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PARTIALLY HYPER INVARIANT SUBSPACES

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Abstract. Between invariance and hyperinvariance are at least two other kinds of partially hyperinvariant subspace.

Introduction

An *invariant subspace* Y for a linear operator T on a vector space X is a linear subspace $Y \subseteq X$ for which

$$T(Y) \subseteq Y: \tag{0.1}$$

when X is a Banach space and T is bounded we usually want $Y = \mathrm{cl}\, Y$. Generally if $T : X \to X$ with $T(Y) \subseteq Y \subseteq X$ we shall write

$$T_Y : Y \to Y, \quad T'_Y : X/Y \to X/Y \tag{0.2}$$

for the induced restriction and quotient operators; thus (a little formally), writing $J_Y : Y \to X$ and $K_Y : X \to X/Y$ for the canonical injection and quotient,

$$K_Y\, J_Y = 0; \quad T\, J_Y = J_Y\, T_Y; \quad K_Y\, T = T'_Y K_Y. \tag{0.3}$$

Barnes [1] calls the pair (T_Y, T'_Y) a "diagonal" for T.

Among the invariant subspaces can be distinguished the *hyperinvariant* subspaces, which are invariant under everything in the *commutant* of the operator we first thought of. In this note we observe that there are at least two other kinds of "partial hyper invariance", lying strictly between invariance and hyperinvariance.

We begin, in §1, by recalling the "three space property" that goes with an invariant subspace: invertibility, and possibly other kinds of non singularity, for an operator, its restriction and the induced quotient, are mutually constrained in a specific way. In §2 we see that "spectral invariance", where the spectrum of the restriction and the quotient are disjoint, means that the subspace is both "reducing", the range of a commuting projection, and "hyperinvariant", invariant under everything that commutes with the operator. In §3 we meet "holomorphic" and "comm-square" invariant subspaces, and in §4 provide the rather simple examples which show that four kinds of invariance are indeed distinct. In §5 we note that for reducing subspaces three of these kinds of invariance coincide, in §6 we look at the role of "platforms" in classifying the Jordan decomposition of nilpotent operators, and in §7 we look at a sort of converse to Lomonosov's lemma, for partially hyperinvariant subspaces.

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1. Invariant subspaces

Recall ([1]; [4] Theorems 3.11.1, 3.11.2)

THEOREM 1. *If $T(Y) \subseteq Y \subseteq X$ then, of the three sets*

$$\sigma(T); \quad \sigma(T_Y); \quad \sigma(T'_Y),$$

each is contained in the union of the other two.

Proof. Here we write

$$\sigma(T) = \{\lambda \in \mathbf{C} : T - \lambda I \text{ is not invertible}\} = \pi(T)_\cup \pi'(T). \tag{1.1}$$

where

$$\begin{aligned} \pi(T) &= \{\lambda \in \mathbf{C} : T - \lambda I \text{ is not one one}\}, \\ \pi'(T) &= \{\lambda \in \mathbf{C} : T - \lambda I \text{ is not onto}\}. \end{aligned} \tag{1.2}$$

The verification of our "three space property" breaks down into simple observations about one-one and onto: notice

$$T_Y, T'_Y \text{ one one} \Longrightarrow T \text{ one one} \Longrightarrow T_Y \text{ one one}; \tag{1.3}$$

$$T_Y, T'_Y \text{ onto} \Longrightarrow T \text{ onto} \Longrightarrow T'_Y \text{ onto}; \tag{1.4}$$

$$T \text{ one one}, T_Y \text{ onto} \Longrightarrow T'_Y \text{ one one}; \tag{1.5}$$

$$T \text{ onto}, T'_Y \text{ one one} \Longrightarrow T_Y \text{ onto}. \tag{1.6}$$

The verification of (1.3)–(1.6) is very simple: observe

$$T_Y \text{ one one} \iff T^{-1}(0)_\cap Y = O; \quad T_Y \text{ onto} \iff Y \subseteq T(Y); \tag{1.7}$$

$$T'_Y \text{ one one} \iff Y \subseteq T^{-1}(Y); \quad T'_Y \text{ onto} \iff X \subseteq Y + T(X) \quad \square \tag{1.8}$$

We remark also, combining (1.3)–(1.6), that as noticed by Barnes ([1] Proposition 4)

$$T_Y,\ T'_Y \text{ invertible} \iff T \text{ invertible and } (T_Y \text{ onto or } T'_Y \text{ one one}). \tag{1.9}$$

2. Spectrally invariant subspaces

From (1.3)–(1.6) it follows

$$\sigma(T) \subseteq \sigma(T_Y)_\cup \sigma(T'_Y) \subseteq \sigma(T)_\cup \big(\sigma(T_Y)_\cap \sigma(T'_Y)\big); \tag{2.1}$$

thus sufficient for

$$\sigma(T) = \sigma(T_Y)_\cup \sigma(T'_Y) \tag{2.2}$$

is that the restriction and quotient spectra are disjoint:

$$\sigma(T_Y)_\cap \sigma(T'_Y) = \emptyset. \tag{2.3}$$

If the condition (2.3) holds we shall call the closed invariant subspace $Y \subseteq X$ *spectrally invariant*, at least when $T \in B(X)$ is bounded. A spectrally invariant subspace is actually a *reducing* subspace:

THEOREM 2. *If* $Y \subseteq X$ *is spectrally invariant for* $T \in B(X)$ *then*

$$Y = P(X) \tag{2.4}$$

with $P = P^2 \in B(X)$ *commuting with* T*.*

Proof. With the help of the functional calculus ([4] Definition 9.7.1) we can define

$$P = f(T) = \frac{1}{2\pi i} \oint_{\sigma(T)} f(z)(zI - T)^{-1} dz \in B(X): \tag{2.5}$$

specifically, with $K = \sigma(T_Y)$ and $K' = \sigma(T'_Y)$, we take $f = \chi_K$ the characteristic function, so also $I - P = \chi_{K'}(T)$. By (2.2) and (1.8) it follows

$$\lambda \notin \sigma(T) \Longrightarrow (T - \lambda I)^{-1}(Y) \subseteq Y, \tag{2.6}$$

and hence, by the Cauchy integral (2.5),

$$P(Y) \subseteq Y. \tag{2.7}$$

Also by (2.2) and (2.5)

$$P_Y = I_Y, \tag{2.8}$$

giving

$$Y \subseteq P(Y) \subseteq P(X). \tag{2.9}$$

To see finally that $P(X) \subseteq Y$ we claim that there is implication

$$P(X) = X \Longrightarrow Y = X: \tag{2.10}$$

for argue

$$\sigma(T) = \sigma(T_Y) \Longrightarrow \sigma(T'_Y) = \emptyset \Longrightarrow X/Y = \{0\} \Longrightarrow Y = X. \tag{2.11}$$

More generally, with $Z = P(X)$, apply (2.10) with $T_Z : P(X) \to P(X)$ in place of $T : X \to X$ $\square$

The simply invariant condition is not necessary for the existence of the projection P, and weaker versions of (2.3) are sufficient [6],[8] for P to be in the double commutant of T . The equivalence (1.9) translates ([1] Proposition 4) as equality

$$\sigma(T_Y)_\cup \sigma(T'_Y) = \sigma(T)_\cup \big(\pi'(T_Y)_\cap \pi(T'_Y)\big). \tag{2.12}$$

3. Hyperinvariant subspaces

Recall $Y = \mathrm{cl}\, Y \subseteq X$ is *hyperinvariant under* $T \in B(X)$ provided

$$\mathrm{comm}(T)Y \subseteq Y, \tag{3.1}$$

where if $K \subseteq B(X)$ we write

$$\text{comm}(K) = \{S \in B(X) : T \in K \Longrightarrow ST = TS\}; \tag{3.2}$$

we identify $T \in B(X)$ with the singleton $K = \{T\}$, and write $\text{comm}^2(T)$ for $\text{comm}(K)$ with $K = \text{comm}(T)$. There are at least two other conditions of interest:

DEFINITION 1. *We shall say that $Y \subseteq X$ is comm-square invariant under $T \in B(X)$ iff*

$$\text{comm}^2(T)Y \subseteq Y, \tag{3.3}$$

and holomorphically invariant under $T \in B(X)$ provided

$$\text{Holo}(T)Y \subseteq Y. \tag{3.4}$$

Evidently

$$(2.3) \Longrightarrow (3.1) \Longrightarrow (3.3) \Longrightarrow (3.4) \Longrightarrow (0.1). \tag{3.5}$$

The first implication is Theorem 2, while the remaining three are obvious. Here of course $S \in \text{Holo}(T)$ means $S = f(T)$ with $f \in \text{Holo}(K)$ with $K = \sigma(T)$, as in (2.5).

Holomorphic invariance (3.4) is equivalent to the "inverse invariance" of (2.6): look at the Cauchy integral. Since the projection P of (2.5) associated with a simply invariant subspace is in the double commutant of T the subspace $P(X)$, and its complement $P^{-1}(0)$, are actually hyperinvariant for T. We might remark that Barnes' observation ([1] Proposition 4) actually shows that there is equality (2.2) whenever $Y \subseteq X$ is holomorphically invariant for T: indeed necessary and sufficient for (3.4) is the inclusion

$$\pi(T_Y') \subseteq \sigma(T).$$

In general however none of the four implications of (3.5) is reversible: to see this recall the *forward and backward shifts* u, v, and the *standard weight* w, on $E = \ell_p$ with $p = 2$ or more generally, given by

$$(ux)_1 = 0, (ux)_{n+1} = x_n; \quad (vx)_n = x_{n+1}; \quad (wx)_n = (1/n)x_n. \tag{3.6}$$

The spectrum of each of the shifts is the closed unit disc $\mathbf{D}$, as is the "onto" spectrum of the forward shift:

$$\pi'(v) = \partial\mathbf{D} \subseteq \mathbf{D} = \sigma(v) = \sigma(u) = \pi'(u). \tag{3.7}$$

The eigenvalues of the backward shift are given by the open disc:

$$\pi(u) = \emptyset; \quad \pi(v) = \text{int}\,\mathbf{D}; \tag{3.8}$$

if $|\lambda| < 1$ then $1 - \lambda u$ is invertible and $v - \lambda = v(1 - \lambda u)$, giving one dimensional eigenspaces

$$(v - \lambda)^{-1}(0) = (1 - \lambda u)^{-1} v^{-1}(0) = (1 - \lambda u)^{-1}(1 - uv)(E). \tag{3.9}$$

Notice also ([2]; [10] Theorem 3.2)

$$\text{comm}(u) = \text{Holo}(u); \quad \text{comm}(v) = \text{Holo}(v); \quad \text{comm}(w) = \text{Holo}(w). \tag{3.10}$$

4. Block triangles

Three formally similar examples serve to distinguish between four kinds of invariant subspace:

EXAMPLE 1. *Invariant does not imply holomorphically invariant.*

Indeed, if we set

$$U = \begin{pmatrix} u & 1-uv \\ 0 & v \end{pmatrix}, \quad V = \begin{pmatrix} v & 0 \\ 1-uv & u \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.1}$$

then $Y = P(X) = E \oplus O \subseteq X$ is invariant under U but not its inverse $V = U^{-1}$.

Note that U and V are in effect the *bilateral shifts*: $VU = I = UV$. This shows that the implication (3.4)$\Longrightarrow$(0.1) does not reverse. We can also show that the implication (3.1)$\Longrightarrow$(3.3) does not reverse:

EXAMPLE 2. *Comm-square invariant does not imply hyperinvariant.*

Here, with

$$\mathbf{u} = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{4.2}$$

we have $\mathbf{u}P - P\mathbf{u} = \mathbf{v}P - P\mathbf{v} = \mathbf{u}Q - Q\mathbf{u} = \mathbf{v}Q - Q\mathbf{v} = O \neq PQ - QP$ and hence (cf. (3.10))

$$P \in \mathrm{comm}(\mathbf{v}) \setminus \mathrm{comm}^2(\mathbf{v}); \quad P \in \mathrm{comm}(\mathbf{u}) \setminus \mathrm{comm}^2(\mathbf{u}). \tag{4.3}$$

Also the subspace $Y = P(X)$ is (3.10) comm-square invariant under $\mathbf{v}$, $\mathbf{u}$ and P but not Q.

We remark also that if either $T = \mathbf{u}$ or $T = \mathbf{v}$ and $Y = P(X)$ then there is equality (2.2) without disjointness (2.3). More delicately the implication (3.3)$\Longrightarrow$(3.4) does not reverse:

EXAMPLE 3. *Holomorphically invariant does not imply comm-square invariant.*

This time (*cf.* [6]), with

$$T = \begin{pmatrix} u & 0 \\ 0 & 1-u \end{pmatrix}, \quad S = \begin{pmatrix} v & 0 \\ 0 & 1-v \end{pmatrix}: \tag{4.4}$$

we have (*cf.* [6]; [10] Theorem 4.23)

$$P \in \mathrm{comm}^2(S) \setminus \mathrm{Holo}(S); \quad P \in \mathrm{comm}^2(T) \setminus \mathrm{Holo}(T). \tag{4.5}$$

Indeed since the spectra of v and $v-1$ are not disjoint the operator P cannot be a holomorphic function of S; on the other hand

$$\begin{pmatrix} a & m \\ n & b \end{pmatrix} \in \mathrm{comm}(S) \Longrightarrow m(1-v) - vm = (1-v)n - nv = 0 \Longrightarrow m = n = 0. \tag{4.6}$$

To check this note that whenever $x \in B(E)$ satisfies $x = vx + xv$ then

$$(1-v)x(1-u^n v^n) = 0 \ (n \in \mathbf{N}) \Longrightarrow x = xuv = xu^2v^2 = xu^3v^3 = \dots \Longrightarrow x = 0.$$

The operators S and T each show that not everything in the double commutant of an operator need be a holomorphic function of it: to find a subspace $Y \subseteq X = E \oplus E$ which is invariant under the operator S but not the operator P we follow an idea of Warren Wogen, and focus on an eigenvalue $\lambda \in \mathbf{C}$ common to both v and $1-v$. If $0 < |\lambda| < 1$ and $|\lambda'| = |1-\lambda| < 1$ then by (3.9) the two dimensional subspace

$$W = (S-\lambda I)^{-1}(0) = \begin{pmatrix} (v-\lambda)^{-1}(0) \\ (1-v-\lambda)^{-1}(0) \end{pmatrix} = \begin{pmatrix} (1-\lambda u)^{-1}(1-uv)E \\ (1-\lambda' u)^{-1}(1-uv)E \end{pmatrix} \tag{4.7}$$

is (hyper)invariant under S, and therefore also invariant under P. However, as noticed by Wogen, the one dimensional subspace

$$Y = \begin{pmatrix} (1-\lambda u)^{-1} & 0 \\ 0 & (1-\lambda' u)^{-1} \end{pmatrix} \begin{pmatrix} 1-uv \\ 1-uv \end{pmatrix} E \tag{4.8}$$

is (holomorphically) invariant under S but not invariant under P.

5. Reduction

When the invariant subspace $Y = P(X)$ is the range of a projection $P = P^2 : X \to X$, then ([10] Theorem 0.1,0.2; [4] Theorem 2.5.3)

$$TY \subseteq Y \Longleftrightarrow TP = PTP; \tag{5.1}$$

$$P \in \mathrm{comm}(T) \Longrightarrow \mathrm{comm}^2(T)Y \subseteq Y; \tag{5.2}$$

$$P \in \mathrm{comm}^2(T) \Longrightarrow \mathrm{comm}(T)Y \subseteq Y. \tag{5.3}$$

Neither of the implications (5.2) or (5.3) are reversible: it is easy to alter the null space of P without changing its range. For "reducing" subspaces however, in which both the range and the null space are invariant, three kinds of invariance coalesce:

THEOREM 3. *If $P = P^2 \in B(X)$ and $T \in B(X)$ then*

$$\begin{aligned} P(X),\ P^{-1}(0)\ \textit{invariant} &\Longleftrightarrow P \in \mathrm{comm}(T) \\ &\Longrightarrow P(X),\ P^{-1}(0)\ \textit{comm-square invariant} \end{aligned} \tag{5.4}$$

and

$$P \in \mathrm{comm}^2(T) \Longleftrightarrow P(X),\ P^{-1}(0)\ \textit{hyperinvariant} \tag{5.5}$$

Proof. The first implication of (5.4), both ways, follows from (5.1) applied to both P and $I-P$, and then implies the second, and also implies (5.5) $\square$

The spectrally invariant condition is genuinely stronger than hyperinvariant:

EXAMPLE 4. *Hyperinvariant and reducing do not together imply spectrally invariant.*

Indeed the subspace W of (4.7) is hyperinvariant, and also reducing, but not spectrally invariant, under S. Alternatively, with $R = v \oplus wu$, $R^{-1}(0)$ is hyperinvariant, but not simply invariant, under R.

EXAMPLE 5. *Neither of hyperinvariant nor reducing implies the other.*

The subspace $P(X) = E \oplus O$ of Example 2 is not only comm-square invariant but also reducing, and not hyperinvariant, for each of the operators **u** and **v**. The same subspace $P(X)$ is also hyperinvariant, but not reducing, for the operator Q of (4.2). Alternatively for the standard weight w of (3.6) with $p = \infty$. the closure of its range is c_0, therefore ([4] Theorem 5.10.2) uncomplemented, and necessarily hyperinvariant.

We offer a curious characterization of invariance under a projection, which was involved in an unsuccessful attempt to uncover the subspace of Example 3:

THEOREM 5. *If $P = P^2 \in B(X)$ and $Y \subseteq X$ then the following are equivalent:*

$$PY \subseteq Y; \tag{5.6}$$

$$(I-P)Y \subseteq Y; \tag{5.7}$$

$$Y \subseteq (Y_\cap PX) + (Y_\cap P^{-1}0). \tag{5.8}$$

Proof. The equivalence of (5.6) and (5.7) is clear. If (5.8) holds then

$$PY \subseteq P(Y_\cap PX) + P(Y_\cap P^{-1}0) = P(Y_\cap PX) \subseteq Y$$

and similarly $(I-P)Y \subseteq Y$. Conversely if both (5.6) and (5.7) hold then

$$Y = (P + (I-P))Y \subseteq PY + (I-P)Y \subseteq (Y_\cap PY) + (Y_\cap P^{-1}0) \qquad \square$$

6. Jordan

Observe that if $T : X \to X$ has an invariant subspace giving an onto restriction and a one-one quotient then it is *self exact*:

$$T_Y \ onto\ , T'_Y \ one\ one \Longrightarrow T\ selfexact\ , \tag{6.1}$$

in the sense [H2] that

$$T^{-1}(0) \subseteq T(X)\ : \tag{6.2}$$

simply observe (cf [Mü] Theorem 14.21)

$$T^{-1}(0) \subseteq T^{-1}(Y) \subseteq Y \subseteq T(Y) \subseteq T(X)\ .$$

Since the assumptions about the restriction and the quotient are transmitted to all powers T^n it follows that, further,

$$T_Y \ onto\ , T'_Y \ one\ one \Longrightarrow T\ hyperexact\ , \tag{6.3}$$

in the sense [H2] that for arbitrary $m,n \in \mathbf{N}$

$$T^{-n}(0) \subseteq T^m(X) . \tag{6.4}$$

For example the condition (6.4) distinguishes those "zero jump" Fredholm operators whose nullity is continuous at the origin. When the restriction T_Y is self exact we shall call the subspace $Y \subseteq X$ a *subplatform* [HHC] Definition 1):

DEFINITION 2. *A subplatform for $T : X \to X$ is a subspace for which*

$$T^{-1}(0)_\cap Y \subseteq T(Y) \subseteq Y , \tag{6.5}$$

and a platform is a maximal subplatform. A coplatform for the subplatform $Y \subseteq X$ is a subspace $Y' \subseteq X$ for which

$$T(Y') \subseteq Y' \ ; \ Y_\cap Y' = O \ ; \ Y + Y' = X , \tag{6.6}$$

compatible provided

$$Y' \subseteq T^{-1}(0) . \tag{6.7}$$

Zorn's condition is easily checked for subplatforms, and $O = \{0\}$ is always a subplatform. Platforms can be harnessed in a classification of nilpotent operators ([HHC] Theorems 2,3):

THEOREM 6. *If $T : X \to X$ is strictly nilpotent, in the sense that*

$$T^2 = 0 , \tag{6.8}$$

and if $Y \subseteq X$ is a subplatform for T then for arbitrary $x \in X \setminus Y$ there is implication

$$Tx \notin Y \Longrightarrow Y + \mathbf{C}x + \mathbf{C}Tx \text{ is a subplatform for } T \Longrightarrow x \notin Y + T^{-1}(0) . \tag{6.9}$$

A subplatform Y for strictly nilpotent T has coplatforms, compatible iff Y is a platform.

Theorem 6 shows that a strictly nilpotent operator $T = T_1 \oplus T_0$ is the direct sum of a self exact operator T_1 and a zero operator T_0; more generally if $T^{n+1} = 0$ then the same argument shows that

$$T = T_1 \oplus T_0 \text{ with } T_1^{-1}(0) \subseteq T^n(X_1) \ , \ T_0^n = 0 . \tag{6.10}$$

7. Lomonosov

Suppose $Y \subseteq X$ is a (closed) subspace, with corresponding quotient X/Y, and consider maps $U : X/Y \to Y$: then, with $S = J_Y U K_Y : X \to X$,

$$TY \subseteq Y \Longrightarrow STS = 0. \tag{6.1}$$

This is clear from (0.3):

$$STS = J_Y U K_Y T J_Y U K_Y = J_Y U K_Y J_Y T_Y U K_Y = 0.$$

Naturally (6.1) is rather trivial if either $Y = O$ or $Y = X$. Lomonosov's Lemma ([10] Lemma 8.22) says that if a subalgebra $A \subseteq B(X)$ has no non trivial (closed) invariant subspaces and if $0 \neq K \in K(X) \subseteq B(X)$ is a non trivial compact operator then

$$\exists R \in A: \ (I + RK)^{-1}(0) \neq \{0\}. \tag{6.2}$$

This applies, if $T \in B(X)$, to each of the algebras

$$\mathrm{Alg}(T); \quad \mathrm{Holo}(T); \quad \mathrm{comm}^2(T); \quad \mathrm{comm}(T). \tag{6.3}$$

Garimella, Hrynkiv and Sourour ([3] Theorem 2.2) have a sort of converse to this:

THEOREM 7. *If $A \subseteq B(X)$ has a non trivial invariant subspace Y and if $T \in A$ satisfies*

$$0 \notin \mathrm{cvx}\, \sigma(T), \tag{6.4}$$

then there is $S \in B(X)$ for which

$$\dim(ST + TS)(X) = 1 \tag{6.5}$$

and

$$R \in A \Longrightarrow I + RS \in B(X)^{-1}. \tag{6.6}$$

Proof. By a little cosmetic surgery [3] we can arrange that the spectrum of T lies in the left half plane,

$$\sigma(T) \subseteq \{\mathrm{Re}(z) < 1\},$$

which ([10] Theorem 0.8) implies the same for the spectrum of the restriction T_Y and the quotient T'_Y, which in turn means that the multiplications L_{T_Y} and $-R_{T'_Y}$ have disjoint spectra. Now some joint spectral theory ([10] Theorem 0.12; [4] Theorem 11.6.5) guarantees that the mapping

$$U \mapsto T_Y U + U T'_Y \ (BL(X/Y, Y) \to BL(X/Y, Y))$$

is invertible. Thus if $V \in BL(X/Y, Y)$ is arbitrary then there is $U \in BL(X/Y, Y)$ for which $V = T_Y U + U T'_Y$, and provided Y is non trivial we can, using the Hahn-Banach theorem, arrange that V is of rank 1 : now we take

$$S = J_Y U K_Y. \tag{6.7}$$

Evidently $ST + TS = J_Y V K_Y$ is of rank 1, giving (6.5), while by (6.1)

$$R \in A \Longrightarrow (RS)^2 = 0, \tag{6.8}$$

giving invertibility (6.6) $\square$

Conversely ([3] Theorem 2.3) the conditions (6.5) and (6.6) guarantee that $T \in B(X)$ has a non trivial invariant subspace: remark that

$$S = \int_{t=0}^{\infty} e^{tT}(ST + TS)e^{tT}\,dt \tag{6.9}$$

is compact. Thus if $T \in A$ without a non trivial invariant subspace then (6.2) will apply, contradicting (6.6).

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