{jm_i}{n-m_i}+O(\frac{1}{n})\right]\\ &= m_i x = f_i(x),\end{aligned}$$

$$\begin{aligned}&\lim_{n\to\infty}(n-m_i)^2[x_n^{(i)j}-\frac{j}{n-m_i}]\\ &= \lim_{n\to\infty}(n-m_i)^2\left[\frac{j}{n-m_i}-\frac{jc_i}{2(n-m_i)^3\pi^2}+\frac{\int_0^{x_n^{(i)j}} q(t)dt}{2(n-m_i)^2\pi^2}-\frac{j}{n-m_i}+O\left(\frac{1}{n^3}\right)\right]\\ &= \lim_{n\to\infty}\left[-\frac{jc_i}{2(n-m_i)\pi^2}+\frac{\int_0^{x_n^{(i)j}} q(t)dt}{2\pi^2}\right]+O\left(\frac{1}{n}\right)\\ &= -\frac{c_i x}{2\pi^2}+\frac{\int_0^x q(t)dt}{2\pi^2}=g_i(x)\end{aligned}$$

for the case $i=2,4$. $\square$

Proof of Theorem 2.2. Now for $i\in\{1,2,3,4\}$ and given a nodal subset $X_0^{(i)}$, by Theorem 2.1 we can build up the reconstruction formulae.

Formulae (2.6) and (2.7) can be derived from (2.3) and (2.5) stepwise. We obtain the following procedure.

For the case $i=1,3$:

Step 1. Take $x=1$, we obtain

$$m=\begin{cases} f_1(1) & (i=1);\\ f_3(1)+\frac{1}{2} & (i=3).\end{cases}$$

Step 2. Take $x=0$, it follows $\cot\alpha=\pi^2 g_i(0)$.

Step 3. Take $x=1$, then it yields $A_m=\pi^2 g_1(1)$, $A_{m-1}=-\pi^2 g_3(1)$.

Step 4. By taking derivatives we obtain

$$q(x)-\overline{q}=\begin{cases} 2\pi^2\frac{d}{dx}g_1(x)-2\cot\alpha-2A_m & (i=1);\\ 2\pi^2\frac{d}{dx}g_3(x)-2\cot\alpha+2A_{m-1} & (i=3);\end{cases}$$

For the $i=2,4$:

Step 1. Take $x=1$, we obtain

$$m=\begin{cases} f_2(1)+\frac{1}{2} & (i=2);\\ f_4(1)-1 & (i=4).\end{cases}$$

Step 2. Take $x = 1$, then it yields $A_m = \pi^2 g_2(1)$, $A_{m-1} = -\pi^2 g_4(1)$.
Step 4. By taking derivatives we obtain

$$q(x) - \overline{q} = \begin{cases} 2\pi^2 \frac{d}{dx} g_2(x) - 2A_m & (i = 2); \\ 2\pi^2 \frac{d}{dx} g_4(x) + 2A_{m-1} & (i = 4). \end{cases}$$

Thus these formulae are constructed. Since each nodal data only determine a set of reconstruction formulae which only depend on nodal data, the uniqueness holds obviously. □

Proof of Theorem 2.3. We now proceed with the proof of theorem in the case $\alpha\tilde{\alpha} \neq 0$. The other cases are treated similarly.

Since

$$-\phi''(x,\lambda) + q(x)\phi(x,\lambda) = \lambda\phi(x,\lambda), \quad -\widetilde{\phi}''(x,\lambda) + \widetilde{q}(x)\widetilde{\phi}(x,\lambda) = \lambda\widetilde{\phi}(x,\lambda),$$
$$\phi(0,\lambda) = \widetilde{\phi}(0,\lambda) = 1, \quad \phi'(0,\lambda) = \cot\alpha, \quad \widetilde{\phi}'(0,\lambda) = \cot\widetilde{\alpha},$$

from the boundary conditions (1.2) it follows that

$$\int_0^1 Q(x)\phi(x,\lambda)\widetilde{\phi}(x,\lambda)dx = \phi'(1,\lambda)\widetilde{\phi}(1,\lambda) - \phi(1,\lambda)\widetilde{\phi}'(1,\lambda) - (\cot\alpha - \cot\widetilde{\alpha}), \tag{3.11}$$

where $Q(x) = q(x) - \widetilde{q}(x)$. Using (3.11), the boundary conditions (1.2) and $q(x) = \widetilde{q}(x)$ a.e. on $(b,1)$ we obtain

$$\int_0^b Q(x)\phi(x,\lambda_n^{(i)})\widetilde{\phi}(x,\lambda_n^{(i)})dx + (\cot\alpha - \cot\widetilde{\alpha}) = 0, \quad \lambda_n^{(i)} \in \Omega. \tag{3.12}$$

Since $\phi(x,\lambda)$ is the solution of the equation (1.1) satisfying the initial conditions $\phi(0,\lambda) = 1$ and $\phi'(0,\lambda) = \cot\alpha$, there exists a bounded function $K(x,t)$ (independent of λ) such that [12]

$$\phi(x,\lambda)\widetilde{\phi}(x,\lambda) = \frac{1}{2} + \frac{1}{2}\cos(2kx) + \frac{1}{2}\int_0^x K(x,t)\cos(2kt)dt, \tag{3.13}$$

where $k = \sqrt{\lambda}$. Substituting (3.13) into (3.12), we calculate

$$(\cot\alpha - \cot\widetilde{\alpha}) + \frac{1}{2}\int_0^b Q(x)dx = 0 \tag{3.14}$$

and

$$\int_0^b [Q(x) + \int_x^b K(x,t)Q(t)dt]\cos\left(2\sqrt{\lambda_n^{(i)}}x\right)dx = 0, \quad \lambda_n^{(i)} \in \Omega; \tag{3.15}$$

consequently, by the completeness of the system $\left\{\cos\left(2\sqrt{\lambda_n^{(i)}}x\right)\right\}_{n\in\Lambda}$ in $L^2(0,b)$, it yields

$$Q(x) + \int_x^b K(x,t)Q(t)dt = 0 \text{ a.e. on } (0,b). \tag{3.16}$$

But this homogeneous Volterra integral equation has only the trivial solution it follows that $Q(x)=0$ a.e. on $(0,b)$, i.e., $q(x)=\widetilde{q}(x)$ for almost all $x\in[0,b]$. From (3.14) the equality $\alpha=\widetilde{\alpha}$ becomes obvious. $\square$

To prove Theorem 2.4 we need a Lemma.

LEMMA 3.3. [5, 27] *Fix i, n and j. Let $x_n^{(i)j}=\widetilde{x}_n^{(i)j}$, $x_n^{(i)j+1}=\widetilde{x}_n^{(i)j+1}$, and let $q(x)-\overline{q}=\widetilde{q}(x)-\overline{\widetilde{q}}$ a.e. on $(x_n^{(i)j},x_n^{(i)j+1})$. Then $\lambda_n^{(i)}-\overline{q}=\widetilde{\lambda}_n^{(i)}-\overline{\widetilde{q}}$.*

Proof. Fix i, n and j. On the interval $(x_n^{(i)j},x_n^{(i)j+1})$ we consider two the Dirichlet boundary value problems

$$-y''(x)+[q(x)-\overline{q}]\,y(x)=[\lambda-\overline{q}]\,y(x) \tag{3.17}$$

with the boundary conditions

$$y(x_n^{(i)j})=y(x_n^{(i)j+1})=0$$

and

$$-y''(x)+\left[\widetilde{q}(x)-\overline{\widetilde{q}}\right]y(x)=\left[\widetilde{\lambda}_n-\overline{\widetilde{q}}\right]y(x) \tag{3.18}$$

with the boundary conditions

$$y(x_n^{(i)j})=y(x_n^{(i)j+1})=0.$$

The function $y_n(x)=\phi\left(x,\lambda_n^{(i)}\right)$ is an eigenfunction for the problem (3.17) with the eigenvalue $\lambda_n^{(i)}-\overline{q}$. Since $\phi\left(x,\lambda_n^{(i)}\right)$ has no zeros $x\in(x_n^{(i)j},x_n^{(i)j+1})$, the Sturm oscillation theorem implies that $\lambda_n^{(i)}-\overline{q}$ is the first eigenvalue for the problem (3.17), and $\phi\left(x,\lambda_n^{(i)}\right)$ is the first eigenfunction for the problem (3.17) with the potential $q(x)-\overline{q}$. Similarly, $\widetilde{\lambda}_n^{(i)}-\overline{\widetilde{q}}$ is the first eigenvalue for the problem (3.18), and $\widetilde{\phi}\left(x,\widetilde{\lambda}_n^{(i)}\right)$ is the first eigenfunction for the problem (3.18) with the potential $\widetilde{q}(x)-\overline{\widetilde{q}}$. Since $q(x)-\overline{q}=\widetilde{q}(x)-\overline{\widetilde{q}}$ a.e. on $(x_n^{(i)j},x_n^{(i)j+1})$, therefore, $\lambda_n^{(i)}-\overline{q}=\widetilde{\lambda}_n^{(i)}-\overline{\widetilde{q}}$. $\square$

Proof of Theorem 2.4. Since $X_0^{(i)}=\widetilde{X}_0^{(i)}$, from Theorem 2.1 it follows that $f_i(x)=\widetilde{f}_i(x)$ and $g_i(x)=\widetilde{g}_i(x)$ for $x\in(b,1)$. Using (2.6) and (2.7) we obtain $q(x)=\widetilde{q}(x)$ on $(b,1)$. By Lemma 3.3, $\lambda_{m(n)}^{(i)}=\widetilde{\lambda}_{m(n)}^{(i)}$ for $m(n)\in\Lambda_{X_0^{(i)}}$. Thus, from (2.1), the equality $\alpha=\widetilde{\alpha}$ holds. Applying Theorem 2.3 we conclude that $q(x)=\widetilde{q}(x)$ a.e. on $(0,1)$. $\square$

Proof of Theorem 2.5. We now proceed with the proof of theorem in the case $\alpha\tilde{\alpha}\neq 0$. The other cases are treated similarly.

First, since $X_0^{(i)}\subset X^{(i)}\bigcap(b,1)$ is a dense on $(b,1)$ twin subset of nodal points and $X_0^{(i)}=\widetilde{X}_0^{(i)}$, by Theorem 2.2 we obtain $q(x)=\widetilde{q}(x)$ on $(b,1)$. By Lemma 3.3, $\lambda_{m(n)}^{(i)}=\widetilde{\lambda}_{m(n)}^{(i)}$ for $m(n)\in\Lambda_{X_0^{(i)}}$. Thus, from (2.1), the equality $\alpha=\widetilde{\alpha}$ holds.

Since

$$-\phi''(x,\lambda)+q(x)\phi(x,\lambda)=\lambda\phi(x,\lambda),\ -\widetilde{\phi}''(x,\lambda)+\widetilde{q}(x)\widetilde{\phi}(x,\lambda)=\lambda\widetilde{\phi}(x,\lambda),$$
$$\phi(0,\lambda)=\widetilde{\phi}(0,\lambda)=1,\ \phi'(0,\lambda)=\widetilde{\phi}'(0,\lambda)=\cot\alpha,$$

from the boundary conditions (1.2) and $q(x)=\widetilde{q}(x)$ on $(b,1)$ it follows that

$$G(k)=\int_0^b Q(x)\phi(x,\lambda)\widetilde{\phi}(x,\lambda)dx=\phi'(1,\lambda)\widetilde{\phi}(1,\lambda)-\phi(1,\lambda)\widetilde{\phi}'(1,\lambda), \tag{3.19}$$

where $k=\sqrt{\lambda}$ and $Q(x)=q(x)-\widetilde{q}(x)$. Using (3.19) we obtain

$$G(s_{m(n)}^{(i)})=0,\ s_{m(n)}^{(i)}=\sqrt{\lambda_{m(n)}^{(i)}}. \tag{3.20}$$

Next, we will show that $G(k)=0$ on the whole k-plane.

From (3.13) we see that the entire function $G(k)$ is a function of exponential type $\leqslant 2b$. One has

$$|G(k)|\leqslant Ce^{2br|\sin\theta|} \tag{3.21}$$

for some positive constant C, $k=\sqrt{\lambda}=re^{i\theta}$, $|\Im\sqrt{\lambda}|=r|\sin\theta|$ and $\theta=\arg\sqrt{\lambda}$.

Define an indicator of the function $G(k)$ by

$$h(\theta)=\limsup_{r\to\infty}\frac{\ln|G(re^{i\theta})|}{r}. \tag{3.22}$$

From (3.21) and (3.22) one obtains the following estimate

$$h(\theta)\leqslant 2b|\sin\theta|. \tag{3.23}$$

It is known [17] that for any entire function $G(k)$ of exponential type, not identically zero, one has

$$\liminf_{r\to\infty}\frac{n(r)}{r}\leqslant\frac{1}{2\pi}\int_0^{2\pi}h(\theta)d\theta, \tag{3.24}$$

where $n(r)$ is the number of zeros of $G(k)$ in the disk $|k|\leqslant r$. By (3.23),

$$\frac{1}{2\pi}\int_0^{2\pi}h(\theta)d\theta\leqslant\frac{b}{\pi}\int_0^{2\pi}|\sin\theta|d\theta=\frac{4b}{\pi}. \tag{3.25}$$

From the assumption and the known asymptotic expression (2.1) of the eigenvalues $\lambda_n^{(i)}$, for the number of zeros of $G(k)$ in the disk $|k|\leqslant r$ we have the estimate

$$n(r)\geqslant 2\sum_{\frac{n\pi}{\sigma}(1+o(1))<r}1=\frac{2\sigma r}{\pi}[1+o(1)],\ r\to\infty. \tag{3.26}$$

Since $\sigma > 2b$, we get

$$\lim_{n\to\infty} \frac{n(r)}{r} \geqslant \frac{2\sigma}{\pi} > \frac{4b}{\pi} \geqslant \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta. \tag{3.27}$$

The inequalities (3.24) and (3.27) imply that $G(k) \equiv 0$ on the whole k-plane.

Repeating the proof of Theorem 2.3, we have $q(x) = \widetilde{q}(x)$ for almost all $x \in [0,b]$. $\square$

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