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SAMUEL MULTIPLICITIES AND BROWDER SPECTRUM OF OPERATOR MATRICES

SHIFANG ZHANG AND JUNDE WU

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Abstract. In this paper, we first point out that the necessity of Theorem 4 in [8] does not hold under the given condition and present a revised version with a little modification. Then we show that the definitions of some classes of semi-Fredholm operators, which use the language of algebra and first introduced by X. Fang in [8], are equivalent to that of some well-known operator classes. For example, the concept of shift-like semi-Fredholm operator on Hilbert space coincide with that of upper semi-Browder operator. For applications of Samuel multiplicities we characterize the sets of $\bigcap_{C\in B(K,H)}\sigma_{ab}(M_C)$, $\bigcap_{C\in B(K,H)}\sigma_{sb}(M_C)$ and $\bigcap_{C\in B(K,H)}\sigma_{b}(M_C)$, respectively, where $M_C=\begin{pmatrix} A & C\\ 0 & B\end{pmatrix}$ denotes a 2-by-2 upper triangular operator matrix acting on the Hilbert space $H\oplus K$.

1. Introduction

Throughout this paper, let H and K be separable infinite dimensional complex Hilbert spaces and $B(H,K)$ the set of all bounded linear operators from H into K, when $H=K$, we write $B(H,H)$ as $B(H)$. For $A\in B(H)$, $B\in B(K)$ and $C\in B(K,H)$, we have $M_C=\begin{pmatrix} A & C\\ 0 & B\end{pmatrix}\in B(H\oplus K)$. For $T\in B(H,K)$, let $R(T)$ and $N(T)$ denote the range and kernel of T, respectively, and denote $\alpha(T)=\dim N(T)$, $\beta(T)=\dim K/R(T)$. If $T\in B(H)$, the ascent $asc(T)$ of T is defined to be the smallest nonnegative integer k which satisfies that $N(T^k)=N(T^{k+1})$. If such k does not exist, then the ascent of T is defined as infinity. Similarly, the descent $des(T)$ of T is defined as the smallest nonnegative integer k for which $R(T^k)=R(T^{k+1})$ holds. If such k does not exist, then $des(T)$ is defined as infinity, too. If the ascent and the descent of T are finite, then they are equal (see [3]). For $T\in B(H)$, if $R(T)$ is closed and $\alpha(T)<\infty$, then T is said to be a upper semi-Fredholm operator, if $\beta(T)<\infty$, which implies that $R(T)$ is closed, then T is said to be a lower semi-Fredholm operator. If $T\in B(H)$ is either upper or lower semi-Fredholm operator, then T is said to be a semi-Fredholm operator. If both $\alpha(T)<\infty$ and $\beta(T)<\infty$, then T is said to be a Fredholm operator. For a semi-Fredholm operator T, its index ind (T) is defined by ind $(T)=\alpha(T)-\beta(T)$.

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In this paper, the sets of invertible operators, left invertible operators and right invertible operators on H are denoted by $G(H), G_l(H)$ and $G_r(H)$, respectively, the sets of all Fredholm operators, upper semi-Fredholm operators and lower semi-Fredholm operators on H are denoted by $\Phi(H)$, $\Phi_+(H)$ and $\Phi_-(H)$, respectively, the sets of all Browder operators, upper semi-Browder operators and lower semi-Browder operators on H are defined, respectively, by

$$\Phi_b(H) := \{T \in \Phi(H) : asc(T) = des(T) < \infty\},$$
$$\Phi_{ab}(H) := \{T \in \Phi_+(H) : asc(T) < \infty\},$$
$$\Phi_{sb}(H) := \{T \in \Phi_-(H) : des(T) < \infty\}.$$

Moreover, for $T \in B(H)$, we introduce its corresponding spectra as following [19]:

the spectrum: $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin G(H)\}$,
the left spectrum: $\sigma_l(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin G_l(H)\}$,
the right spectrum: $\sigma_r(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin G_r(H)\}$,
the essential spectrum: $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi(H)\}$,
the upper semi-Fredholm spectrum: $\sigma_{SF+}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_+(X)\}$,
the lower semi-Fredholm spectrum: $\sigma_{SF-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_-(X)\}$,
the Browder spectrum: $\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_b(H)\}$,
the upper semi-Browder spectrum: $\sigma_{ab}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_{ab}(X)\}$,
the lower semi-Browder spectrum: $\sigma_{sb}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_{sb}(X)\}$.

For a semi-Fredholm operator $T \in B(H)$, its shift Samuel multiplicity $s\_mul(T)$ and backward shift Samuel multiplicity $b.s\_mul(T)$ are defined ([5-8]), respectively, by

$$s\_mul(T) = \lim_{k\to\infty} \frac{\beta(T^k)}{k},$$

$$b.s\_mul(T) = \lim_{k\to\infty} \frac{\alpha(T^k)}{k}.$$

Moreover, it has been proved that $s\_mul(T), b.s.\_mul(T) \in \{0,1,2,\ldots,\infty\}$ and $\operatorname{ind}(T) = b.s.\_mul(T) - s\_mul(T)$. These two invariants refine the Fredholm index and can be regarded as the stabilized dimension of the kernel and cokernel [8].

DEFINITION 1.1. ([8]) A semi-Fredholm operator $T \in B(H)$ is called a pure shift semi-Fredholm operator if T has the form $T = U^n P$, where $n \in \mathbf{N}$ or $n = \infty$, U is the unilateral shift, and P is a positive invertible operator. Analogously, T is called a pure backward shift semi-Fredholm operator if its adjoint T^* is a pure shift semi-Fredholm operator. Here U^∞ denotes the direct sum of countably (infinite) many copies of U.

DEFINITION 1.2. ([8]) A semi-Fredholm operator $T \in B(H)$ is called a shift-like semi-Fredholm operator if $b.s.\_mul(T) = 0$; T is called a shift semi-Fredholm operator if $N(T) = 0$. Analogous concepts for backward shifts can also be defined. T is called a stationary semi-Fredholm operator if $b.s.\_mul(T) = 0$ and $s\_mul(T) = 0$.

It follows from Definition 1.1 that T is a shift semi-Fredholm operator iff T is a left invertible operator, and that T is a backward shift semi-Fredholm operator iff T is a right invertible operator.

In ([8], Theorem 4 and Corollary 18), Fang gave the following 4×4 upper-triangular representation theorem: An operator $T \in B(H)$ is semi-Fredholm iff T can be decomposed into the following form with respect to some orthogonal decomposition $H = H_1 \oplus H_2 \oplus H_3 \oplus H_4$,

$$T = \begin{pmatrix} T_1 & * & * & * \\ 0 & T_2 & * & * \\ 0 & 0 & T_3 & * \\ 0 & 0 & 0 & T_4 \end{pmatrix},$$

where $\dim H_4 < \infty$, T_1 is a pure backward shift semi-Fredholm operator, T_2 is invertible, T_3 is a pure shift semi-Fredholm operator, T_4 is a finite nilpotent operator. Moreover, $\operatorname{ind}(T_1) = b.s.\_mul(T)$ and $\operatorname{ind}(T_3) = -s\_mul(T)$.

The following example shows that the representation theorem is not accurate.

EXAMPLE 1.3. Let H be the direct sum of countably many copies of $\ell^2 := \ell^2(\mathbf{N})$, that is, the elements of H are the sequences $\{x_j\}_{j=1}^{\infty}$ with $x_j \in \ell^2$ and $\sum_{j=1}^{\infty} \|x_j\|^2 < \infty$. Let V be the unilateral shift on ℓ^2, i.e.,

$$V : \ell^2 \to \ell^2, \quad \{z_1, z_2, \ldots\} \mapsto \{0, z_1, z_2, \ldots\},$$

and the operators T_1 and T_3 be defined by

$$T_1 : H \to H, \quad \{x_1, x_2, \ldots\} \mapsto \{V^* x_1, V^* x_2, \ldots\}$$

and

$$T_3 : H \to H, \quad \{x_1, x_2, \ldots\} \mapsto \{V x_1, V x_2, \ldots\}.$$

Now, we consider the operator

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix} : H \oplus H \to H \oplus H.$$

Note that T_1 is a pure backward shift semi-Fredholm operator, T_3 is a pure shift semi-Fredholm operator, so T satisfies the conditions of Fang's 4×4 triangular representation theorem, but, since $\alpha(T_1) = \alpha(T) = \beta(T) = \dim(H/R(T_3)) = \infty$, so T is not a semi-Fredholm operator.

Now, we can prove the following improved 4×4 upper-triangular representation theorem:

THEOREM 1.4. *An operator $T \in B(H)$ is semi-Fredholm iff T can be decomposed into the following form with respect to some orthogonal decomposition $H = H_1 \oplus H_2 \oplus H_3 \oplus H_4$,*

$$T = \begin{pmatrix} T_1 & * & * & * \\ 0 & T_2 & * & * \\ 0 & 0 & T_3 & * \\ 0 & 0 & 0 & T_4 \end{pmatrix},$$

where $\dim H_4 < \infty$, T_1 *is a pure backward shift semi-Fredholm operator,* T_2 *is invertible,* T_3 *is a pure shift semi-Fredholm operator and* $\min\{\text{ind}\,(T_1), -\text{ind}\,(T_3)\} < \infty$, T_4 *is a finite nilpotent operator. Moreover,*

(1) $\text{ind}\,(T_1) = b.s.\_mul(T)$, $\text{ind}\,(T_3) = -s\_mul(T)$;

(2) $\text{ind}\,(T) = +\infty$ *iff* $\text{ind}\,(T_1) = +\infty$;

(3) $\text{ind}\,(T) = -\infty$ *iff* $\text{ind}\,(T_3) = -\infty$;

(4) $\text{ind}\,(T)$ *is finite iff both of* $\text{ind}\,(T_1)$ *and* $\text{ind}\,(T_3)$ *are finite.*

Theorem 1.4 can be described as 3×3 triangular representation form which may be more convenient for the study of operator theory, that is,

THEOREM 1.5. *An operator* $T \in B(H)$ *is semi-Fredholm if and only if* T *can be decomposed into the following form with respect to some orthogonal decomposition* $H = H_1 \oplus H_2 \oplus H_3$

$$T = \begin{pmatrix} T_1 & T_{12} & T_{13} \\ 0 & T_2 & T_{23} \\ 0 & 0 & T_3 \end{pmatrix} : H_1 \oplus H_2 \oplus H_3 \to H_1 \oplus H_2 \oplus H_3,$$

where $\dim H_3 < \infty$, T_1 *is a right invertible operator,* T_3 *is a finite, nilpotent operator,* T_2 *is a left invertible operator, and* $\min\{\text{ind}\,(T_1), -\text{ind}\,(T_2)\} < \infty$. *Moreover,* $\text{ind}\,(T_1) = \alpha(T_1) = b.s.\_mul(T)$, $\text{ind}\,(T_2) = -\beta(T_2) = -s\_mul(T)$ *and* $\text{ind}\,(T) = \alpha(T_1) - \beta(T_2)$.

The next lemma is useful for the proofs of our results below, especially in Section 2.

LEMMA 1.6. [19] *Let* $A \in B(H)$, $B \in B(K)$ *and* $C \in B(K,H)$.

(1) *If* $A \in \Phi_b(H)$, *then* $B \in \Phi_{ab}(K)$ *iff* $M_C \in \Phi_{ab}(H \oplus K)$ *for some* $C \in B(K,H)$.

(2) *If* $M_C \in \Phi_{ab}(H \oplus K)$ *for some* $C \in B(K,H)$, *then* $A \in \Phi_{ab}(H)$.

(3) *If* $A \in \Phi_{ab}(H)$ *and* $B \in \Phi_{ab}(K)$, *then* $M_C \in \Phi_{ab}(H \oplus K)$ *for any* $C \in B(K,H)$.

(4) *If* $B \in \Phi_b(K)$, *then* $A \in \Phi_{ab}(H)$ *iff* $M_C \in \Phi_{ab}(H \oplus K)$ *for some* $C \in B(K,H)$; $A \in \Phi_{sb}(H)$) *iff* $M_C \in \Phi_{sb}(H \oplus K)$ *for some* $C \in B(K,H)$.

(5) *If* $M_C \in \Phi_b(H \oplus K)$ *for some* $C \in B(K,H)$, *then* $A \in \Phi_{ab}(H)$ *and* $B \in \Phi_{sb}(K)$.

(6) *If two of* A, B *and* M_C *are Browder, then so is the third.*

PROPOSITION 1.7. *Let* $T \in B(H)$. *Then* T *is upper semi-Browder iff* T *can be decomposed into the following form with respect to some orthogonal decomposition* $H = H_1 \oplus H_2$,

$$T = \begin{pmatrix} T_1 & T_{12} \\ 0 & T_2 \end{pmatrix},$$

where $\dim(H_1) < \infty$, T_1 *is nilpotent,* T_2 *is left invertible, and* $\beta(T_2) = s\_mul(T) = -\operatorname{ind}(T)$.

Proof. Necessity. Suppose that T is upper semi-Browder. Then we can assume $p = asc(T) < \infty$. Let $H_1 = N(T^p)$. Note that T is upper semi-Fredholm, so $\dim H_1 < \infty$. Let $H = H_1 \oplus H_1^{\perp}$, we have

$$T = \begin{pmatrix} T_1 & T_{12} \\ 0 & T_2 \end{pmatrix} : H_1 \oplus H_1^{\perp} \to H_1 \oplus H_1^{\perp}.$$

That T_1 is nilpotent is clear. Moreover, since the fact that $\dim H_1 < \infty$ implies $T_1 \in \Phi_b(H_1)$, it follows from Lemma 1.6 (1) that $T_2 \in \Phi_{ab}(H_1^{\perp})$. A direct calculation shows that T_2 is injective, thus, T_2 is left invertible. From Theorem 1.5, it is clear that $\beta(T_2) = s\_mul(T) = \operatorname{ind}(T_2)$.

Sufficiency follows from Lemma 1.6 immediately. □

PROPOSITION 1.8. *Let* $T \in B(H)$. *Then* T *is lower semi-Browder iff* T *can be decomposed into the following form with respect to some orthogonal decomposition* $H = H_1 \oplus H_2$,

$$T = \begin{pmatrix} T_1 & T_{12} \\ 0 & T_2 \end{pmatrix},$$

where $\dim(H_2) < \infty$, T_1 *is right invertible,* T_2 *is nilpotent, and* $\alpha(T_1) = b.s.\_mul(T) = \operatorname{ind}(T)$.

Proof. Necessity. If T is lower semi-Browder, then we can assume $p = des(T) < \infty$. Denote $H_1 = R(T^p)$ and $H_2 = H_1^{\perp}$. Note that T^p is lower semi-Browder, so $\dim H_2 < \infty$. Let $H = H_1 \oplus H_2$, we have

$$T = \begin{pmatrix} T_1 & T_{12} \\ 0 & T_2 \end{pmatrix} : H_1 \oplus H_2 \to H_1 \oplus H_2.$$

That T_1 is surjective and $T_2^p = 0$ is evident. Note that $\dim H_2 < \infty$ implies $T_2 \in \Phi_b(H_2)$, it follows from Lemma 1.6 that $T_1 \in \Phi_{sb}(H_1)$, and so T_1 is right invertible. From Theorem 1.5, we have $\alpha(T_1) = \operatorname{ind}(T_1) = b.s.\_mul(T)$.

Sufficiency follows from Lemma 1.6. □

Combining Theorem 1.5, Propositions 1.7 and 1.8, we have the following theorem immediately.

THEOREM 1.9. *Let* $T \in B(H)$. *Then*

(1) T *is a shift-like semi-Fredholm operator iff* T *is an upper semi-Browder operator.*

(2) T *is a backward shift-like semi-Fredholm operator iff* T *is a lower semi-Browder operator.*

(3) T *is a stationary semi-Fredholm operator iff* T *is a Browder operator.*

2. Applications of Samuel multiplicities

In ([8-12]), Fang studied Samuel multiplicities and presented some applications. In this section, by using Samuel multiplicities, we characterize the sets $\bigcap_{C\in B(K,H)} \sigma_{ab}(M_C)$, $\bigcap_{C\in B(K,H)} \sigma_{sb}(M_C)$ and $\bigcap_{C\in B(K,H)} \sigma_{b}(M_C)$ completely, where $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is a 2×2 upper triangular operator matrix defined on $H\oplus K$. For the study advances of 2×2 upper triangular operator matrix, see ([1-4], [13-19]).

First, note that if $T\in B(H)$, then T is bounded below iff T is left invertible, thus, Theorem 1 of [14] can be rewritten as follows:

PROPOSITION 2.1. *[14]. For any given $A\in B(H)$ and $B\in B(K)$, M_C is left invertible for some $C\in B(K,H)$ iff A is left invertible and*

$$\begin{cases} a(B)\leqslant \beta(A) & \text{if } R(B) \text{ is closed,} \\ \beta(A)=\infty & \text{if } R(B) \text{ is not closed.} \end{cases}$$

PROPOSITION 2.2. [4] *For any given $A\in B(H)$ and $B\in B(K)$,*

$$\bigcap_{C\in B(K,H)} \sigma(M_C) = \sigma_l(A)\cup\sigma_r(B)\cup\{\lambda\in\mathbb{C} : \alpha(B-\lambda)\neq\beta(A-\lambda)\}. \tag{1}$$

One of the main results in this section is:

THEOREM 2.3. *For any given $A\in B(H)$ and $B\in B(K)$, $M_C\in\Phi_{ab}(H\oplus K)$ for some $C\in B(K,H)$ iff $A\in\Phi_{ab}(H)$ and*

$$\begin{cases} s\_mul(A)=\infty & \text{if } B\notin\Phi_+(K), \\ b.s.\_mul(B)\leqslant s\_mul(A) & \text{if } B\in\Phi_+(K). \end{cases}$$

Proof. We first claim that if $B\notin\Phi_+(K)$, then

$$M_C\in\Phi_{ab}(H\oplus K) \text{ for some } C\in B(K,H) \Leftrightarrow A\in\Phi_{ab}(H) \text{ and } s\_mul(A)=\infty. \tag{2}$$

To do this, suppose $M_C\in\Phi_{ab}(H\oplus K)$. Then from Lemma 1.6 we have $A\in\Phi_{ab}(H)$. If $s\_mul(A)<\infty$, then $A\in\Phi(H)$, since $\mathrm{ind}(A)=\alpha(A)-\beta(A)=b.s.\_mul(A)-s\_mul(A)$. Hence it is easy to show that $B\in\Phi_+(K)$, which is in a contradiction. Thus, $s\_mul(A)=\infty$.

Conversely, suppose that $A\in\Phi_{ab}(H)$ and $s\_mul(A)=\infty$, which implies $\beta(A)=\infty$. It follows from Proposition 1.7 that A can be decomposed into the following form with respect to some orthogonal decomposition $H=H_1\oplus H_2$

$$A=\begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix},$$

where $\dim(H_1)<\infty$, A_1 is nilpotent, and A_2 is a left invertible operator. Noting that $\beta(A)=\infty$, we have $\beta(A_2)=\infty$. Hence it follows from Lemma 2.1 that there exists some $C_0 \in B(K,H_2)$ such that $\begin{pmatrix} A_2 & C_0 \\ 0 & B \end{pmatrix}$ is left invertible. Now consider operator

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} A_1 & A_{12} & 0 \\ 0 & A_2 & C_0 \\ 0 & 0 & B \end{pmatrix},$$

where $C = \begin{pmatrix} 0 \\ C_0 \end{pmatrix} \in B(K,H)$. By Lemma 1.6, it is easy to check that $M_C \in \Phi_{ab}(H \oplus K)$.

Next, We claim that if $B \in \Phi_+(K)$, then

$$M_C \in \Phi_{ab}(H \oplus K) \text{ for some } C \in B(K,H) \Leftrightarrow A \in \Phi_{ab}(H) \text{ and } b.s.\_mul(B) \leqslant s\_mul(A). \tag{3}$$

To this end, suppose $M_C \in \Phi_{ab}(H \oplus K)$, which implies $A \in \Phi_{ab}(H)$. By Proposition 1.8, we have that A can be decomposed into the following form with respect to some orthogonal decomposition $H = H_1 \oplus H_2$

$$A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix},$$

where $\dim(H_1) < \infty$, A_1 is nilpotent, A_2 is a left invertible operator, and $\beta(A_2) = s\_mul(A)$. Since the assumption that $B \in \Phi_+(K)$, using Theorem 1.5, we know that B can be decomposed into the following form with respect to some orthogonal decomposition $K = K_1 \oplus K_2 \oplus K_3$

$$B = \begin{pmatrix} B_1 & * & * \\ 0 & B_2 & * \\ 0 & 0 & B_3 \end{pmatrix},$$

where $\dim K_3 < \infty$, B_1 is a right invertible operator, B_2 is a left invertible operator, B_3 is a finite, nilpotent operator, and the parts marked by $*$ can be any operators. Moreover, $\mathrm{ind}(B_1) = \alpha(B_1) = b.s.\_mul(B)$, $ind(B_2) = -\beta(B_2) = -s\_mul(B_1)$ and $\mathrm{ind}(B) = \alpha(B_1) - \beta(B_2)$. Therefore, M_C can be rewritten as the following form

$$M_C = \begin{pmatrix} A_1 & A_{12} & C_{11} & C_{12} & C_{13} \\ 0 & A_2 & C_{21} & C_{32} & C_{23} \\ 0 & 0 & B_1 & * & * \\ 0 & 0 & 0 & B_2 & * \\ 0 & 0 & 0 & 0 & B_3 \end{pmatrix} : H_1 \oplus H_2 \oplus K_1 \oplus K_2 \oplus K_3 \to H_1 \oplus H_2 \oplus K_1 \oplus K_2 \oplus K_3.$$

Noting that $\dim(H_1) < \infty$ and $\dim(K_3) < \infty$, we have $A_1 \in \Phi_b(H_1)$ and $B_3 \in \Phi_b(K_3)$. Consequently, Lemma 1.6 leads to

$$\begin{pmatrix} A_2 & C_{21} & C_{32} \\ 0 & B_1 & * \\ 0 & 0 & B_2 \end{pmatrix} \in \Phi_{ab}(H_2 \oplus K_1 \oplus K_2),$$

which implies

$$\begin{pmatrix} A_2 & C_{21} \\ 0 & B_1 \end{pmatrix} \in \Phi_{ab}(H_2 \oplus K_1).$$

Now we shall prove that

$$\beta(A_2) \geqslant \alpha(B_1).$$

If $\beta(A_2) = \infty$, the above inequality obviously holds. On the other hand, if $\beta(A_2) < \infty$, then $A_2 \in \Phi(H_2)$, and hence $B_1 \in \Phi_+(K_1)$. Thus,

$$0 \geqslant \text{ ind}\left(\begin{pmatrix} A_2 & C_{21} \\ 0 & B_1 \end{pmatrix}\right) = \text{ ind}(A_2) + \text{ ind}(B_1) = -\beta(A_2) + \alpha(B_1),$$

that is,

$$\alpha(B_1) \leqslant \beta(A_2).$$

Therefore,

$$b.s.\_mul(B) \leqslant s\_mul(A).$$

Conversely, suppose $A \in \Phi_{ab}(H)$, $B \in \Phi_+(K)$ and $b.s.\_mul(B) \leqslant s\_mul(A)$. Similar to the above arguments, we have

$$A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} : H_1 \oplus H_2 \mapsto H_1 \oplus H_2$$

and

$$B = \begin{pmatrix} B_1 & * & * \\ 0 & B_2 & * \\ 0 & 0 & B_3 \end{pmatrix} : K_1 \oplus K_2 \oplus K_3 \mapsto K_1 \oplus K_2 \oplus K_3,$$

where $\dim(H_1) < \infty$, A_1 is nilpotent, A_2 is a left invertible operator; $\dim K_3 < \infty$, B_1 is a right invertible operator, B_2 is a left invertible operator, B_3 is a finite, nilpotent operator, and the parts marked by $*$ can be any operators. Moreover, $\beta(A_2) = s\_mul(A)$ and $\alpha(B_1) = b.s.\_mul(B)$. Since the assumption that $b.s.\_mul(B) \leqslant s\_mul(A)$, we have $\alpha(B_1) \leqslant \beta(A_2)$. It follows from Lemma 2.1 that there exists a left invertible operator $\widetilde{C} \in B(K_1, H_2)$ such that

$$\begin{pmatrix} A_2 & \widetilde{C} \\ 0 & B_1 \end{pmatrix} \in B(H_2 \oplus K_1) \text{ is left invertible.}$$

Consider operator

$$\begin{aligned} M_C &= \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : H \oplus K \to H \oplus K \\ &= \begin{pmatrix} A_1 & A_{12} & 0 & 0 & 0 \\ 0 & A_2 & \widetilde{C} & 0 & 0 \\ 0 & 0 & B_1 & * & * \\ 0 & 0 & 0 & B_2 & * \\ 0 & 0 & 0 & 0 & B_3 \end{pmatrix} : H_1 \oplus H_2 \oplus K_1 \oplus K_2 \oplus K_3 \to H_1 \oplus H_2 \oplus K_1 \oplus K_2 \oplus K_3, \end{aligned}$$

where $C = \begin{pmatrix} 0 & 0 & 0 \\ \widetilde{C} & 0 & 0 \end{pmatrix} \in B(K_1 \oplus K_2 \oplus K_3, H_1 \oplus H_2)$. Using Lemma 1.6 , it is easy to see that $M_C \in \Phi_{ab}(H \oplus K)$. $\square$

By duality, we have

THEOREM 2.4. *For any given $A \in B(H)$ and $B \in B(K)$, $M_C \in \Phi_{sb}(H \oplus K)$ for some $C \in B(K,H)$ iff $B \in \Phi_{sb}(K)$ and*

$$\begin{cases} b.s.\_mul(B) = \infty & \text{if } A \notin \Phi_-(H) \\ b.s.\_mul(B) \geqslant s\_mul(A) & \text{if } A \in \Phi_-(H) \end{cases}$$

From Theorems 2.3 and 2.4, we obtain the following two corollaries, concerning perturbations of the upper semi-Browder spectrum and lower semi-Browder spectrum, respectively.

COROLLARY 2.5. *For any given $A \in B(H)$ and $B \in B(K)$, we have*

$$\bigcap_{C \in B(K,H)} \sigma_{ab}(M_C) = \sigma_{ab}(A) \cup \{\lambda \in \mathbb{C} : \lambda \in \sigma_{SF+}(B) \text{ and } s.\_mul(A-\lambda) < \infty\} \cup$$

$$\{\lambda \in \Phi(A) \cap \Phi_+(B) : b.s.\_mul(B-\lambda) > s.\_mul(A-\lambda)\}.$$

COROLLARY 2.6. *For any given $A \in B(H)$ and $B \in B(K)$, we have*

$$\bigcap_{C \in B(K,H)} \sigma_{sb}(M_C) = \sigma_{sb}(B) \cup \{\lambda \in \mathbb{C} : \lambda \in \sigma_{SF-}(A) \text{ and } b.s.\_mul(B-\lambda) < \infty\} \cup$$

$$\{\lambda \in \Phi(B) \cap \Phi_-(A) : b.s.\_mul(B-\lambda) < s.\_mul(A-\lambda)\}.$$

THEOREM 2.7. *For any given $A \in B(H)$ and $B \in B(K)$, the following statements are equivalent:*

(1) $M_C \in \Phi_b(H \oplus K)$ *for some* $C \in B(K,H)$;

(2) $A \in \Phi_{ab}(H)$, $B \in \Phi_{sb}(K)$ *and* $b.s.\_mul(B) = s\_mul(A)$;

(3) $A \in \Phi_{ab}(H)$, $B \in \Phi_{sb}(K)$ *and* $\alpha(A) + \alpha(B) = \beta(A) + \beta(B)$.

Proof. $(1) \Rightarrow (2)$. Suppose that $M_C \in \Phi_b(H \oplus K)$. Then from Lemma 1.6, we have $A \in \Phi_{ab}(H)$ and $B \in \Phi_{sb}(K)$. Using Propositions 1.7 and 1.8, we have

$$M_C = \begin{pmatrix} A_1 & A_{12} & C_{11} & C_{12} \\ 0 & A_2 & C_{21} & C_{32} \\ 0 & 0 & B_1 & B_{12} \\ 0 & 0 & 0 & B_2 \end{pmatrix} : H_1 \oplus H_2 \oplus K_1 \oplus K_2 \to H_1 \oplus H_2 \oplus K_1 \oplus K_2,$$

where $\dim(H_1) < \infty$, A_1 is nilpotent, A_2 is a left invertible operator, $\dim K_2 < \infty$, B_1 is a right invertible operator, B_2 is a finite, nilpotent operator. Moreover,

$$\beta(A_2) = s.\_mul(A) \text{ and } \alpha(B_1) = b.s.\_mul(B).$$

In addition, it follows from Lemma 1.6 that

$$\begin{pmatrix} A_2 & C_{21} \\ 0 & B_1 \end{pmatrix} \in \Phi_b(H_2 \oplus K_1).$$

Note the well-known fact that if $M_C \in \Phi(H \oplus K)$, then $A \in \Phi(H)$ if and only if $B \in \Phi(K)$. Thus, if $\beta(A_2) = \infty$, then $B_1 \notin \Phi(K_1)$, and so $\beta(A_2) = \alpha(B_1) = \infty$ since that B_1 is right invertible. Otherwise, if $\beta(A_2) < \infty$, then both A_2 and B_1 are Fredholm. Consequently,

$$0 = \operatorname{ind}(\begin{pmatrix} A_2 & C_{21} \\ 0 & B_1 \end{pmatrix}) = \operatorname{ind}(A_2) + \operatorname{ind}(B_1) = -\beta(A_2) + \alpha(B_1),$$

that is, $\beta(A_2) = \alpha(B_1)$. Therefore, $s.\_mul(A) = b.s.\_mul(B)$.

$(2) \Rightarrow (1)$. Suppose that $A \in \Phi_{ab}(H)$, $B \in \Phi_{sb}(K)$ and that $s.\_mul(A) = b.s.\_mul(B)$. Then from Proposition 1.7 we have that A can be decomposed into the following form with respect to some orthogonal decomposition $H = H_1 \oplus H_2$

$$A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix},$$

where and $\dim(H_1) < \infty$, A_1 is nilpotent, and A_2 is a left invertible operator. By Proposition 1.8, $B \in B(K)$ can be decomposed into the following form with respect to some orthogonal decomposition $K = K_1 \oplus K_2$

$$B = \begin{pmatrix} B_1 & B_{12} \\ 0 & B_2 \end{pmatrix},$$

where $\dim(K_2) < \infty$, B_1 is a right invertible operator, and B_2 is nilpotent. Moreover, $s.\_mul(A) = \beta(A_2)$ and $b.s.\_mul(B) = \alpha(B_1)$. Since the assumption that $s.\_mul(A) = b.s.\_mul(B)$, $\alpha(B_1) = \beta(A_2)$. Thus, we conclude from Theorem 1.5 that there exists some operator $C_{12} \in B(K_1, H_2)$ such that $\begin{pmatrix} A_2 & C_{21} \\ 0 & B_1 \end{pmatrix}$ is invertible. Define $C \in B(K,H)$ as follows:

$$C = \begin{pmatrix} 0 & 0 \\ C_{12} & 0 \end{pmatrix}.$$

By Lemma 1.6, it no hard to prove that $M_C \in \Phi_b(H \oplus K)$.

(2) is equal to (3). For this, it is sufficient to prove that if $A \in \Phi_{ab}(H)$ and $B \in \Phi_{sb}(K)$, then

$$\alpha(A) + \alpha(B) = \beta(A) + \beta(B) \text{ if and only if } b.s.\_mul(B) = s\_mul(A),$$

which follows from Propositions 1.7 and 1.8 immediately. This completes the proof. $\square$

In [1], Cao has proved the equivalence of (1) and (3) of Theorem 2.7 by a different method, which seems to be more complicated.

The next corollary immediately follows from Theorem 2.7.

COROLLARY 2.8.. *For any given $A \in B(H)$ and $B \in B(K)$, we have*

$$\begin{aligned} &\bigcap_{C \in G(K,H)} \sigma_b(M_C) \\ &= \sigma_{ab}(A) \cup \sigma_{sb}(B) \cup \{\lambda \in \Phi_{ab}(A) \cap \Phi_{sb}(B) : b.s.\_mul(B-\lambda) \neq s\_mul(A-\lambda)\} \\ &= \sigma_{ab}(A) \cup \sigma_{sb}(B) \cup \{\lambda \in \mathbb{C} : \alpha(A-\lambda) + \alpha(B-\lambda) \neq \beta(A-\lambda) + \beta(B-\lambda)\}. \end{aligned}$$

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Shifang Zhang
Department of Mathematics
Zhejiang University
Hangzhou 310027, P.R. China
and
School of Mathematics and Computer Science
Fujian Normal University
Fuzhou 350007, P.R. China
e-mail: shifangzhangfj@163.com

Junde Wu
Department of Mathematics
Zhejiang University
Hangzhou 310027, P R. China
e-mail: wjd@zju.edu.cn