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THE FLOW APPROACH FOR WAVES IN NETWORKS

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Abstract. We present a "non-standard method" to treat wave equations on networks, leading to a transport process on the doubled directed graph. From the node conditions, we derive a flow governed by a certain adjacency matrix which, in particular, builds the bridge to the theory of difference operators. This approach provides the fundament for a powerful method to examine (boundary-)controllability and to prove stability results for damped and delay-damped networks of wave equations.

1. Introduction

Wave equations on networks appear in models of multiple-link flexible structures such as bridges, robot arms, solar panels, etc.. On the other hand, their study combines many different mathematical disciplines such as functional analysis, graph theory, number theory and control theory. J. von Below together with F. Ali Mehmeti and S. Nicaise have been pioneers in studying these processes (see, e.g., [3], [31], [32], [33], [34], [35], [36], [4]). Further references are publications by J. Lagnese, G. Leugering and E. Schmidt (see [23], [19], [20], [21], [24], [29]).

In this paper we want to contribute to this research and are inspired by an approach by B. Dorn, K.-J. Engel, M. Kramar-Fijavvz, R. Nagel and E. Sikolya. In a series of papers ([18], [25], [10], [9], [8] [11], [12]) they propose a semigroup approach to flows in (in)finite networks. More precisely, they consider the transport of material along the edges of a directed graph described by the linear transport equation

$$\dot{y}_j(t,x) = c_j\, y_j'(t,x), \quad x \in [0,1], \quad t \geqslant 0. \tag{1.1}$$

Here, the function $y_j(t,\cdot)$ describes the distribution of material on the j-th edge at time t, $c_j > 0$ is the speed of propagation on the j-th edge and the time and spatial derivatives are denoted by the symbols "$\cdot$" and "$\,'\,$", respectively. To describe the flow behavior in the vertices the authors introduced a *(weighted) adjacency matrix* $\mathbb{B}$ for the underlying graph. The entry $\mathbb{B}_{ij}$ describes the proportion of the total incoming material flowing from the j-th into the i-th edge. Then the (vector of) inflows $y(t,0)$ at time t is distributed to the outgoing edges according to the weights and the structure of the graph coded in the matrix $\mathbb{B}$, i.e., the outflows are determined by

$$y(t,1) = \mathbb{B}y(t,0), \quad t \geqslant 0. \tag{1.2}$$

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The abstract operator corresponding to (1.1) with boundary conditions (1.2) is a *difference operator*

$$A := \operatorname{diag}\left(c_j \frac{d}{dx}\right)_{j=1}^{m}, \quad D(A) := \left\{g \in \mathrm{W}^{1,p}([0,1],\mathbb{C}^m) \mid \; g(1) = \mathbb{B}g(0)\right\},$$

as studied by the author in [16]. Essential properties for the generated C_0-semigroup such as type, spectrum and asymptotics can be characterized in terms of the matrix $\mathbb{B}$. In this paper we show how the wave equation on a network can be interpreted as a flow and consequently be modeled by means of a difference operator. To do so, we present an "unconventional" reduction method to treat waves as flows in networks.

1.1. The reduction from waves to flows

To illustrate our approach, let us consider a star-shaped configuration of three vibrating strings of length one as depicted in Figure 1(a). We model this by a directed graph consisting of the three edges e_1, e_2, e_3 and four vertices $\mathrm{v}_1,\dots,\mathrm{v}_4$, where the star center v_1 is the common vertex of all three edges. Moreover, the orientation of the edges is chosen such that the head of e_j is in v_1 for $j = 1,2,3$ (compare Section 2.1). If we parameterize the edges by the interval $[0,1]$ such that position zero is in the star center and rescale all physical constants to one, the vibrations of the strings can be described by a system of wave equations

$$\ddot{z}_j(t,x) = z_j''(t,x), \;\; x \in [0,1],\; t \geqslant 0,\; j = 1,2,3, \tag{1.3}$$

supplemented by an initial displacement and initial velocity of the strings

$$\begin{cases} z_j(0,x) &= z_j^0(x), \\ \dot{z}_j(0,x) &= v_j^0(x), \end{cases} \qquad x \in [0,1], \;\; j = 1,2,3. \tag{1.4}$$

Here $z_j(t,x)$ denotes the vertical displacement of the j-th string at position x and time t. The usual approach to treat such second order systems is to reduce them to a first order system with vertical displacement $(z_1,\dots,z_3)$ and velocity $(\dot{z}_1,\dots,\dot{z}_3)$ as state variables, see e.g. [1], [2], [23], [38]. We will use a different reduction. The form of the energy of this system, consisting of the kinetic and potential part on each edge, i.e.,

$$E(t) = \frac{1}{2}\sum_{j=1}^{3}\int_0^1 \left(\left|\dot{z}_j(t,s)\right|^2 + \left|z_j'(t,s)\right|^2\right) ds, \tag{1.5}$$

indicates that the variables *velocity* $(\dot{z}_j)$ and *deformation* (z_j') should be used for the modeling (see also [22]). This is also a reasonable state formulation for a string since its characterizing property is its shape rather than the vertical position where this shape is attained. From the identity

$$\left|\dot{z}_j(t,s)\right|^2 + \left|z_j'(t,s)\right|^2 = \frac{1}{2}\left(\left|\dot{z}_j(t,s) + z_j'(t,s)\right|^2 + \left|\dot{z}_j(t,s) - z_j'(t,s)\right|^2\right)$$

we are then led to the state variables (see also [14], [27], [28])

$$\begin{cases} y_j(t,x) & := \frac{1}{2}\left[\dot{z}_j(t,x) + z_j'(t,x)\right], \\ y_{j+3}(t,x) & := \frac{1}{2}\left[\dot{z}_j(t,1-x) - z_j'(t,1-x)\right], \end{cases} \qquad j = 1,2,3. \tag{1.6}$$

Formally, if $z = (z_1, z_2, z_3)$ satisfies the wave system (1.3), then the function $y = (y_1, \dots, y_6)$ satisfies a *transport system*

$$\dot{y}_j(t,x) = y_j'(t,x), \;\; x \in [0,1], \; t \geqslant 0, \; j = 1,\dots,6, \tag{1.7}$$

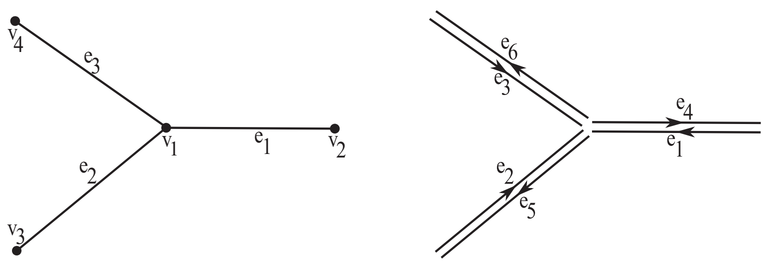
with initial data

$$\begin{cases} y_j(0,x) & = \frac{1}{2}\left[v_j^0(x) + (z_j^0)'(x)\right], \\ y_{j+3}(0,x) & = \frac{1}{2}\left[v_j^0(1-x) - (z_j^0)'(1-x)\right], \end{cases} \qquad x \in [0,1], \; j = 1,2,3. \tag{1.8}$$

Conversely, if $y = (y_1, \dots, y_6)$ satisfies (1.7) with initial data (1.8), we obtain

$$z_j(t,x) := \int_0^t \left[y_j(s,x) + y_{j+3}(s,1-x)\right] ds + z_j^0(x), \;\; j = 1,2,3$$

fulfilling the wave equation (1.3). So, via the above substitution, solutions of the wave equation correspond to solutions of the transport equation and vice versa. In particular, the substitution transforms the wave process on a single edge into a transport process on two edges. The connections between these edges will become clear through the boundary conditions.



(a) From the wave process... (b) ...to the transport process

Figure 1:

1.2. The boundary conditions

We start from the original star-shaped graph and add boundary conditions to describe the behavior of the wave process in the vertices. In the star center we require Kirchhoff's law

$$\sum_{j=1}^{3} z_j'(t,0) = 0, \; t \geqslant 0, \tag{1.9}$$

meaning that the common force is zero, and the "continuity" conditions

$$\dot{z}_j(t,0) = \dot{z}_{j+1}(t,0), \quad t \geqslant 0, \; j = 1,2, \tag{1.10}$$

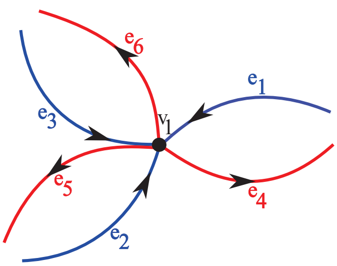
meaning that the original difference in displacement of the strings will be kept fixed in time. In the outer vertices v_2, v_3 we choose Neumann and in vertex v_4 Dirichlet-type boundary conditions, i.e.,

$$\begin{cases} z_j'(t,1) = 0, & t \geqslant 0, \; j = 1,2, \\ \dot{z}_3(t,1) = 0, & t \geqslant 0. \end{cases} \tag{1.11}$$

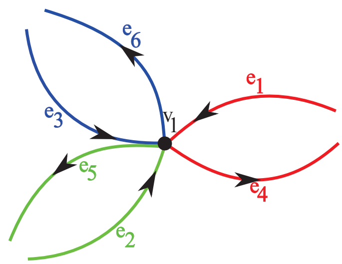
Now the task is to reformulate these conditions, using substitution (1.6), i.e., for the resulting edges in Figure 1(b). In what follows we will call a directed edge e_j *incoming edge* in a vertex v_i if its head is in v_i, and an *outgoing edge* of v_i if its tail is in v_i.
First, the Kirchhoff condition transforms to

$$\sum_{j=1}^{3} y_{j+3}(t,1) = \sum_{j=1}^{3} y_j(t,0), \quad t \geqslant 0, \tag{1.12}$$

which is a Kirchhoff law for the resulting transport process. More precisely, the condition means that there should be a vertex connecting the tails of e_4, e_5, e_6 with the heads of e_1, e_2, e_3 and that the total incoming flow through the edges e_1, e_2, e_3 must be equal to the total outgoing flow into the edges e_4, e_5, e_6. This is indicated in Figure 2(a).



(a) Kirchhoff law for the transport process



(b) Continuity BC for the transport process

Figure 2:

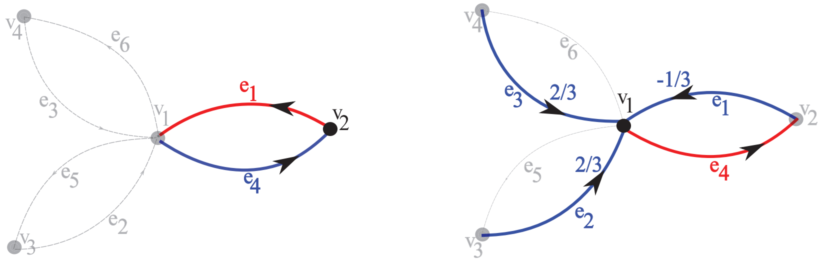
If we proceed in this manner, the continuity condition (1.10) yields

$$y_{j+3}(t,1) + y_j(t,0) = y_{j+4}(t,1) + y_{j+1}(t,0), \quad j = 1,2. \tag{1.13}$$

This condition is depicted in Figure 2(b) and means that the blue flow must equal the green and the red flow in vertex v_1. Finally, the Neumann and Dirichlet boundary conditions transform to

$$\begin{cases} y_j(t,1) = y_{j+3}(t,0), & t \geqslant 0, \; j = 1,2, \\ y_3(t,1) = -y_6(t,0), & t \geqslant 0, \end{cases}$$

for the transport process, i.e., there should be a vertex connecting the tail of e_j and the head of e_{j+3} for $j = 1,2,3$. Furthermore, the outgoing flow into edge e_j equals the incoming flow through edge e_{j+3} for $j = 1,2$ and the outgoing flow into edge e_3 equals the negative of the incoming flow through edge e_6. This is indicated for vertex v_2 in Figure 3(a).



(a) Flow through vertex v_2 (b) Flow through vertex v_1

Figure 3: The flow relations

The key observation is now that (1.12) and (1.13) can be combined to determine the outgoing flow of each individual edge e_4, e_5 and e_6 in terms of the incoming flows in vertex v_1. If we express the resulting conditions in matrix form, we obtain

$$\begin{pmatrix} y_1(t,1) \\ y_2(t,1) \\ y_3(t,1) \\ y_4(t,1) \\ y_5(t,1) \\ y_6(t,1) \end{pmatrix} = \underbrace{\left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & & 1 & \\ 0 & 0 & 0 & & & -1 \\ \hline -1/3 & 2/3 & 2/3 & 0 & 0 & 0 \\ 2/3 & -1/3 & 2/3 & 0 & 0 & 0 \\ 2/3 & 2/3 & -1/3 & 0 & 0 & 0 \end{array}\right)}_{=:\mathbb{B}} \begin{pmatrix} y_1(t,0) \\ y_2(t,0) \\ y_3(t,0) \\ y_4(t,0) \\ y_5(t,0) \\ y_6(t,0) \end{pmatrix}.$$

For example, the fourth row in the matrix condition means that in v_1 the outgoing flow into edge e_4 equals $(-1/3)$ of the incoming flow through edge e_1 and $2/3$ of the incoming flow through edges e_2 and e_3 (see Figure 3(b)).
Consequently, we can interpret the above wave system as a *flow on the doubled graph* governed by the matrix $\mathbb{B}$, which is in fact a weighted adjacency matrix of the (doubled) line graph (compare Section 2.1). In particular, it can be examined using difference operators.

In this paper we develop the above approach for a general network of wave equations incorporating a possible damping in the vertices. In a subsequent paper it will be shown that this approach allows to examine (boundary-) controllability and to prove stability results for damped and delay-damped networks of wave equations. In particular, results of A. Borichev, Y. Tomilov ([6]), K.-J. Engel et al. ([12]) and the author ([16]) can be applied to treat these problems.

2. The flow approach to waves on networks

2.1. Some notions from graph theory

We consider a finite network of vibrating strings. In order to model the network structure and to formulate the boundary conditions it is convenient to use some graph-theoretical notions. As a standard reference we refer to the monographs [5] or [7]. More precisely, the network is modeled by a directed (simple, i.e., no loops and multiple edges) graph G=G(V,E) consisting of the set of vertices $\mathrm{V} := \{\mathrm{v}_i \mid i = 1,\dots,n\}$ and the set of edges $\mathrm{E} := \{\mathrm{e}_j \mid j = 1,\dots,m\}$. In contrast to the usual notation, head and tail of an edge will also be denoted by $\mathrm{e}_j(0)$ and $\mathrm{e}_j(1)$, respectively. Moreover, we define

$$\Gamma(\mathrm{v}_i) := \left\{ j \in \{1,\dots,m\} \;\middle|\; \mathrm{e}_j(1) = \mathrm{v}_i \text{ or } \mathrm{e}_j(0) = \mathrm{v}_i \right\}$$

to be the *index set of adjacent edges to a vertex* v_i and the *degree of a vertex* is

$$\deg(\mathrm{v}_i) := |\Gamma(\mathrm{v}_i)|.$$

We do not allow loose vertices in the graph, i.e., $\deg(\mathrm{v}_i) \geqslant 1$ for all $i = 1,\dots,n$. Moreover, the set of vertices is divided into *inner vertices* V_{in}, i.e., vertices with $\deg(\mathrm{v}_i) > 1$ and *outer vertices* $\mathrm{V}_{\mathrm{out}}$, i.e., vertices with $\deg(\mathrm{v}_i) = 1$. In order to allow different boundary conditions in the outer vertices, we divide them into two disjoint sets $\mathrm{V}_{\mathrm{out}}^{\mathrm{D}}$ and $\mathrm{V}_{\mathrm{out}}^{\mathrm{N}}$, representing vertices with Dirichlet- or Neumann-type boundary conditions. Similar as in [15], we define some graph matrices representing the relations between vertices and edges.

DEFINITION 2.1. The *outgoing* and *incoming incidence matrices* $\Phi^- := (\phi_{ij}^-)_{n\times m}$ and $\Phi^+ := (\phi_{ij}^+)_{n\times m}$ are defined, respectively, by

$$\phi_{ij}^- := \begin{cases} 1, \ \mathrm{e}_j(1) = \mathrm{v}_i, \\ \quad \mathrm{v}_i \in \mathrm{V}_{\mathrm{in}} \dot{\cup} \mathrm{V}_{\mathrm{out}}^{\mathrm{N}}, \\ 0, \ \text{else}, \end{cases} \quad \text{and} \quad \phi_{ij}^+ := \begin{cases} 1, \ \mathrm{e}_j(0) = \mathrm{v}_i, \\ \quad \mathrm{v}_i \in \mathrm{V}_{\mathrm{in}} \dot{\cup} \mathrm{V}_{\mathrm{out}}^{\mathrm{N}}, \\ 0, \ \text{else}. \end{cases}$$

Moreover, a matrix $A \in M_{n\times m}(\mathbb{C})$ satisfying $\phi_{ij}^- = 0 \Leftrightarrow A_{ij} = 0$ will be called a *weighted outgoing incidence matrix* and it will be denoted by Φ_ω^-. *Weighted incoming incidence matrices* Φ_ω^+ are defined analogously.
We also need an adjacency matrix of the graph representing the relations between edges separately.

DEFINITION 2.2. A matrix $\mathbb{B} \in M_{m\times m}(\mathbb{C})$ is called a *weighted (transposed) adjacency matrix (of the line graph)* if it fulfills

$$(\mathbb{B})_{ij} \neq 0 \Longrightarrow \exists \ \text{ a vertex } \mathrm{v}_k \text{ with } \mathrm{e}_i(1) = \mathrm{v}_k = \mathrm{e}_j(0).$$

An examples of such a matrix is, e.g., $\mathbb{B} = (\Phi_\omega^-)^\top \Phi^+$ (see [30]).

One of our main tools is the so-called *doubled graph* G_{d} associated to our graph G. This graph results from the original one as follows.

(i) Double every edge e_j between two vertices v_i and v_k of the original graph and index the new edge by e_{j+m}.

(ii) Reverse the orientation of e_{j+m} by exchanging its head and its tail and connect the vertices v_i and v_k also via e_{j+m}.

The new graph $\mathrm{G_d} = \mathrm{G_d}(\mathrm{V}, \mathrm{E_d})$ consists of the set of edges $\mathrm{E_d} := \{\mathrm{e}_j | \ j = 1, \ldots, 2m\}$ and the same set of vertices as the original graph. Its incidence matrices are

$$\begin{aligned} \Phi^- &= (\Phi_o^- | \Phi_o^+), & \Phi^+ &= (\Phi_o^+ | \Phi_o^-), \\ \Phi_\omega^- &= ((\Phi_\omega^-)_o | (\Phi_\omega^+)_o), & \Phi_\omega^+ &= ((\Phi_\omega^+)_o | (\Phi_\omega^-)_o), \end{aligned}$$

where the subscript "o" indicates that the matrix is associated to the original graph G and the " $|$ "-notation is used for 1×2 block matrices.

2.2. The basic model

The (static) graph G introduced above is now used to model the network of vibrating strings. These strings are free to vibrate transversally to the plane with respect to the restoring forces due to tension. The length of the j-th string is l_j and so we parameterize the edges of the graph by the intervals $[0, l_j]$ such that position zero is at the head of each edge, i.e., in $\mathrm{e}_j(0)$. If we assume the physical parameters mass density and Young's modulus to be independent of the spatial variable and rescaled to one, the vibrations of the strings can be described by a system of (one-dimensional) wave equations

$$\ddot{z}_j(t,x) = z_j''(t,x), \quad x \in [0, l_j], \quad t \geqslant 0, \quad j = 1, \ldots, m, \tag{2.1}$$

supplemented by an initial displacement and velocity of the strings

$$\begin{cases} z_j(0,x) = z_j^0(x), \\ \dot{z}_j(0,x) = v_j^0(x), \end{cases} \qquad x \in [0, l_j], \quad j = 1, \ldots, m. \tag{2.2}$$

Here, the elements z_j are (complex-valued) functions defined on $\mathbb{R}_+ \times [0, l_j]$ and $z_j(t,x)$ is the vertical displacement of the j-th string at position $x \in [0, l_j]$ and time $t \geqslant 0$.

Additionally, we describe the interaction between the strings by supplementing the system with boundary conditions. As already indicated in the introduction, these will be formulated in terms of the *energy variables* appearing in (1.5), i.e., as linear combinations of velocity and deformation at the boundaries of the strings. We first impose the "continuity" conditions

$$\dot{z}_j(t, \mathrm{v}_i) = \dot{z}_l(t, \mathrm{v}_i), \quad \mathrm{v}_i \in \mathrm{V_{in}}, \quad j, l \in \Gamma(\mathrm{v}_i), \quad t \geqslant 0, \tag{2.3}$$

meaning that the velocities of the strings in a common vertex are equal. Especially, if the displacement of two adjacent edges in a common vertex is initially equal, it will remain equal in time (i.e., the strings are "tied" together in that vertex). Note that we used the suggestive notation $z_j(t, \mathrm{v}_i)$ if $\mathrm{e}_j(0) = \mathrm{v}_i$ or $\mathrm{e}_j(1) = \mathrm{v}_i$ for the values of the function $z_j(t, \cdot)$ in 0 or l_j.

In the outer vertices V_{out}^{D} we assign Dirichlet type boundary conditions, but to us it seems to be more natural to require the velocity in the vertices V_{out}^{D} to be zero rather than the actual vertical position (compare also, e.g., [37]), i.e.,

$$\dot{z}_j(t, v_i) = 0, \quad v_i \in V_{out}^{D}, \quad j \in \Gamma(v_i), \quad t \geqslant 0.$$

This means that the strings are fixed at some (perhaps different) height in these vertices rather than fixed at position zero.
To write the boundary conditions in a more convenient way we use the above graph structure matrices and the vector notation

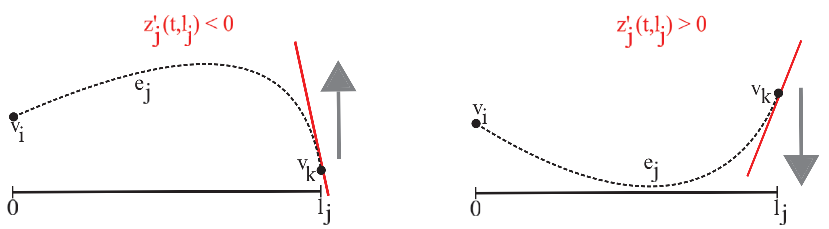
$$z(t,x) := \begin{pmatrix} z_1(t,x_1) \\ \vdots \\ z_m(t,x_m) \end{pmatrix} \quad \text{for } x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in \prod_{j=1}^{m} [0, l_j], \quad l := (l_1, \ldots, l_m).$$

We claim that the Dirichlet and continuity conditions are fulfilled if

$$\forall\, t \geqslant 0 \;\; \exists\, d(t) \in \mathbb{C}^n \;\; \text{s.t.} \begin{pmatrix} (\Phi^+)^\top \\ (\Phi^-)^\top \end{pmatrix} d(t) = \begin{pmatrix} \dot{z}(t,0) \\ \dot{z}(t,l) \end{pmatrix}. \tag{2.4}$$

This follows since $e_j(0) = v_i \in V_{out}^{D}$ implies, by Definition 2.1 of the matrix $(\Phi^+)^\top$, that the j-th row contains only zeros. Analogously, if $e_j(1) = v_i \in V_{out}^{D}$, then the j-th row of the matrix $(\Phi^-)^\top$ contains only zeros. Moreover, the continuity condition (2.3) also follows, because given a vertex $v_i \in V_{in}$, the matrix $(\Phi^+|\Phi^-)^\top$ assigns the value d_i to all coordinates where either $\dot{z}_j(t,0) = \dot{z}_j(t, v_i)$ or $\dot{z}_j(t, l_j) = \dot{z}_j(t, v_i)$. The remaining rows yield no further restrictions.

Finally, we impose a Kirchhoff law in the inner vertices and Neumann boundary conditions in the set V_{out}^{N}. However, we want to incorporate the possibility to damp the system in each of these vertices. Therefore, we introduce for each vertex $v_i \in V_{in} \dot{\cup} V_{out}^{N}$ the *damping constant* $\alpha_{v_i} \in \mathbb{R}$. A negative value will describe a "gain" of energy, while a positive value $\alpha_{v_i} > 0$ corresponds to damping the system.



(a) $z_j'(t, l_j) < 0$ indicates this form (b) $z_j'(t, l_j) > 0$ indicates this form

Figure 4: The damping method.

Let us briefly describe our damping method. Consider a single string modeled by an edge between two vertices v_i and v_k, where v_k is an outer vertex (see Figure 4).

Assume that we are able to observe the deformation of the string at position l_j and that our aim is to bring the string to rest. Observing a negative slope $z'_j(t,l_j) < 0$ indicates that the string has a deformation as depicted in Figure 4(a). This should be balanced by an upwards evolution in that vertex. Conversely, a positive slope $z'_j(t,l_j) > 0$ indicates that the string has a deformation as depicted in Figure 4(b), which should be balanced by a downwards evolution in that vertex. Therefore a reasonable damping in vertex v_k is

$$z'_j(t,l_j) = -\alpha_{\mathrm{v}_k}\dot{z}_j(t,l_j), \quad t \geqslant 0.$$

Note that the same idea of damping applied to vertex v_i leads to

$$z'_j(t,0) = \alpha_{\mathrm{v}_i}\dot{z}_j(t,0), \quad t \geqslant 0.$$

To generalize this damping concept to inner vertices, fix $\mathrm{v}_i \in \mathrm{V}_{\mathrm{in}}$ and choose an adjacent edge e_k. Without loss of generality we assume that $\mathrm{e}_j(0) = \mathrm{v}_i$ for all $j \in \Gamma(\mathrm{v}_i)$. Then we impose the damping condition

$$\sum_{j\in\Gamma(\mathrm{v}_i)} z'_j(t,0) = \alpha_{\mathrm{v}_i}\dot{z}_k(t,0), \quad t \geqslant 0.$$

To model this condition, for each vertex $\mathrm{v}_i \in \mathrm{V}_{\mathrm{in}}\dot{\cup}\mathrm{V}^{\mathrm{N}}_{\mathrm{out}}$ *one single adjacent edge of* v_i, denoted by $\mathrm{e}_{\mathrm{v}_i}$, is chosen (representing e_k above). This information together with the damping constants is collected in the following matrices.

DEFINITION 2.3. Let α_{v_i} be the damping constant and $\mathrm{e}_{\mathrm{v}_i}$ the associated edge for $\mathrm{v}_i \in \mathrm{V}_{\mathrm{in}}\dot{\cup}\mathrm{V}^{\mathrm{N}}_{\mathrm{out}}$. Then the *damping matrices* $\Psi^- = \left(\psi^-_{ij}\right)$ and $\Psi^+ = \left(\psi^+_{ij}\right)$ are defined by

$$\psi^-_{ij} := \begin{cases} \alpha_{\mathrm{v}_i}, & \mathrm{v}_i \in \mathrm{V}_{\mathrm{in}}\dot{\cup}\mathrm{V}^{\mathrm{N}}_{\mathrm{out}}, \\ & \mathrm{e}_j = \mathrm{e}_{\mathrm{v}_i}, \\ & \mathrm{e}_j(1) = \mathrm{v}_i, \\ 0, & \text{else,} \end{cases} \quad \text{and} \quad \psi^+_{ij} := \begin{cases} \alpha_{\mathrm{v}_i}, & \mathrm{v}_i \in \mathrm{V}_{\mathrm{in}}\dot{\cup}\mathrm{V}^{\mathrm{N}}_{\mathrm{out}}, \\ & \mathrm{e}_j = \mathrm{e}_{\mathrm{v}_i}, \\ & \mathrm{e}_j(0) = \mathrm{v}_i, \\ 0, & \text{else.} \end{cases}$$

Then we can formulate the (damped) Kirchhoff and Neumann boundary conditions as

$$\Phi^+ z'(t,0) - \Phi^- z'(t,l) = \Psi^+\dot{z}(t,0) + \Psi^-\dot{z}(t,l), \quad t \geqslant 0. \tag{2.5}$$

Observe that for $\alpha_{\mathrm{v}_i} = 0$, (2.5) yields a *usual* Kirchhoff law or Neumann boundary condition for vertex v_i.

These considerations lead to the following system

$$\begin{cases} \ddot{z}_j(t,x) = z''_j(t,x), & x \in [0,l_j],\ t \geqslant 0, \\ \forall\, t \geqslant 0\ \ \exists\, d(t) \in \mathbb{C}^n \text{ s.t. } (\Phi^+|\Phi^-)^\top d(t) = (\dot{z}(t,0),\dot{z}(t,l))^\top, & \\ \Phi^+ z'(t,0) - \Phi^- z'(t,l) = \Psi^+\dot{z}(t,0) + \Psi^-\dot{z}(t,l),\ t \geqslant 0, & \\ z_j(0,x) = z^0_j(x), & x \in [0,l_j], \\ \dot{z}_j(0,x) = v^0_j(x), & x \in [0,l_j], \end{cases} \tag{2.6}$$

for $j = 1,\dots,m$. We will examine system (2.6) on existence and uniqueness of *classical solutions* defined as follows.

DEFINITION 2.4. A function $z:\mathbb{R}_+\to\prod_{j=1}^{m}\mathrm{L}^2([0,l_j],\mathbb{C})$ is a *classical solution* of (2.6) if

(i) $z\in\mathrm{C}^2\left(\mathbb{R}_+,\prod_{j=1}^{m}\mathrm{H}^2([0,l_j],\mathbb{C})\right)$ and

(ii) z satisfies (2.6).

Moreover, the system is well-posed if there exists a unique classical solution for initial data $(z^0,v^0)\in\prod_{j=1}^{m}\mathrm{H}^2([0,l_j],\mathbb{C})\times\prod_{j=1}^{m}\mathrm{H}^1([0,l_j],\mathbb{C})$ satisfying boundary conditions (2.4) and (2.5).

Finally, we introduce the *weighted incidence matrices* for this problem.

DEFINITION 2.5. Let $\alpha_{\mathrm{v}_i}\neq-\deg(\mathrm{v}_i)$ for all $\mathrm{v}_i\in\mathrm{V}_{\mathrm{in}}\dot{\cup}\mathrm{V}_{\mathrm{out}}^{\mathrm{N}}$. Then the *weighted outgoing* and *incoming incidence matrices* $\Phi_\omega^-:=(\omega_{ij}^-)_{n\times m}$ and $\Phi_\omega^+:=(\omega_{ij}^+)_{n\times m}$ are defined by

$$\omega_{ij}^-:=\begin{cases}\frac{2}{\deg(\mathrm{v}_i)+\alpha_{\mathrm{v}_i}}, & \mathrm{e}_j(1)=\mathrm{v}_i,\\ & \mathrm{v}_i\in\mathrm{V}_{\mathrm{in}}\dot{\cup}\mathrm{V}_{\mathrm{out}}^{\mathrm{N}},\\ 0, & \text{else,}\end{cases}\qquad \omega_{ij}^+:=\begin{cases}\frac{2}{\deg(\mathrm{v}_i)+\alpha_{\mathrm{v}_i}}, & \mathrm{e}_j(0)=\mathrm{v}_i,\\ & \mathrm{v}_i\in\mathrm{V}_{\mathrm{in}}\dot{\cup}\mathrm{V}_{\mathrm{out}}^{\mathrm{N}},\\ 0, & \text{else.}\end{cases}$$

The assumption $\alpha_{\mathrm{v}_i}\neq-\deg(\mathrm{v}_i)$ for all $\mathrm{v}_i\in\mathrm{V}_{\mathrm{in}}\dot{\cup}\mathrm{V}_{\mathrm{out}}^{\mathrm{N}}$ will be justified at the end of Section 2.5. In the next section we will proceed with the modeling of system (2.6).

2.3. The reduction from waves to flows

As motivated in the introduction, we do not follow the standard approach using as state variables vertical displacement (z_j) and velocity $(\dot{z}_j)$ to model system (2.6), but rather choose state variables that reflect the energy of the system.

DEFINITION 2.6. For every $z_j\in\mathrm{C}^1\left(\mathbb{R}_+,\mathrm{H}^1([0,l_j],\mathbb{C})\right)$ $(j=1,\dots,m)$ we define the *state variables* y_j, $y_{j+m}\in\mathrm{C}(\mathbb{R}_+,\mathrm{L}^2([0,1],\mathbb{C}))$ by

$$\begin{cases} y_j(t,x) &:= \frac{1}{2}\left[\dot{z}_j(t,l_jx)+z_j'(t,l_jx)\right],\\ y_{j+m}(t,x) &:= \frac{1}{2}\left[\dot{z}_j(t,l_j(1-x))-z_j'(t,l_j(1-x))\right],\end{cases}\qquad x\in[0,1],\ t\geqslant 0.$$

Moreover, for initial data $(z_j^0,v_j^0)\in\mathrm{H}^1([0,l_j],\mathbb{C})\times\mathrm{L}^2([0,l_j],\mathbb{C})$ we define the *associated initial data*

$$\begin{cases} y_j^0(x) &:= \frac{1}{2}\left[v_j^0(l_jx)+(z_j^0)'(l_jx)\right],\\ y_{j+m}^0(x) &:= \frac{1}{2}\left[v_j^0(l_j(1-x))-(z_j^0)'(l_j(1-x))\right],\end{cases}$$

for $x \in [0,1]$, $j = 1,\dots,m$ and the *speed* constants

$$c_j = c_{j+m} := \frac{1}{l_j} \quad \text{for } j = 1,\dots,m.$$

REMARK 2.7. Note that for a function $z_j \in \mathrm{C}^1\left(\mathbb{R}_+, \mathrm{H}^1([0,l_j],\mathbb{C})\right)$ we have

$$\begin{aligned} \dot{z}_j(t,x) &= y_j\left(t,\tfrac{x}{l_j}\right) + y_{j+m}\left(t,1-\tfrac{x}{l_j}\right), \\ z_j'(t,x) &= y_j\left(t,\tfrac{x}{l_j}\right) - y_{j+m}\left(t,1-\tfrac{x}{l_j}\right), \end{aligned} \qquad x \in [0,l_j], \quad t \geqslant 0.$$

We now rewrite the wave system in terms of the state variables (y_j). As indicated in the introduction, the state variables follow a transport process.

PROPOSITION 2.8. *If $z_j \in \mathrm{C}^2(\mathbb{R}_+, \mathrm{H}^2([0,l_j],\mathbb{C}))$ satisfy the wave system (*2.1*), then $y_j \in \mathrm{C}^1(\mathbb{R}_+, \mathrm{H}^1([0,1],\mathbb{C}))$ fulfill the transport system*

$$\dot{y}_j(t,x) = c_j y_j'(t,x), \quad x \in [0,1], \quad t \geqslant 0, \tag{2.7}$$

for $j = 1,\dots,2m$.

Proof. In the following, we denote the derivation operator by $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$, respectively, in contrast to the notation of derived functions $\dot{z}$ or z'. A simple calculation yields

$$\begin{aligned} \dot{y}_j(t,x) &= \frac{1}{2}\left(\frac{\partial}{\partial t}\dot{z}_j(t,l_jx) + \frac{\partial}{\partial t}z_j'(t,l_jx)\right) \\ &= \frac{1}{2}\left(z_j''(t,l_jx) + \frac{1}{l_j}\frac{\partial}{\partial x}\frac{\partial}{\partial t}z_j(t,l_jx)\right) \\ &= \frac{1}{2}\left(\frac{\partial}{\partial x}\frac{1}{l_j}z_j'(t,l_jx) + \frac{1}{l_j}\frac{\partial}{\partial x}\dot{z}_j(t,l_jx)\right) \\ &= c_j y_j'(t,x) \end{aligned}$$

for $j = 1,\dots,m$, using the underlying wave equation (2.1). For $j = m+1,\dots,2m$ we obtain

$$\begin{aligned} \dot{y}_{j+m}(t,x) &= \frac{1}{2}\left(\frac{\partial}{\partial t}\dot{z}_j(t,l_j(1-x)) - \frac{\partial}{\partial t}z_j'(t,l_j(1-x))\right) \\ &= \frac{1}{2}\left(z_j''(t,l_j(1-x)) + \frac{\partial}{\partial x}\frac{\partial}{\partial t}\frac{1}{l_j}z_j(t,l_j(1-x))\right) \\ &= \frac{1}{2l_j}\left(\frac{\partial}{\partial x}\dot{z}_j(t,l_j(1-x)) - \frac{\partial}{\partial x}z_j'(t,l_j(1-x))\right) \\ &= c_{j+m}y_{j+m}'(t,x). \quad \square \end{aligned}$$

To model the transport process, we associate to each edge e_j in the original graph two new edges $\hat{\mathrm{e}}_j$ and $\hat{\mathrm{e}}_{j+m}$. They are parameterized by the interval $[0,1]$ such that 0 corresponds to the head of the individual edge. We put some "material" on the edges and assume that it is transported according to (2.7). So with our choice of parametrization for the edges, the "material" is transported with speed c_j from the tail of the edge to its head. The relations between these edges will be incorporated when reformulating the boundary conditions.

2.4. From waves to flows – reformulation of boundary conditions

In order to show how the transport processes of the above doubled edges are linked we now reformulate the boundary conditions (2.4) and (2.5) using the state variables (y_j).

In what follows we will be concerned with the doubled graph $\mathrm{G_d}$ (introduced in Section 2.1) and all appearing incidence and adjacency matrices are associated to this graph. Graph matrices corresponding to the original graph G are indicated by a subscript "o".

The matrix $\mathbb{B} \in M_{2m\times 2m}(\mathbb{C})$ defined by

$$\begin{aligned}\mathbb{B} &:= \left(\Phi_\omega^-\right)^\top \cdot \Phi^+ - \begin{pmatrix} 0_m & \mathrm{Id}_m \\ \mathrm{Id}_m & 0_m \end{pmatrix} \\ &= \begin{pmatrix} (\Phi_\omega^-)_o^\top \\ (\Phi_\omega^+)_o^\top \end{pmatrix} \left(\Phi_o^+ | \Phi_o^-\right) - \begin{pmatrix} 0_m & \mathrm{Id}_m \\ \mathrm{Id}_m & 0_m \end{pmatrix}\end{aligned}$$

and the *damping matrices* associated to the doubled graph

$$\Psi^- = (\Psi_o^- | \Psi_o^+) \qquad \text{and} \qquad \Psi^+ = (\Psi_o^+ | \Psi_o^-)$$

play an important role in the reformulation of the boundary conditions. Recall that $\left(\Phi_\omega^-\right)^\top \Phi^+$ defines a weighted (transposed) adjacency matrix of the doubled graph $\mathrm{G_d}$ (see Section 2.1). Then, by Definition 2.2, $\mathbb{B}$ is also a weighted (transposed) adjacency matrix. This matrix helps to reformulate the boundary conditions by means of a difference equation.

PROPOSITION 2.9. *For* $z \in \mathrm{C}^1\left(\mathbb{R}_+, \prod_{j=1}^m \mathrm{H}^1([0,l_j],\mathbb{C})\right)$ *let* y *be as in Definition* 2.6 *and assume that* $\alpha_{\mathrm{v}_i} \neq -\deg(\mathrm{v}_i)$ *for all* $\mathrm{v}_i \in \mathrm{V_{in}} \dot{\cup} \mathrm{V_{out}^N}$. *Consider the following assertions.*

(i) $z \in \mathrm{C}^1\left(\mathbb{R}_+, \prod_{j=1}^m \mathrm{H}^2([0,l_j],\mathbb{C})\right)$ *satisfies* (2.4) *and* (2.5).

(ii) $y \in \mathrm{C}\left(\mathbb{R}_+, \mathrm{H}^1([0,1],\mathbb{C}^{2m})\right)$ *satisfies*

(a) $$\Phi^+ y(t,0) - \Phi^- y(t,1) = \Psi^+ y(t,0) + \Psi^- y(t,1), \quad t \geqslant 0,$$

(b) $$\forall\, t \geqslant 0\ \exists\, d(t) \in \mathbb{C}^n \text{ s.t. } \left(\Phi^+\right)^\top d(t) = y(t,0) + \begin{pmatrix} 0_m & \mathrm{Id}_m \\ \mathrm{Id}_m & 0_m \end{pmatrix} y(t,1).$$

(iii) $y \in \mathrm{C}\left(\mathbb{R}_+, \mathrm{H}^1([0,1],\mathbb{C}^{2m})\right)$ *satisfies*

$$y(t,1) = \mathbb{B}y(t,0), \quad t \geqslant 0. \tag{2.8}$$

Then the implications (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) hold.

Proof.
"(i)$\Rightarrow$(ii)": We denote by $\Pi_1 : \mathbb{C}^{2m} \to \mathbb{C}^m$ the projection onto the first m coordinates and by Π_2 the projection onto the coordinates $m+1,\dots,2m$. If we use Remark 2.7 we obtain

$$\begin{aligned}
\Phi^+ y(t,0) - \Phi^- y(t,1) &= \Phi_o^+ \left(\Pi_1 y(t,0) - \Pi_2 y(t,1)\right) - \Phi_o^- \left(\Pi_1 y(t,1) - \Pi_2 y(t,0)\right) \\
&= \Phi_o^+ z'(t,0) - \Phi_o^- z'(t,l) \\
&= \Psi_o^+ \dot{z}(t,0) + \Psi_o^- \dot{z}(t,l) \\
&= \Psi_o^+ \left(\Pi_1 y(t,0) + \Pi_2 y(t,1)\right) + \Psi_o^- \left(\Pi_1 y(t,1) + \Pi_2 y(t,0)\right) \\
&= \Psi^+ y(t,0) + \Psi^- y(t,1),
\end{aligned}$$

hence (a) holds. Condition (b) follows from the definition of Φ^+ and the identity

$$\begin{pmatrix} \dot{z}(t,0) \\ \dot{z}(t,l) \end{pmatrix} = y(t,0) + \begin{pmatrix} 0_m & \mathrm{Id}_m \\ \mathrm{Id}_m & 0_m \end{pmatrix} y(t,1).$$

"(ii)$\Rightarrow$(iii)": To prove this implication we use the diagonal $n \times n$ matrices

$$(\mathrm{D}_1)_{ii} := \begin{cases} 1, & \mathrm{v}_i \in \mathrm{V}_{\mathrm{in}} \dot{\cup} \mathrm{V}_{\mathrm{out}}^{\mathrm{N}}, \\ 0, & \text{else}, \end{cases} \qquad (\mathrm{D}_2)_{ii} := \begin{cases} \deg(\mathrm{v}_i), & \mathrm{v}_i \in \mathrm{V}_{\mathrm{in}} \dot{\cup} \mathrm{V}_{\mathrm{out}}^{\mathrm{N}}, \\ 0, & \text{else}. \end{cases}$$

and

$$(\mathrm{D}_3)_{ii} := \begin{cases} \deg(\mathrm{v}_i) + \alpha_{\mathrm{v}_i}, & \mathrm{v}_i \in \mathrm{V}_{\mathrm{in}} \dot{\cup} \mathrm{V}_{\mathrm{out}}^{\mathrm{N}}, \\ 0, & \text{else}. \end{cases}$$

They satisfy

$$\begin{aligned}
\mathrm{D}_1 &= \frac{1}{2}\left(\Phi^- + \Psi^-\right)\left(\Phi_\omega^-\right)^\top, \\
\mathrm{D}_2 &= \Phi^+ \left(\Phi^+\right)^\top, \\
\mathrm{D}_3 &= \mathrm{D}_2 + \Psi^+ \left(\Phi^+\right)^\top.
\end{aligned}$$

Using (b) we derive

$$\begin{aligned}
\left(\Phi_\omega^-\right)^\top \Phi^+ y(t,0) &= \left(\Phi_\omega^-\right)^\top \Phi^+ \left(\left(\Phi^+\right)^\top d(t) - \begin{pmatrix} 0_m & \mathrm{Id}_m \\ \mathrm{Id}_m & 0_m \end{pmatrix} y(t,1)\right) \\
&= \left(\Phi_\omega^-\right)^\top \Phi^+ \left(\Phi^+\right)^\top d(t) - \left(\Phi_\omega^-\right)^\top \Phi^- y(t,1).
\end{aligned}$$

Moreover, assumption (a) yields

$$\begin{aligned}\left(\Phi^-_\omega\right)^\top \Phi^- y(t,1) &= \left(\Phi^-_\omega\right)^\top \left[\Phi^+ y(t,0) - \Psi^+ y(t,0) - \Psi^- y(t,1)\right] \\ &= \left(\Phi^-_\omega\right)^\top \Phi^+ y(t,0) - \left(\Phi^-_\omega\right)^\top \Bigg[\Psi^+ y(t,0) \\ &\quad + \Psi^- \begin{pmatrix} 0_m & \mathrm{Id}_m \\ \mathrm{Id}_m & 0_m \end{pmatrix} \left(\left(\Phi^+\right)^\top d(t) - y(t,0)\right)\Bigg] \\ &= \left(\Phi^-_\omega\right)^\top \Phi^+ y(t,0) - \left(\Phi^-_\omega\right)^\top \left[\Psi^+ y(t,0) + \Psi^+ \left(\left(\Phi^+\right)^\top d(t) - y(t,0)\right)\right] \\ &= \left(\Phi^-_\omega\right)^\top \Phi^+ y(t,0) - \left(\Phi^-_\omega\right)^\top \Psi^+ \left(\Phi^+\right)^\top d(t).\end{aligned}$$

Combining this and using the above identities for the D_i, we obtain

$$\begin{aligned}\left(\Phi^-_\omega\right)^\top \Phi^+ y(t,0) &= \frac{1}{2}\left(\Phi^-_\omega\right)^\top \left(\Phi^+ \left(\Phi^+\right)^\top + \Psi^+ \left(\Phi^+\right)^\top\right) d(t) \\ &= \frac{1}{2}\left(\Phi^-_\omega\right)^\top \left(\mathrm{D}_2 + \Psi^+ \left(\Phi^+\right)^\top\right) d(t) \\ &= \frac{1}{2}\left(\Phi^-_\omega\right)^\top \mathrm{D}_3 \cdot d(t) \\ &= \left(\Phi^-\right)^\top d(t).\end{aligned}$$

This leads to

$$\begin{aligned}\mathbb{B}y(t,0) &= \left(\Phi^-_\omega\right)^\top \Phi^+ y(t,0) - \begin{pmatrix} 0_m & \mathrm{Id}_m \\ \mathrm{Id}_m & 0_m \end{pmatrix} y(t,0) \\ &= \left(\Phi^-\right)^\top d(t) - \begin{pmatrix} 0_m & \mathrm{Id}_m \\ \mathrm{Id}_m & 0_m \end{pmatrix} \left(\left(\Phi^+\right)^\top d(t) - \begin{pmatrix} 0_m & \mathrm{Id}_m \\ \mathrm{Id}_m & 0_m \end{pmatrix} y(t,1)\right) \\ &= y(t,1).\end{aligned}$$

"(iii) ⇒ (ii)": A simple calculation shows that

$$\begin{aligned}\left(\Phi^- + \Psi^-\right) y(t,1) &= \left(\Phi^- + \Psi^-\right)\left(\left(\Phi^-_\omega\right)^\top \Phi^+ - \begin{pmatrix} 0_m & \mathrm{Id}_m \\ \mathrm{Id}_m & 0_m \end{pmatrix}\right) y(t,0) \\ &= (\underbrace{\left(\Phi^- + \Psi^-\right)\left(\Phi^-_\omega\right)^\top}_{=2\mathrm{D}_1} \Phi^+ - \Phi^+ - \Psi^+) y(t,0) \\ &= (2\underbrace{\mathrm{D}_1 \Phi^+}_{=\Phi^+} - \Phi^+ - \Psi^+) y(t,0) \\ &= \left(\Phi^+ - \Psi^+\right) y(t,0).\end{aligned}$$

To verify condition (b), observe that

$$\begin{aligned}\begin{pmatrix} 0_m & \mathrm{Id}_m \\ \mathrm{Id}_m & 0_m \end{pmatrix} y(t,1) + y(t,0) &= \begin{pmatrix} 0_m & \mathrm{Id}_m \\ \mathrm{Id}_m & 0_m \end{pmatrix} \left(\Phi^-_\omega\right)^\top \Phi^+ y(t,0) \\ &= \left(\Phi^+_\omega\right)^\top \Phi^+ y(t,0) \\ &= \left(\Phi^+\right)^\top \Phi^+_\omega y(t,0),\end{aligned}$$

and so we can define $d(t) := \Phi_\omega^+ y(t,0)$. $\square$

If we recall Proposition 2.8 and since $\mathbb{B}$ is a weighted adjacency matrix of the doubled graph $\mathrm{G_d}$, condition (2.8) means that the flow out of a vertex into an outgoing edge is determined by the inflows through the incoming edges. Because of this behavior in the vertices we call the resulting transport process a *flow* on the doubled graph governed by the adjacency matrix $\mathbb{B}$.

2.5. Well-posedness for the wave process

We now introduce the abstract Cauchy problem corresponding to the transport process (2.7) with boundary conditions (2.8). The abstract framework for our examinations is as follows.

DEFINITION 2.10. Let $\alpha_{\mathrm{v}_i} \neq -\deg(\mathrm{v}_i)$ for all $\mathrm{v}_i \in \mathrm{V_{in}} \dot\cup \mathrm{V_{out}^N}$.

(i) The *state space* is the Hilbert space $\mathrm{L}^2([0,1],\mathbb{C}^{2m})$, equipped with the inner product

$$\langle f,g\rangle := \sum_{j=1}^{2m} \frac{1}{c_j}\int_0^1 f_j(s)\overline{g_j(s)}\,ds.$$

(ii) The *system operator* $(A,D(A))$ is

$$A = \operatorname{diag}\left(c_j \frac{\partial}{\partial x}\right)_{j=1}^{2m}, \; D(A) = \left\{ f \in \mathrm{H}^1([0,1],\mathbb{C}^{2m}) \middle| \quad f(1) = \mathbb{B} f(0)\right\}.$$

The operator $(A,D(A))$ is a difference operator and it follows from [16, Thm. 2.6] that it generates a strongly continuous (difference) semigroup denoted by $(T(t))_{t\geqslant 0}$. However, we can prove even more.

THEOREM 2.11.

(i) *If* $\alpha_{\mathrm{v}_i} \neq -\deg(\mathrm{v}_i)$ *for all* $\mathrm{v}_i \in \mathrm{V_{in}} \dot\cup \mathrm{V_{out}^N}$, *the operator* $(A,D(A))$ *is the generator of a* C_0*-semigroup on* $\mathrm{L}^2([0,1],\mathbb{C}^{2m})$.

(ii) *If* $\alpha_{\mathrm{v}_i} = 0$ *for all* $\mathrm{v}_i \in \mathrm{V_{in}} \dot\cup \mathrm{V_{out}^N}$, *i.e., if the wave equation is not damped in the vertices,* $(A,D(A))$ *generates a unitary group on* $\mathrm{L}^2([0,1],\mathbb{C}^{2m})$.

(iii) *If* $\alpha_{\mathrm{v}_i} \geqslant 0$ *for all* $\mathrm{v}_i \in \mathrm{V_{in}} \dot\cup \mathrm{V_{out}^N}$, *then* $(A,D(A))$ *generates a contraction semigroup on* $\mathrm{L}^2([0,1],\mathbb{C}^{2m})$.

Proof. Assertion (i) follows from [16, Thm. 2.6]. To show statement (iii) we use [16, Thm. 2.6(i)], i.e., we only have to show $\|\mathbb{B}\|_2 = \sqrt{r(\mathbb{B}^\top \mathbb{B})} \leqslant 1$. Therefore, it suffices to determine the spectrum of $\mathbb{B}^\top\mathbb{B}$. To that purpose, we introduce the diagonal $n\times n$ matrices

$$(\mathrm{D}_1)_{ii} := \begin{cases} \frac{2}{\deg(\mathrm{v}_i)}, & \mathrm{v}_i \in \mathrm{V}_{\mathrm{in}}\dot{\cup}\mathrm{V}_{\mathrm{out}}^{\mathrm{N}}, \\ 0, & \text{else}, \end{cases} \qquad (\mathrm{D}_2)_{ii} := \begin{cases} \frac{4\deg(\mathrm{v}_i)}{(\deg(\mathrm{v}_i)+\alpha_{\mathrm{v}_i})^2}, & \mathrm{v}_i \in \mathrm{V}_{\mathrm{in}}\dot{\cup}\mathrm{V}_{\mathrm{out}}^{\mathrm{N}}, \\ 0, & \text{else}, \end{cases}$$

and the (weighted) incoming incidence matrix $\tilde{\Phi}_\omega^+ = (\tilde{\omega}_{ij}^+)_{n\times 2m}$ defined by

$$\tilde{\omega}_{ij}^+ := \begin{cases} \frac{-4\alpha_{\mathrm{v}_i}}{(\deg(\mathrm{v}_i)+\alpha_{\mathrm{v}_i})^2}, & \mathrm{e}_j(0) = \mathrm{v}_i, \\ & \mathrm{v}_i \in \mathrm{V}_{\mathrm{in}}\dot{\cup}\mathrm{V}_{\mathrm{out}}^{\mathrm{N}}, \\ 0, & \text{else}. \end{cases}$$

These matrices satisfy

$$\begin{aligned} \Phi_\omega^+ &= \mathrm{D}_1\Phi^+, \\ \mathrm{D}_2 &= \Phi_\omega^- \left(\Phi_\omega^-\right)^\top, \\ \tilde{\Phi}_\omega^+ &= \mathrm{D}_2\Phi^+ - 2\Phi_\omega^+. \end{aligned}$$

Calculating yields

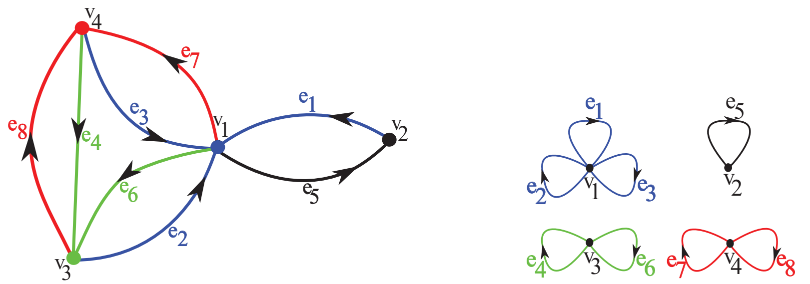
$$\begin{aligned} \mathbb{B}^\top\mathbb{B} &= \left[\left(\Phi^+\right)^\top\Phi_\omega^- - \begin{pmatrix} 0_m & \mathrm{Id}_m \\ \mathrm{Id}_m & 0_m \end{pmatrix}\right]\cdot\left[\left(\Phi_\omega^-\right)^\top\Phi^+ - \begin{pmatrix} 0_m & \mathrm{Id}_m \\ \mathrm{Id}_m & 0_m \end{pmatrix}\right] \\ &= \left(\Phi^+\right)^\top \mathrm{D}_2\Phi^+ - \left(\Phi^+\right)^\top\Phi_\omega^+ - \left(\Phi_\omega^+\right)^\top\Phi^+ + \mathrm{Id}_{2m} \\ &= \left(\mathrm{D}_2\Phi^+\right)^\top\Phi^+ - \left(\Phi^+\right)^\top \mathrm{D}_1\Phi^+ - \left(\Phi_\omega^+\right)^\top\Phi^+ + \mathrm{Id}_{2m} \\ &= \left(\mathrm{D}_2\Phi^+ - 2\Phi_\omega^+\right)^\top\Phi^+ + \mathrm{Id}_{2m} \\ &= \left(\tilde{\Phi}_\omega^+\right)^\top\Phi^+ + \mathrm{Id}_{2m}. \end{aligned}$$

Recall now that the product $(\Phi_\omega^-)^\top\Phi^+$ of incidence matrices always defines a weighted (transposed) adjacency matrix of a line graph. Consequently, the matrix $(\tilde{\Phi}_\omega^+)^\top\Phi^+$ defines a weighted (transposed) adjacency matrix of a line graph $\tilde{\mathrm{G}}_{\mathrm{d}}$ with $\Phi_\omega^- = \tilde{\Phi}_\omega^+$. This graph consists of the same vertex set as G_{d} with the same incoming edges in a vertex. However, all these incoming edges are also outgoing edges in the same vertex, i.e., all edges are loops. Hence, $\tilde{\mathrm{G}}_{\mathrm{d}}$ divides the graph G_{d} in n subgraphs, where each subgraph contains a vertex with its incoming edges in G_{d} realized as loops. An example is depicted in Figure 5.

Since $(\tilde{\Phi}_\omega^+)^\top\Phi^+$ is a weighted adjacency matrix of the line graph of $\tilde{\mathrm{G}}_{\mathrm{d}}$, by Definition 2.2, $(\tilde{\Phi}_\omega^+)^\top\Phi^+ + \mathrm{Id}_{2m}$ is also a (weighted) adjacency matrix of the same graph.

Let us now determine the entries of this matrix. Fix an index $i \in \{1,\dots,2m\}$. Then the edge e_i is an incoming edge in a vertex of G_{d}, say v_k. Assume first that $\mathrm{v}_k \notin \mathrm{V}_{\mathrm{out}}^{\mathrm{D}}$. Since Φ^+ has only one non-zero entry in the i-th column (namely the k-th entry), we can use the definition of $\tilde{\Phi}_\omega^+$ to compute the diagonal entry of the i-th row as

$$(\mathbb{B}^\top\mathbb{B})_{ii} = \tilde{\omega}_{ki}^+ \cdot \phi_{ki}^+ + 1 = \frac{-4\alpha_{\mathrm{v}_k}}{(\deg(\mathrm{v}_k)+\alpha_{\mathrm{v}_k})^2} + 1.$$



(a) A graph $\mathrm{G_d}$ with adjacency matrix $\mathbb{B}$.

(b) Corresponding graph $\tilde{\mathrm{G}}_\mathrm{d}$ with adjacency matrix $\mathbb{B}^\top\mathbb{B}$.

Figure 5: The transformation of a graph $\mathrm{G_d}$ to $\tilde{\mathrm{G}}_\mathrm{d}$

For every other incoming edge e_j of vertex v_k in $\mathrm{G_d}$ the matrix Φ^+ has a (unique) non-zero entry in the j-th column (namely the k-th entry) and so the i-th row of $(\mathbb{B}^\top\mathbb{B})$ has an additional entry

$$(\mathbb{B}^\top\mathbb{B})_{ij} = \tilde{\omega}^+_{kj} \cdot \phi^+_{kj} = \frac{-4\alpha_{\mathrm{v}_k}}{(\deg(\mathrm{v}_k) + \alpha_{\mathrm{v}_k})^2}.$$

All other entries in the i-th row are zero. If e_i is an incoming edge in a vertex $\mathrm{v}_k \in \mathrm{V}^\mathrm{D}_\mathrm{out}$ the only non-zero entry is $(\mathbb{B}^\top\mathbb{B})_{ii} = 1$ because the i-th column of Φ^+ contains only zeros in that case.

Since the graph $\tilde{\mathrm{G}}_\mathrm{d}$ is divided in n independent subgraphs, we can order the columns and rows in the matrix $\mathbb{B}^\top\mathbb{B}$ block-wise with respect to the vertices. More precisely, using elementary row and column permutations we can transform the matrix into the block form

$$\mathbb{B}^\top\mathbb{B} \sim \begin{pmatrix} B_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & B_n \end{pmatrix},$$

where B_k is the $\deg(\mathrm{v}_k) \times \deg(\mathrm{v}_k)$ matrix

$$B_k := \left(-\frac{4\alpha_{\mathrm{v}_k}}{(\deg(\mathrm{v}_k) + \alpha_{\mathrm{v}_k})^2} \right)_{i,j=1}^{\deg(\mathrm{v}_k)} + \mathrm{Id}_{\deg(\mathrm{v}_k)}.$$

Here, we set $\alpha_{\mathrm{v}_k} := 0$ for $\mathrm{v}_k \in \mathrm{V}^\mathrm{D}_\mathrm{out}$. Since the eigenvalues of these blocks are known, this representation yields

$$\sigma(\mathbb{B}^\top\mathbb{B}) = \begin{cases} \left\{ 1 - \frac{4\deg(\mathrm{v}_i)\alpha_{\mathrm{v}_i}}{(\deg(\mathrm{v}_i)+\alpha_{\mathrm{v}_i})^2} \,\middle|\, \mathrm{v}_i \in \mathrm{V} \right\} \cup \{1\}, & \text{if } \exists\, \mathrm{v}_i \text{ s.t. } \deg(\mathrm{v}_i) \geqslant 2, \\ \left\{ 1 - \frac{4\deg(\mathrm{v}_i)\alpha_{\mathrm{v}_i}}{(\deg(\mathrm{v}_i)+\alpha_{\mathrm{v}_i})^2} \,\middle|\, \mathrm{v}_i \in \mathrm{V} \right\}, & \text{else.} \end{cases}$$

Since $\alpha_{v_i} \geqslant 0$ by assumption, all eigenvalues of $(\mathbb{B}^\top\mathbb{B})$ have modulus smaller than one and so $\mathbb{B}$ is contractive and (iii) follows.

For statement (ii), by [16, Thm. 2.6(iii)] it suffices to show that $\mathbb{B}$ is unitary if $\alpha_{v_i} = 0$ for all $v_i \in V_{in}\dot{\cup}V_{out}^N$. However, this follows immediately from the explicit form of $\mathbb{B}^\top\mathbb{B}$. □

Finally, we consider these results in the context of system (2.6) and show how its classical solutions and well-posedness can be obtained via our flow approach.

THEOREM 2.12. *If* $\alpha_{v_i} \neq -\deg(v_i)$ *for all* $v_i \in V_{in}\dot{\cup}V_{out}^N$, *system (2.6) is well-posed.*

More precisely, let $(z_j^0, v_j^0) \in H^2([0,l_j],\mathbb{C}) \times H^1([0,l_j],\mathbb{C})$ *satisfy boundary conditions (2.4) and (2.5) and let* y^0 *be the associated initial data. Then there exists a unique classical solution to the wave system and is given by*

$$z_j(t,x) = \int_0^t \left(\left[T(s)y^0\right]_j \left(\tfrac{x}{l_j}\right) + \left[T(s)y^0\right]_{j+m} \left(1 - \tfrac{x}{l_j}\right)\right) ds + z_j^0(x) \tag{2.9}$$

for $x \in [0,l_j]$, $t \geqslant 0$ *and* $j = 1,\dots,m$.

Proof. To show that z_j given by (2.9) is a classical solution for the wave system, observe that by Proposition 2.9(i)⇒(iii), the function y^0 belongs to $D(A)$. Consequently,

$$\mathbb{R}_+ \ni t \mapsto y(t,\cdot) := T(t)y^0$$

is the unique classical solution to the transport equation (2.7) with boundary conditions (2.8) and initial condition y^0 (see [13, Chap. II, Prop. 6.2]). Then we differentiate z_j twice with respect to time and obtain

$$\begin{aligned}\frac{\partial^2}{\partial t^2} z_j(t,x) &= \frac{\partial}{\partial t}\left[y_j\left(t,\tfrac{x}{l_j}\right) + y_{j+m}\left(t, 1 - \tfrac{x}{l_j}\right)\right] \\ &= \frac{1}{l_j}\left(y_j'\left(t,\tfrac{x}{l_j}\right) + y_{j+m}'\left(t, 1 - \tfrac{x}{l_j}\right)\right),\end{aligned}$$

where we used that y_j and y_{j+m} satisfy the transport equation (2.7). Differentiating z_j with respect to the spatial variable gives

$$\begin{aligned}\frac{\partial^2}{\partial x^2} z_j(t,x) &= \frac{\partial}{\partial x}\int_0^t \frac{\partial}{\partial x}\left[y_j\left(s,\tfrac{x}{l_j}\right) + y_{j+m}\left(s, 1 - \tfrac{x}{l_j}\right)\right] ds + (z_j^0)''(x) \\ &= \frac{\partial}{\partial x}\int_0^t \left(\dot{y}_j\left(s,\tfrac{x}{l_j}\right) - \dot{y}_{j+m}\left(s, 1 - \tfrac{x}{l_j}\right)\right) ds + (z_j^0)''(x) \\ &= \frac{\partial}{\partial x}\left[y_j\left(t,\tfrac{x}{l_j}\right) - y_{j+m}\left(t, 1 - \tfrac{x}{l_j}\right)\right. \\ &\qquad \left. - \underbrace{\left(y_j^0\left(\tfrac{x}{l_j}\right) - y_{j+m}^0\left(1 - \tfrac{x}{l_j}\right)\right)}_{=(z_j^0)'(x)}\right] + (z_j^0)''(x) \\ &= \frac{1}{l_j}\left(y_j'\left(t,\tfrac{x}{l_j}\right) + y_{j+m}'\left(t, 1 - \tfrac{x}{l_j}\right)\right),\end{aligned}$$

and so z_j satisfies the wave equation for all $j = 1, \dots, m$. Moreover, these calculations show that

$$\begin{cases} z_j'(t,x) = y_j\left(t, \frac{x}{l_j}\right) - y_{j+m}\left(t, 1 - \frac{x}{l_j}\right), \\ \dot{z}_j(t,x) = y_j\left(t, \frac{x}{l_j}\right) + y_{j+m}\left(t, 1 - \frac{x}{l_j}\right). \end{cases} \tag{2.10}$$

Since $y(t,\cdot) \in D(A)$, condition (2.8) holds for all $t \geqslant 0$. Using the equalities (2.10) we obtain that z satisfies the boundary conditions (2.4), (2.5) for the wave equation. Since z also satisfies the initial conditions (2.2), it is a classical solution for the wave system.

For the uniqueness suppose that w is another classical solution of the wave system. By Proposition 2.8 the variables

$$\begin{cases} y_j^w(t,x) := \frac{1}{2}\left[\dot{w}_j(t, l_j x) + w_j'(t, l_j x)\right], \\ y_{j+m}^w(t,x) := \frac{1}{2}\left[\dot{w}_j(t, l_j(1-x)) - w_j'(t, l_j(1-x))\right], \end{cases} \qquad x \in [0,1],\ t \geqslant 0$$

satisfy the transport equation (2.7) and by Proposition 2.9

$$y^w(t,1) = \mathbb{B} y^w(t,0) \quad \text{for all } t \geqslant 0.$$

Consequently, y^w is also a classical solution of the transport system and must coincide with $t \mapsto y(t,\cdot) = T(t)y^0$. But then, by definition of y^w and by equations (2.10), we have

$$w' = z' \quad \text{and} \quad \dot{w} = \dot{z},$$

and so $w = z$ follows, since both solutions have the same initial data (z^0, v^0). □

At the end of this section we want to justify our assumption $\alpha_{\mathrm{v}_i} \neq -\deg(\mathrm{v}_i)$ for $\mathrm{v}_i \in \mathrm{V}_{\mathrm{in}} \dot\cup \mathrm{V}_{\mathrm{out}}^{\mathrm{N}}$. Consider the following example of a single string with Neumann boundary conditions in vertex v_1 and a damping with damping constant $\alpha_{\mathrm{v}_2} = -1$ in vertex v_2.

(a) String damped in vertex v_2 (b) Damping in the flow setting

Figure 6:

In our setting this system is described by

$$\begin{cases} \ddot{z}(t,x) = z''(t,x), & x \in [0,1], \quad t \geqslant 0, \\ z'(t,0) = 0, & t \geqslant 0, \\ z'(t,1) = \dot{z}(t,1), & t \geqslant 0. \end{cases}$$

We reformulate this system in terms of the state variables $(y_j)_{j=1}^2$ from Definition 2.6 and the corresponding transport process. The Neumann boundary condition becomes

$$y_2(t,1) = y_1(t,0), \quad t \geqslant 0,$$

while the damping condition transforms to

$$y_2(t,0) = 0, \quad t \geqslant 0.$$

In matrix form we obtain

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{=:K} \begin{pmatrix} y_1(t,1) \\ y_2(t,1) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{=:L} \begin{pmatrix} y_1(t,0) \\ y_2(t,0) \end{pmatrix}$$

and the operator corresponding to this transport process is

$$A = \operatorname{diag}\left(\frac{d}{dx}\right)_{j=1}^2, \quad D(A) = \left\{ g \in \mathrm{H}^1([0,1],\mathbb{C}^2) | \;\; Kg(1) = Lg(0) \right\}.$$

Assume that there is some material on edge e_1. Then it is transported with speed one to vertex v_1 where it simply flows through. Afterwards it is transported on edge e_2 towards vertex v_2 where the boundary conditions tells us that nothing flows in. This is a very strange physical behavior and in fact, it can be shown that the operator $(A, D(A))$ does not generate a C_0-semigroup (e.g., by [17, Chap. III, Thm. 1.10]).

3. Outlook

In a subsequent paper we will use the presented approach to treat control and stability questions for wave equations on networks.

(i) We will show that the energy of a wave system corresponds directly to the norm of the corresponding difference semigroup. Using recent results by A. Borishev, Y. Tomilov and the author ([6], [16]), this allows to characterize exponential and polynomial stability.

(ii) We combine an idea of S. Nicaise and J. Valein ([26]) with our approach and show how delay-damped networks of wave equations can be modeled by means of a difference operator. Moreover, we prove stability results for delay-damped star-shaped networks.

(iii) We show how our approach can be combined with abstract boundary control systems as studied in [12]. More precisely, a node-controlled wave equation transforms into a node-controlled flow on the doubled graph. Furthermore the theory of difference operators can be used to find the maximal reachable deformations of the wave network.

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