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ON SPLITTINGS OF MATRICES AND NONNEGATIVE GENERALIZED INVERSES

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Abstract. The authors introduce a new type of matrix splitting generalizing the notion of B-splitting and study its relationships with nonnegativity of the Moore-Penrose inverse and the group inverse.

1. Introduction

A real $n \times n$ matrix A is called *monotone* (or a matrix of "*monotone kind*") if $Ax \geqslant 0 \Rightarrow x \geqslant 0$. Here, $y \geqslant 0$ for $(y_1, y_2, \ldots, y_n)^T = y \in \mathbb{R}^n$ means that $y_i \geqslant 0$ for all $i = 1, 2, \ldots, n$. This notion was introduced by Collatz, who showed that A is monotone if and only if A^{-1} exists and $A^{-1} \geqslant 0$, where the latter denotes that all the entries of A^{-1} are nonnegative. The book by Collatz [7] has details of how monotone matrices arise naturally in the study of finite difference approximation methods for certain elliptic partial differential equations. The problem of characterizing monotone (also referred to as inverse positive) matrices has been extensively dealt with in the literature. Motivated by Collatz's result, Mangasarian [11] extended the concept of monotone matrices to the rectangular case, and proved that a rectangular matrix is monotone if and only if it has a nonnegative left inverse. The books by Berman and Plemmons [5], and Varga [18] give an excellent account of many of these characterizations. The former also presents several extensions to generalized inverses.

Much effort also has been devoted to characterizing inverse positive matrices in terms of the so-called splittings of the matrix concerned. For a real $n \times n$ matrix A, a decomposition $A = U - V$ is called a splitting if U is invertible. Associated with the splitting, one is interested in the convergence of the iterative method $x^{k+1} = U^{-1}Vx^k + U^{-1}b$, $k = 0, 1, 2, \cdots$ for numerically solving the linear system of equations $Ax = b$, $b \in \mathbb{R}^m$. It is well known that this iterative scheme converges to a solution of $Ax = b$ if and only if spectral radius of $U^{-1}V$ is strictly less than 1, for any initial vector x^0. Standard iterative methods like the Jacobi, Gauss-Seidel and successive over-relaxation methods arise from different choices of U and V. Below, we briefly review the most important types of splittings which have been studied in the literature. Schröder [17] and Varga [18] proposed the notion of a regular splitting as follows: $A = U - V$ is called a regular

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splitting if U is invertible, $U^{-1} \geqslant 0$ and $V \geqslant 0$. For any regular splitting $A = U - V$, they demonstrated that A is inverse positive if and only if $U^{-1}V$ has spectral radius strictly less than one. Ortega and Rheinboldt [14] proposed the notion of a weak regular splitting: $A = U - V$ is called a weak regular splitting if U is invertible, $U^{-1} \geqslant 0$ and $U^{-1}V \geqslant 0$. They show that in fact, for any weak regular splitting $A = U - V$, A is inverse positive if and only if $U^{-1}V$ has spectral radius strictly less than one. In both the cases, the iterative method arising from the splitting converges to a solution of the system $Ax = b$, for any initial vector x^0. It is noteworthy to point out the fact that the three methods mentioned above belong to either of the two types of splittings defined here. We refer to the book by Varga for the precise details.

More recently, Peris [15] studied what are called positive splittings. Specifically, a splitting $A = U - V$ is known as a positive splitting if $U \geqslant 0$ and $V \geqslant 0$. He came up with the following characterization: An invertible matrix A has a nonnegative inverse if and only if for any positive splitting $A = U - V$, there exist a vector $x > 0$ (meaning that all the components are positive) and a scalar $\mu \in [0,1)$ such that $Vx = \mu Ux$. He also gave a characterization of inverse positivity in terms of a subclass of positive splittings, namely B-splittings. In what follows, we say that $A = U - V$ is a B-splitting of A if U is invertible, $VU^{-1} \geqslant 0$, and $Ux \geqslant 0,\ Ax \geqslant 0 \Rightarrow x \geqslant 0$ for all $x \in \mathbb{R}^n$. In a result most pertinent to the present work, Peris showed that if A is inverse positive, then *there exists a B-splitting $A = U - V$* with VU^{-1} having spectral radius strictly less than one. In this connection, let us point out that Weber ([19] and [20]) generalized the work of Peris for bounded linear operators over certain ordered Banach spaces. Generalizations of inverse positivity to nonnegativity of the Moore-Penrose inverse and their relationships to the concept of proper splittings were studied mainly by Berman, Plemmons and Neumann ([2], [3] and [5]). We defer the discussion on proper splittings and nonnegative Moore-Penrose inverses to the section on main results. Analogous to the nonsingular case, proper splittings lead to the iteration scheme: $x^{k+1} = U^{\dagger}Vx^k + U^{\dagger}b,\ k = 0,1,2,\cdots$. It is shown in [3] that for a proper splitting, the spectral radius of $U^{\dagger}V$ is strictly less than 1 if and only if the above scheme converges to $A^{\dagger}b$, the least squares solution of minimum norm of the system $Ax = b$.

The objective of this work is to show how Peris' results can be extended to the case of the Moore-Penrose inverse and the group inverse. Our approach here has been inspired and guided mainly by the aforementioned recent work of Weber, [20]. We hasten to add that while Weber studied the infinite dimensional situation, our frame work is the finite dimensional real Euclidean space. We believe that the results presented here should lead to a similar theory in the infinite dimensional case, too. The organization of this paper is as follows. In Section 2, we fix our notation and discuss preliminary notions and results that will be used in the sequel. Section 3 presents the main results for the Moore-Penrose inverse. In Section 4, the corresponding group inverse results are stated. Closing remarks in the final Section propose three problems of interest for future work. A preliminary report of the results presented here appears in [13].

2. Preliminaries

Let $\mathbb{R}^n$ denote the n dimensional real Euclidean space and $\mathbb{R}^n_+$ denote the nonnegative orthant in $\mathbb{R}^n$. Let $int(\mathbb{R}^n_+)$ denote the set of all interior points of $\mathbb{R}^n_+$. For a matrix $A \in \mathbb{R}^{m\times n}$, the set of all $m \times n$ matrices of reals, we denote the range space, the null space and the transpose of A by $R(A)$, $N(A)$ and A^T, respectively. Let K,L be complementary subspaces of $\mathbb{R}^n$, i.e., $K \oplus L = \mathbb{R}^n$. Then $P_{K,L}$ denotes the (not necessarily orthogonal) projection of $\mathbb{R}^n$ onto K along L. So we have $P^2_{K,L} = P_{K,L}$, $R(P_{K,L}) = K$ and $N(P_{K,L}) = L$. If in addition $K \perp L$, $P_{K,L}$ will be replaced by P_K. In such case, we also have $P^T_K = P_K$. For $A \in \mathbb{R}^{m\times n}$, $A \geqslant 0$ means all the entries of A are nonnegative. For $A,B \in \mathbb{R}^{m\times n}$, $A \leqslant B$ denotes that $B-A \geqslant 0$. The spectral radius of $A \in \mathbb{R}^{n\times n}$, denoted by $\rho(A)$ is defined by $\rho(A) = \max_{1\leqslant i\leqslant n} |\lambda_i|$ where $\lambda_1,\lambda_2,\cdots,\lambda_n$ are the eigenvalues of A. A factorization $A = BC$ of $A \in \mathbb{R}^{m\times n}$ is called a full-rank factorization if $B \in \mathbb{R}^{m\times r}$ and $C \in \mathbb{R}^{r\times n}$ with rank $B =$rank $C =$rank $A = r$.

The Moore-Penrose inverse of matrix $A \in \mathbb{R}^{m\times n}$, denoted by $A^\dagger$ is the unique solution of the equations: $A = AXA$, $X = XAX$, $(AX)^T = AX$ and $(XA)^T = XA$. The group inverse of a matrix $A \in \mathbb{R}^{n\times n}$, denoted by $A^\#$ (if it exists), is the unique matrix X satisfying $A = AXA$, $X = XAX$ and $AX = XA$. If A is nonsingular, then of course, we have $A^{-1} = A^\dagger = A^\#$. Next, we collect some well known properties of $A^\dagger$ and $A^\#$ ([1]) which will be frequently used in this paper: $R(A^T) = R(A^\dagger)$; $N(A^T) = N(A^\dagger)$; $AA^\dagger = P_{R(A)}$; $A^\dagger A = P_{R(A^T)}$; $R(A) = R(A^\#)$; $N(A) = N(A^\#)$; $AA^\# = P_{R(A),N(A)}$. In particular, if $x \in R(A^T)$ then $x = A^\dagger Ax$ and if $x \in R(A)$ then $x = A^\# Ax$.

Next, we list certain results to be used in the sequel. The first result is well known, for instance refer to [1].

LEMMA 2.1. *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. The system $Ax = b$ has a solution if and only if $AA^\dagger b = b$. In that case, the general solution is given by $x = A^\dagger b + z$ for some $z \in N(A)$.*

The next result characterizes the "reverse order law" for the Moore-Penrose inverse.

THEOREM 2.2. (Theorem 1, [8]) *Let X and Y be arbitrary matrices such that XY is defined. Then $(XY)^\dagger = Y^\dagger X^\dagger$ if and only if $X^\dagger XYY^TX^T = YY^TX^T$ and $YY^\dagger X^TXY = X^TXY$.*

The following results are finite dimensional versions of corresponding results which hold in Banach spaces.

THEOREM 2.3. (Theorem 3.16, [18]) *Let $X \in \mathbb{R}^{n\times n}$ and $X \geqslant 0$. Then $\rho(X) < 1$ if and only if $(I-X)^{-1}$ exists and $(I-X)^{-1} = \sum_{k=0}^{\infty} X^k \geqslant 0$.*

THEOREM 2.4. (Theorem 25.4, [9]) *Suppose that $C,B \in \mathbb{R}^{n\times n}$ with $C \leqslant B$, B^{-1} exists, and $B^{-1} \geqslant 0$. Then C^{-1} exists and $C^{-1} \geqslant 0$ if and only if $C\mathbb{R}^n_+ \cap int(\mathbb{R}^n_+) \neq \phi$.*

3. Main Results

In this section we present our main results (Theorem 3.8 and Theorem 3.10). Let $A \in \mathbb{R}^{m\times n}$. A decomposition $A = U - V$ is called a proper splitting of A [3] if $R(A) = R(U)$ and $N(A) = N(U)$. The following theorem is similar to the corresponding result (Theorem 1) in [3]. It collects some of the properties of a proper splitting.

THEOREM 3.1. *Let $A = U - V$ be a proper splitting. Then*
(a) $AA^\dagger = UU^\dagger$; $A^\dagger A = U^\dagger U$.
(b) $A = (I - VU^\dagger)U$.
(c) $I - VU^\dagger$ is invertible.
(d) $A^\dagger = U^\dagger(I - VU^\dagger)^{-1}$.

Proof.
(a) $AA^\dagger = P_{R(A)} = P_{R(U)} = UU^\dagger$. The second identity is similar.

(b) Since $N(A) = N(U)$, it follows that $N(A) \subseteq N(V)$, that is $R(V^T) \subseteq R(A^T) = R(U^T)$. Hence $U^\dagger UV^T = V^T$ so that $V = VU^\dagger U$ and so $A = U - V = U - VU^\dagger U = (I - VU^\dagger)U$.

(c) Let $(I - VU^\dagger)x = 0$. Then $x = VU^\dagger x \in R(V) \subseteq R(A) = R(U)$. So $x = UU^\dagger x$ and hence $x = VU^\dagger x = (U - A)U^\dagger x = UU^\dagger x - AU^\dagger x = x - AU^\dagger x$. So, $AU^\dagger x = 0$. Thus, $U^\dagger x \in N(A) = N(U)$ and so $x = UU^\dagger x = 0$. Hence $I - VU^\dagger$ is invertible.

(d) Set $X = I - VU^\dagger$, $Y = U$. Then $A = XY$, $X^\dagger = X^{-1}$ and $Y^\dagger = U^\dagger$. We have $R(X^T XY) = R((I - VU^\dagger)^T(I - VU^\dagger)U) = R((I - VU^\dagger)^T A) = R(A - (U^\dagger)^T V^T A) \subseteq R(A) = R(U)$ and so $YY^\dagger X^T XY = X^T XY$. Also, since $X^\dagger X = I$, we have $X^\dagger XYY^T X^T = YY^T X^T$. By Theorem 2.2, we have $A^\dagger = U^\dagger(I - VU^\dagger)^{-1}$. $\square$

It is known that [2] any $A \in \mathbb{R}^{m\times n}$ of rank r has a full-rank factorization of the form

$$A = P\begin{pmatrix} I \\ C \end{pmatrix} A_{11} \begin{pmatrix} I & B \end{pmatrix} Q,$$

where A_{11} and I are of order $r\times r$, P and Q are permutation matrices of order m and n, $C \in \mathbb{R}^{(m-r)\times r}$ and $B \in \mathbb{R}^{r\times(n-r)}$. The next result shows how to construct proper splittings using the factorization as above.

THEOREM 3.2. (Theorem 1, [2]) *Let A be factorized as above. Then $A = U - V$ is a proper splitting if and only if $U = P\begin{pmatrix} I \\ C \end{pmatrix} M_{11} \begin{pmatrix} I & B \end{pmatrix} Q$, where M_{11} is a nonsingular matrix of order $r\times r$.*

Using this theorem, we next derive a simpler method of constructing a proper splitting. This result is not new. However, we have not found a proof in the literature.

THEOREM 3.3. *Let $A \in \mathbb{R}^{m\times n}$ and $A = FG$ be a full-rank factorization. Then the splitting $A = U - V$ is proper if and only if $U = FSG$ (and $V = U - A$) for some nonsingular S. In this case $\rho(VU^\dagger) = \rho(I - S^{-1})$.*

Proof. *Necessity.* From Theorem 3.2, we have $U = FSG$ where $F = P\begin{pmatrix} I \\ C \end{pmatrix}$, $G = A_{11}\begin{pmatrix} I & B \end{pmatrix}Q$ and $S = M_{11}A_{11}^{-1}$ is nonsingular. The fact that $U^\dagger = G^\dagger S^{-1}F^\dagger$ is easily verified. Now $VU^\dagger = (U-A)G^\dagger S^{-1}F^\dagger = (FSG-FG)G^\dagger S^{-1}F^\dagger = FF^\dagger - FS^{-1}F^\dagger = F(I-S^{-1})F^\dagger$. To prove $\rho(VU^\dagger) = \rho(I-S^{-1})$, it is enough to show that $VU^\dagger$ and $I-S^{-1}$ have the same nonzero eigenvalues. Let λ be any nonzero eigenvalue of $VU^\dagger$ and x be a corresponding eigenvector. Then $F(I-S^{-1})F^\dagger x = VU^\dagger x = \lambda x$. Pre-multiplying by $F^\dagger$ and using the fact $F^\dagger F = I$, we have $(I-S^{-1})w = \lambda w$ where $F^\dagger x = w \neq 0$. (If $F^\dagger x = 0$, then $x \in N(F^\dagger) = N(A^T)$. Also $x \in R(V) \subseteq R(A)$. So $x = 0$, a contradiction). Thus λ is an eigenvalue of $I-S^{-1}$. On the other hand, let μ be a nonzero eigenvalue of $I-S^{-1}$ and y be a corresponding eigenvector. Then $(I-S^{-1})y = \mu y$ so that $F^\dagger Fy - S^{-1}F^\dagger Fy = \mu F^\dagger Fy$. If $Fy = 0$, then pre-multiplying $(I-S^{-1})y = \mu y$ by F, we get $FS^{-1}y = 0$ yielding $0 = S^{-1}y = F^\dagger FS^{-1}y$, so that $y = 0$, a contradiction. Thus $z = Fy \neq 0$. We then have $\mu F^\dagger z = \mu F^\dagger Fy = \mu y = (I-S^{-1})F^\dagger z$. Thus $VU^\dagger z = F(I-S^{-1})F^\dagger z = \mu FF^\dagger z = \mu FF^\dagger Fy = \mu z$. Thus $VU^\dagger$ and $I-S^{-1}$ have the same nonzero eigenvalues. It now follows that $\rho(VU^\dagger) = \rho(I-S^{-1})$.

Sufficiency. For a full-rank factorization $A = FG$, let $U = FSG$ for some nonsingular S and $V = U - A$. We must show that the splitting $A = U - V$ is proper, i.e., $R(A) = R(U)$ and $N(A) = N(U)$. Let $x \in N(A)$. Then $Ax = 0$ so that $FGx = 0$. We then have $Gx = 0$, since F is of full-column rank. We then have $Ux = FSGx = 0$, showing that $N(A) \subseteq N(U)$. Retracing the above steps and using the invertibility of S, it follows that $N(U) \subseteq N(A)$. If $A = FG$ is a full-rank factorization of A, then $A^T = G^T F^T$ is a full-rank factorization of A^T. The fact that $N(A^T) = N(U^T)$ now follows similarly. This, in turn means that $R(A) = R(U)$. $\square$

The next result is similar to a well known theorem of Berman and Plemmons (Theorem 3, [3]) and is included for ready reference and completeness.

THEOREM 3.4. *Let $A \in \mathbb{R}^{m\times n}$ and $A = U - V$ be a proper splitting of A. If $U^\dagger \geqslant 0$, $VU^\dagger \geqslant 0$ and $\rho(VU^\dagger) < 1$, then $A^\dagger \geqslant 0$.*

Proof. Let $U^\dagger \geqslant 0$, $VU^\dagger \geqslant 0$ and $\rho(VU^\dagger) < 1$. The existence of $(I - VU^\dagger)^{-1}$ is guaranteed by $A = U - V$ being a proper splitting or by invoking the condition $\rho(VU^\dagger) < 1$. The latter also ensures that the representation $(I - VU^\dagger)^{-1} = \sum_{j=0}^{\infty}(VU^\dagger)^j$ holds. As $VU^\dagger \geqslant 0$, it follows that $(I - VU^\dagger)^{-1} \geqslant 0$. The proof is complete by recalling that $A^\dagger = U^\dagger(I - VU^\dagger)^{-1} \geqslant 0$. $\square$

REMARK 3.5. It can be also shown that if $A = U - V$ is a proper splitting of A with $U^\dagger \geqslant 0$, $VU^\dagger \geqslant 0$ and $A^\dagger \geqslant 0$, then $\rho(VU^\dagger) < 1$. This is similar in spirit to the nonsingular case mentioned in the introduction. We propose a definition for the splitting as above. A splitting is called a weak pseudo regular splitting if it is a proper splitting such that $U^\dagger \geqslant 0$ and $VU^\dagger \geqslant 0$. We can now say that if A has a weak pseudo regular splitting $A = U - V$, then $\rho(VU^\dagger) < 1$ if and only if $A^\dagger \geqslant 0$.

At this juncture, we would like to draw the attention of the reader to the works of Climent et. al. [6] and Yimin Wei et. al. [21] & [22]. In the first article the authors discuss the nonnegativity of the Moore-Penrose inverse of full-column rank matrices, under the assumption of the existence of a proper splitting. In the next two, the authors study splittings more general than proper splittings and also consider nonnegativity of generalized inverses satisfying certain specific properties.

The notion of B-splitting, due to Peris, was mentioned in the introduction. We propose an extension called $B_\dagger$-splitting, in what follows. This splitting is also different from the weak pseudo regular splitting.

DEFINITION 3.6. A proper splitting $A = U - V$ of A is called a $B_\dagger$-splitting if it satisfies the following conditions:

(i) $U \geqslant 0$,
(ii) $V \geqslant 0$,
(iii) $VU^\dagger \geqslant 0$, and
(iv) $Ax, Ux \in \mathbb{R}^m_+ + N(A^T)$ and $x \in R(A^T) \Rightarrow x \geqslant 0$.

The next example shows a matrix allowing a $B_\dagger$-splitting, but not a B-splitting.

EXAMPLE 3.7. The matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ does not allow a B-splitting. However, by taking $U = 2A$ and $V = A$, we obtain a $B_\dagger$-splitting.

The class of matrices having all entries 1 always have a $B_\dagger$-splitting. Clearly, any B-splitting is a $B_\dagger$-splitting.

Berman and Plemmons (Theorem 3, [4]) have shown the following equivalence. Let $A \in \mathbb{R}^{m\times n}$. Then $A^\dagger \geqslant 0$ if and only if $Ax \in \mathbb{R}^m_+ + N(A^T)$ and $x \in R(A^T) \Rightarrow x \geqslant 0$. In the first main result of this article given next, we provide other equivalences, including one involving a $B_\dagger$-splitting.

THEOREM 3.8. *Let $A \in \mathbb{R}^{m\times n}$. Consider the following statements.*

(a) $A^\dagger \geqslant 0$.
(b) $Ax \in \mathbb{R}^m_+ + N(A^T)$ *and* $x \in R(A^T) \Rightarrow x \geqslant 0$.
(c) $\mathbb{R}^m_+ \subseteq A\mathbb{R}^n_+ + N(A^T)$.
(d) There exists $x^0 \in \mathbb{R}^n_+$ *and* $z^0 \in N(A^T)$ *such that* $Ax^0 + z^0 \in int(\mathbb{R}^m_+)$.

Then we have (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

Suppose that A has a $B_\dagger$-splitting. Then each of the above is equivalent to the following:

(e) $\rho(VU^\dagger) < 1$.

Proof. (a) $\Leftrightarrow$ (b): This is the result of Berman and Plemmons mentioned above.

(b) $\Rightarrow$ (c): Let $p \in \mathbb{R}^m_+$ and $q = A^\dagger p$. Then by Lemma 2.1, $p = Aq + r$, $r \in N(A^T)$ so that $Aq = p - r \in \mathbb{R}^m_+ + N(A^T)$. Also $q \in R(A^T)$. Hence $q \in \mathbb{R}^n_+$ and so $p \in A\mathbb{R}^n_+ + N(A^T)$.

(c) $\Rightarrow$ (d): Let $u^0 \in int(\mathbb{R}^m_+)$. Then there exist $x^0 \in \mathbb{R}^n_+$ and $z^0 \in N(A^T)$ such that $u^0 = Ax^0 + z^0$. Thus $Ax^0 + z^0 \in int(\mathbb{R}^m_+)$.

(d) $\Rightarrow$ (e): Since $A = U - V$ is a $B_\dagger$-splitting, we have $U \geqslant 0$, $V \geqslant 0$, $VU^\dagger \geqslant 0$, $R(A) = R(U)$ and $N(A) = N(U)$ (Condition (iv) will not be used in the proof). So $I - VU^\dagger \leqslant I$. Set $C = I - VU^\dagger$ and $B = I$. Then $C \leqslant B$, B^{-1} exists and $B^{-1} \geqslant 0$. We show that there exists a vector $w^0 \in \mathbb{R}^m_+$ such that $Cw^0 \in int(\mathbb{R}^m_+)$. It would then follow from Theorem 2.4 that C^{-1} exists and $C^{-1} \geqslant 0$. By (d), there exists $x^0 \in \mathbb{R}^n_+$ and $z^0 \in N(A^T)$ such that $Ax^0 + z^0 \in int(\mathbb{R}^m_+)$. Set $w^0 = Ux^0 + z^0$. Then $w^0 = (A + V)x^0 + z^0 = Ax^0 + z^0 + Vx^0$. Since $V \geqslant 0$ and $x^0 \in \mathbb{R}^n_+$, we have $Vx^0 \in \mathbb{R}^m_+$. Also $Ax^0 + z^0 \in \mathbb{R}^m_+$. Thus $w^0 \in \mathbb{R}^m_+$. Further, $Cw^0 = (I - VU^\dagger)w^0 = (I - VU^\dagger)(Ux^0 + z^0) = (I - VU^\dagger)Ux^0 + (I - VU^\dagger)z^0 = Ax^0 + z^0 \in int(\mathbb{R}^m_+)$, where we have used the fact that $z^0 \in N(A^T) = N(U^T) = N(U^\dagger)$. Thus $(I - VU^\dagger)^{-1} = C^{-1} \geqslant 0$. By Theorem 2.3, it now follows that $\rho(VU^\dagger) < 1$.

(e) $\Rightarrow$ (b): Suppose that $Ax \in \mathbb{R}^m_+ + N(A^T)$. We demonstrate that $Ux \in \mathbb{R}^m_+ + N(A^T)$. It would then follow from condition (iv) of a $B_\dagger$-splitting, that $x \geqslant 0$. Let $Ax = p + q$, where $p \in \mathbb{R}^m_+$ and $q \in N(A^T)$. Then $Ux = (I - VU^\dagger)^{-1}Ax = (I - VU^\dagger)^{-1}(p + q) = r + s$, where $r = (I - VU^\dagger)^{-1}p$ and $s = (I - VU^\dagger)^{-1}q$. Since $\rho(VU^\dagger) < 1$ and $VU^\dagger \geqslant 0$, it follows from Theorem 2.3, that $I - VU^\dagger$ is invertible and that $(I - VU^\dagger)^{-1} \geqslant 0$. Thus $r \in \mathbb{R}^m_+$. Also, $q \in N(A^T) = N(A^\dagger)$ yields $0 = A^\dagger q = U^\dagger(I - VU^\dagger)^{-1}q = U^\dagger s$, showing that $s \in N(U^T) = N(A^T)$. Hence $Ux \in \mathbb{R}^m_+ + N(A^T)$, as was required to prove. $\square$

We obtain Peris' result as a particular case. For a vector $y \in \mathbb{R}^m$, $y > 0$ denotes $y \in int(\mathbb{R}^m_+)$.

COROLLARY 3.9. (Theorem 4, [15]) *Let A be a square matrix such that $A = U - V$ is a B-splitting. Then the following conditions are equivalent:*

(a) A is positively invertible.

(b) $\rho(VU^{-1}) < 1$.

(c) There exists some $x \geqslant 0$ such that $Ax > 0$.

Presently, it is not known if all matrices possess $B_\dagger$-splittings. However, we are able to prove its existence for a class of matrices satisfying certain conditions. This is given below.

THEOREM 3.10. *Suppose that $A^\dagger \geqslant 0$ and $R(A) \cap int(\mathbb{R}^m_+) \neq \phi$. Further, let $A^\dagger A \geqslant 0$. Then A possesses a $B_\dagger$-splitting $A = U - V$ such that $\rho(VU^\dagger) < 1$.*

Proof. Let $0 \neq p \in R(A) \cap int(\mathbb{R}^m_+)$ and $q \in int(\mathbb{R}^n_+)$. Set $E = pq^T \in \mathbb{R}^{m \times n}$. For $\alpha > 0$, define $W = \frac{1}{\alpha + q^T A^\dagger p} EA^\dagger$. Then $W \in \mathbb{R}^{m \times m}$ and $W \geqslant 0$. Note that $R(W) \subseteq R(A)$ and $R(W^T) \subseteq R(A)$. Let $\lambda \neq 0$ satisfy $Wu = \lambda u$. Then $\frac{1}{\alpha + q^T A^\dagger p} pq^T A^\dagger u = \lambda u$. So $\lambda u = \beta p$ with $\beta = \frac{q^T A^\dagger u}{\alpha + q^T A^\dagger p}$. Thus $\beta = \frac{\beta q^T A^\dagger p}{\lambda(\alpha + q^T A^\dagger p)}$ and so $0 < \lambda = \frac{q^T A^\dagger p}{\alpha + q^T A^\dagger p} < 1$. Thus $\rho(W) < 1$. By Theorem 2.3, $(I - W)^{-1}$ exists and $(I - W)^{-1} = \sum_{k=0}^{\infty} W^k \geqslant 0$. Using induction, it can be shown that $W^{k+1} = \lambda^k W$, $k \geqslant 1$. Also $(I - W)^{-1}A = A + \frac{1}{1-\lambda} WA = A + \frac{1}{\alpha} EA^\dagger A$. Now, we choose α such that $\frac{1}{\alpha} \geqslant \eta > \max|a_{ij}|$, where $A = (a_{ij})$. Then

$-A \leqslant \eta E$. Post-multiplying by $A^\dagger A \geqslant 0$, we then have $\eta EA^\dagger A \geqslant -AA^\dagger A = -A$ and $(I-W)^{-1}A = A + \frac{1}{\alpha}EA^\dagger A \geqslant A + \eta EA^\dagger A \geqslant 0$. Now set $U = (I-W)^{-1}A$ and $V = WU$. Then $U = A + \frac{1}{\alpha}EA^\dagger A = A + \frac{1}{\alpha}EP_{R(A^T)}$. So $U \geqslant 0$ and $V \geqslant 0$. Let $x^0 \in R(U)$; $x^0 = Uy^0$. Then $(I-W)x^0 = Ay^0$. Since $R(W) \subseteq R(A)$, so we have $x^0 \in R(A)$, and then $R(U) \subseteq R(A)$. Also, rank $A =$ rank U since $(I-W)^{-1}A = U$. Hence $R(A) = R(U)$. Also, $(I-W)^{-1}A = U$ implies that $N(A) = N(U)$. Thus, $A = (I-W)U = U - WU = U - V$ is a positive proper splitting. Also, $R(W^T) \subseteq R(A) = R(U)$ so that $UU^\dagger W^T = W^T$. So $W = WUU^\dagger = VU^\dagger \geqslant 0$. As we have noted earlier, $A^\dagger \geqslant 0$ is equivalent to the statement: $Ax \in \mathbb{R}^m_+ + N(A^T), x \in R(A^T) \ \Rightarrow\ x \geqslant 0$. Thus, even without the extra assumption $Ux \in \mathbb{R}^m_+ + N(A^T)$, the last condition of a $B_\dagger$-splitting is satisfied. Finally, by Theorem 3.8, since $A = U - V$ is a $B_\dagger$-splitting, it follows that $\rho(VU^\dagger) < 1$. $\square$

The next two examples show that the converse of Theorem 3.10 is not true.

EXAMPLE 3.11. Let $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $A^\dagger \geqslant 0$. Set $U = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $V = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Thus $A = U - V$ is a $B_\dagger$-splitting. We have $A^\dagger A \geqslant 0$ but $R(A) \cap int(\mathbb{R}^3_+) = \phi$.

EXAMPLE 3.12. Let $A = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \end{pmatrix}$. Then $A^\dagger \geqslant 0$. Set $U = \begin{pmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$ and $V = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. Thus $A = U - V$ is a $B_\dagger$-splitting. We have $R(A) \cap int(\mathbb{R}^2_+) \neq \phi$ but $A^\dagger A \ngeqslant 0$.

COROLLARY 3.13. (Theorem 5, [15]) *A square matrix A is positively invertible if and only if A allows a B-splitting $A = U - V$ with $\rho(VU^{-1}) < 1$.*

Proof. If $A^{-1} \geqslant 0$, then the hypotheses of Theorem 3.10 trivially hold. The conclusion now follows. Conversely, if A allows a B-splitting, then it trivially allows a $B_\dagger$-splitting. The conclusion that $A^{-1} \geqslant 0$ now follows from Theorem 3.8. $\square$

4. The group inverse analogue

In this section, we collect results for the group inverse, analogous to the case of the Moore-Penrose inverse. Since all the proofs are almost verbatim, we simply state these results.

THEOREM 4.1. *Let $A = U - V$ be a proper splitting of $A \in \mathbb{R}^{n\times n}$. Suppose that $A^\#$ exists. Then*

(a) $U^\#$ exists.
(b) $AA^\# = UU^\# = U^\# U$.
(c) $A = (I - VU^\#)U$.

(d) $I - VU^{\#}$ *is invertible.*
(e) $A^{\#} = U^{\#}(I - VU^{\#})^{-1}$.

THEOREM 4.2. *Let* $A \in \mathbb{R}^{n\times n}$. *Suppose that* $A^{\#}$ *exists and* $A = U - V$ *be a proper splitting of* A. *If* $U^{\#} \geqslant 0$, $VU^{\#} \geqslant 0$ *and* $\rho(VU^{\#}) < 1$, *then* $A^{\#} \geqslant 0$.

DEFINITION 4.3. A proper splitting of $A = U - V$ is called a $B_{\#}$-splitting if it satisfies the following conditions:
(i) $U \geqslant 0$,
(ii) $V \geqslant 0$,
(iii) $U^{\#}$ exists and $VU^{\#} \geqslant 0$, and
(iv) $Ax, Ux \in \mathbb{R}^n_+ + N(A)$ and $x \in R(A) \Rightarrow x \geqslant 0$.

THEOREM 4.4. *Let* $A \in \mathbb{R}^{n\times n}$. *Consider the following statements.*
(a) $A^{\#}$ *exists and* $A^{\#} \geqslant 0$.
(b) $Ax \in \mathbb{R}^n_+ + N(A)$ *and* $x \in R(A) \Rightarrow x \geqslant 0$.
(c) $\mathbb{R}^n_+ \subseteq A\mathbb{R}^n_+ + N(A)$.
(d) There exists $x \in \mathbb{R}^n_+$ *and* $z \in N(A)$ *such that* $Ax + z \in int(\mathbb{R}^n_+)$.
Then we have (a) $\Leftrightarrow$ *(b)* $\Rightarrow$ *(c)* $\Rightarrow$ *(d).*
Suppose that A *has a* $B_{\#}$*-splitting. Then each of the above is equivalent to the following:*
(e) $\rho(VU^{\#}) < 1$.

THEOREM 4.5. *Suppose that* $A^{\#}$ *exists,* $A^{\#} \geqslant 0$ *and* $R(A) \cap int(\mathbb{R}^n_+) \neq \phi$. *Further, let* $A^{\#}A \geqslant 0$. *Then* A *possesses a* $B_{\#}$*-splitting* $A = U - V$ *such that* $\rho(VU^{\#}) < 1$.

5. Concluding Remarks

Let us recall from the discussion in the introduction that the most frequently used iterative methods such as the Jacobi, the Gauss-Seidel and the successive over-relaxation are particular instances of regular splitting or weak regular splittings. Hence those methods provide standard examples of *existence of these splittings*. However, as the class of splittings gets smaller, like the positive splittings, one needs to *demonstrate the existence* of these specific splittings, typically under the assumption of inverse positivity of the matrix concerned. Peris' result mentioned in the introduction is an example of this situation. One of our main results (Theorem 3.10) is another instance of such a demonstration. This result, to the best of our knowledge, appears to be rather uncommon of its kind (in the singular case) in so far as splittings with certain nonnegativity restrictions are concerned. Seemingly, there are only two other similar results in the literature. One result is that of Plemmons (Theorem 4, [16]), where however, the existence of the specific splitting is trivial. The second instance wherein the existence of the splitting concerned is nontrivial, was given by Meyer and Plemmons (Theorem 2, [12]). Let us also point out the historical relevance of this second result. It leads to the definition of what are called MP-matrices in the literature (apparently named after the authors of the cited paper). In this background, let us mention that the following interesting problem with regard to Theorem 3.4 (and Remark 3.5) appears to be open:

PROBLEM 1. Let $A \in \mathbb{R}^{m\times n}$ with $A^{\dagger} \geqslant 0$. Does there exist a nontrivial weak pseudo regular splitting for A?

We conclude by posing two other questions, one of them on the existence, again in the context of a splitting. First, we prove the following result (Theorem 5.1), more general than the result proved in [10] (Theorem 3.5). The present proof uses only the Perron-Frobenius theorem and is much simpler.

THEOREM 5.1. *Let $A \in \mathbb{R}^{m\times n}$ with $0 \neq A^{\dagger} \geqslant 0$. If $A = U - V$ is a splitting with $U \geqslant 0$, $V \geqslant 0$ and $R(U) \subseteq R(A)$, then there exist $0 \neq x \in \mathbb{R}^n_+ \cap R(A^T)$, $\mu \in [0,1)$ such that $Vx = \mu Ux$.*

Proof. Let $A = U - V$ be a splitting with $U \geqslant 0$ and $R(U) \subseteq R(A)$. As $A^{\dagger} \geqslant 0$ and $U \geqslant 0$, so $A^{\dagger}U \geqslant 0$. By the Perron-Frobenius theorem, we have $A^{\dagger}Ux = \lambda x$ with $\lambda > 0$ and $x \in \mathbb{R}^n_+$. Clearly, $x \in R(A^T)$. Pre-multiplying by A, we get $AA^{\dagger}Ux = \lambda Ax = \lambda Ux - \lambda Vx$. Since $R(U) \subseteq R(A)$, so $AA^{\dagger}Ux = P_{R(A)}Ux = Ux$ and then we have $Vx = \frac{\lambda-1}{\lambda}Ux$. Thus $Vx = \mu Ux$ with $\mu = \frac{\lambda-1}{\lambda} \in [0,1)$. If $\lambda < 1$, then $Vx \leqslant 0$. Since $V \geqslant 0$ and $x \geqslant 0$, this means that $Vx = 0$. Consequently $Ux = 0$ so that $Ax = 0$. But $x \in R(A^T)$. So $x = 0$, a contradiction. Hence $\lambda \geqslant 1$. $\square$

Our second question is whether the converse of Theorem 5.1 is true:

PROBLEM 2. Suppose that, whenever $A = U - V$ is a splitting with $U \geqslant 0$, $V \geqslant 0$ and $R(U) \subseteq R(A)$, there exist $0 \neq x \in \mathbb{R}^n_+ \cap R(A^T)$, $\mu \in [0,1)$ such that $Vx = \mu Ux$. Does it follow that $A^{\dagger} \geqslant 0$?

Our final question asks whether there is a counter part for Theorem 3.10.

PROBLEM 3. Let $A \in \mathbb{R}^{m\times n}$ with $A^{\dagger} \geqslant 0$. Does there exist a splitting of $A = U - V$ with $U \geqslant 0$, $V \geqslant 0$ and $R(U) \subseteq R(A)$?

If a splitting of $A = U - V$ with $U \geqslant 0$, $V \geqslant 0$ and $R(U) \subseteq R(A)$ exists, then the existence of $0 \neq x \geqslant 0$, $\mu \in [0,1)$ such that $Vx = \mu Ux$ is, of course, guaranteed by Theorem 5.1.

We conclude this article with a few remarks on the numerical implementation of $B_{\dagger}$-splitting. Let us point out that $B_{\dagger}$-splitting has not been shown to exist for all matrices. We have shown its existence only for a class of matrices (Theorem 3.5). While this is clearly a disadvantage, we would like to point out that this situation is similar to the case of a weak pseudo regular splitting, as mentioned in the first paragraph of this section. However, for this class of matrices for which a $B_{\dagger}$-splitting exists, the constructive proof shows that one could choose such a splitting with the additional provision that the spectral radius of $VU^{\dagger}$ is as small as we wish. This is a definite numerical merit. A comparison of numerical implementation of random examples of large linear systems employing $B_{\dagger}$-splitting and a proper splitting (Theorem 3.3) shows that the number of iterations in arriving at an approximate solution are almost the same.

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