Operators and Matrices
Volume 6, Number 1 (2012), 181–192

WHEN DOES THE MOORE–PENROSE INVERSE FLIP?

R. E. HARTWIG AND P. PATRÍCIO

(*Communicated by P. Šemrl*)

Abstract. In this paper, we give necessary and sufficient conditions for the matrix $\begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$, over a *-regular ring, to have a Moore-Penrose inverse of four different types, corresponding to the four cases where the zero element can stand. In particular, we study the case where the Moore-Penrose inverse of the matrix flips.

1. Introduction

Let R be a regular *-ring with 1, that is, for all $a \in R$ there exist a^- such that $aa^-a = a$, and with an involutory anti-isomorphism $(\cdot)^*$ on R, such that $(a^*)^* = a$, $(a+b)^* = a^* + b^*$ and $(ab)^* = b^*a^*$.

It is well known [9, Lemma 4], that if the involution on R satisfies the one term *star-cancellation* law

$$SC_1 : a^*a = 0 \ \Rightarrow \ a = 0, \tag{1}$$

then the Moore-Penrose inverse $a^\dagger$ can be defined. It is the unique solution to the four equations

$$\text{(i)} \ \ axa = a, \ \ \text{(ii)} \ \ xax = x, \ \ \text{(iii)} \ \ (ax)^* = ax, \ \ \text{(iv)} \ \ (xa)^* = xa. \tag{2}$$

We say x is a 1-3 inverse of a if it satisfies equations (i) and (iii) above, and y is a 1-4 inverse of a if it satisfies equations (i) and (iv) above. From the well known result due to Urquhart (cf. [1, page 48]), if x and y are a 1-3 and 1-4 inverse of a, respectively, then $a^\dagger = yax$.

We note that regular rings that satisfy SC_1 are exactly those for which all of its elements are Moore-Penrose invertible. Such a ring is said to be a $*$-regular ring. We use $R_{2\times 2}$ to denote the ring of 2×2 matrices over R.

A matrix $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ with coefficients in R is said to be of $(i, j, 0)$ type if the (i, j) entry $(M)_{ij}$ of M is zero.

Mathematics subject classification (2010): 15A09, 16E50, 16W10.
Keywords and phrases: rings, triangular matrices, von Neumann regularity, Moore-Penrose inverse.
Research with financial support provided by the Research Centre of Mathematics of the University of Minho (CMAT) through the FCT Pluriannual Funding Program.

Paper OaM-06-13

In this note we will be interested in the questions of when the matrix $\begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$ has a Moore-Penrose inverse of $(i,j,0)$ type, for $i,j \in \{1,2\}$. In particular, we will address to the case when this inverse has the "flipped" form $\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$. We will repeatedly use Cline's results ([3] and [4]) in order to express the Moore-Penrose inverse of a semi-orthogonal sum and of a column matrix. The expressions derived are simpler when compared with [7].

We only consider the special involution on $R_{2\times 2}$ of the form $\begin{bmatrix} a & c \\ b & d \end{bmatrix}^* = \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix}$.

2. Existence of the Moore-Penrose inverse

Consider the matrix $M = \begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$. In order to guarantee the existence and to be able to give a formula of $M^\dagger$, we assume the following extra conditions on the regular R:

1. $SC_2 : a^*a + b^*b = 0 \Rightarrow a = 0 = b$ (two term star-cancellation)
2. for each $a \in R$, there is $c \in R$ such that $1 + a^*a = c^*c = cc^*$ (square root axiom)

We note the following consequences:

(i) $1 + a^*a$ is a unit for all $a \in R$, that is, R has the symmetry property (see [2, page 9]). Indeed, if R is regular and satisfies SC_2 then it also satisfies SC_1, which in turn implies all its elements are Moore-Penrose invertible. Let $u = 1 + a^*a$. If $ux = 0$, then $x^*x + (ax)^*(ax) = 0$ and hence, using condition SC_2, $x = 0$. Thus u is not a divisor of 0. But $u(1 - u^\dagger u) = 0$ and hence $1 - u^\dagger u = 0$. Likewise $1 - uu^\dagger = 0$ and u is a unit.

(ii) Since $1 + a^*a = cc^* = c^*c$ is a unit, then the square root c must be a unit as well.

(iii) $1 + a^*a + b^*b = c^*c + b^*b = c^*[1 + (bc^{-1})^*(bc^{-1})]c$, which is again a unit.

(iv) If R satisfies SC_2 and is regular, then every 2×2 matrix over R is Moore-Penrose invertible. This follows from the facts that

  (a) SC_2 holds in R if and only if SC_1 holds in $R_{2\times 2}$.

  (b) R is regular if and only if the ring $R_{2\times 2}$ is regular.

(v) The previous item shows that the regularity of the involutory ring R together with SC_2 is sufficient to garantee the existence of $A^\dagger$, for any 2×2 matrix A over R, with respect to the special involution in $R_{2\times 2}$ induced by the involution on R. In the remainder of this paper we will give an expression for the Moore-Penrose inverse of a 2×2 matrix over R, and for this we will need the symmetry of $R_{2\times 2}$.

We note that symmetry of $R_{2\times 2}$ does not follow from R being regular and satisfying SC_2. Indeed, set $R = \mathbb{Z}_7$ which is a field and thus regular. The involution we take is the identity map. The squares are $\{0,1,2,4\}$. It is clear that $x^2+y^2=0 \Rightarrow x=0=y$. That is, SC_2 holds. Now, let $M = \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}$. Then $M^*M = M^TM = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix}$. Hence $I_2 + M^*M = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, which is not invertible.

(vi) In a regular symmetric ring, idempotents e have a Moore-Penrose inverse via $e^\dagger = e^*[1+(e-e^*)(e^*-e)]^{-1}$. Indeed, setting $u = 1+(e-e^*)(e^*-e)$, then u and ee^* commute, and so do u^{-1} and ee^*, u and e^*e commute, and so do u^{-1} and e^*e, and also $u^{-1}ee^*ee = e = ee^*eu^{-1}$. Since $e(e^*u^{-1})$ and $(u^{-1}e^*)e$ are symmetric, and $e(e^*u^{-1})e = u^{-1}ee^*e = e = ee^*eu^{-1} = e(u^{-1}e^*)e$, then e^*u^{-1} is a 1-3 inverse of e and $u^{-1}e^*$ is a 1-4 inverse of e, which lead to $e^\dagger = (u^{-1}e^*)e(e^*u^{-1}) = u^{-1}e^*ee^*u^{-1} = e^*u^{-1}$.

As such the orthogonal projections P_{aR} and P_{Ra} can be defined as $p = (aa^-)(aa^-)^\dagger$ and $q = (a^-a)^\dagger(a^-a)$. It then follows that the Moore-Penrose inverse $a^\dagger = qa^-p$ exists and the SC_1 property follows.

2.1. The Moore-Penrose inverse of a sum

We recall that if $ca^* = 0$, then $a+c$ has a Moore-Penrose inverse, which takes the form

$$(a+c)^\dagger = (1+y^*)(1+yy^*)^{-1}s + u^\dagger, \tag{3}$$

where

$$\begin{aligned} u &= (1-aa^\dagger)c \\ s &= a^\dagger(1-cu^\dagger) \\ y &= a^\dagger c(1-u^\dagger u) = sc. \end{aligned}$$

Indeed, and since $1-y^*y(1+y^*y)^{-1} = (1+y^*y)^{-1}$, $(1+y^*y)^{-1}$ and $1-u^\dagger u$ commute, $y(1-u^\dagger u) = y$, and $(1+yy^*)^{-1} = 1-y(1+y^*y)^{-1}y^*$, then, using [4, Theorem 2],

$$\begin{aligned} (a+c)^\dagger &= a^\dagger - a^\dagger cu^\dagger - a^\dagger c(1-u^\dagger u)(1+y^*y)^{-1}c^*a^{\dagger *}a^\dagger(1-cu^\dagger) + u^\dagger + \\ &\quad +(1-u^\dagger u)(1+y^*y)^{-1}c^*a^{\dagger *}a^\dagger(1-cu^\dagger) \\ &= s - y(1+y^*y)^{-1}c^*a^{\dagger *}s + u^\dagger + (1-u^\dagger u)(1+y^*y)^{-1}c^*a^{\dagger *}s \\ &= s - y(1+y^*y)^{-1}(1-u^\dagger u)c^*a^{\dagger *}s + u^\dagger + (1-u^\dagger u)(1-y^*y(1+y^*y)^{-1})c^*a^{\dagger *}s \\ &= s - y(1+y^*y)^{-1}y^*s + u^\dagger + y^*s + y^*y(1+y^*y)^{-1}c^*a^{\dagger *}s \\ &= u^\dagger + (1-y(1+y^*y)^{-1}y^*)s + y^*(1-y(1+y^*y)^{-1}y^*)s \\ &= u^\dagger + (1+yy^*)^{-1}s + y^*(1+yy^*)^{-1}s \\ &= (1+y^*)(1+yy^*)^{-1}s + u^\dagger \end{aligned}$$

Moreover, we also have, from [3, Theorem 2] (also from [7, Lemma 2]),

$$\begin{bmatrix} a \\ b \end{bmatrix}^{\dagger} = [\xi a^*, \xi b^*] \text{ and } \begin{bmatrix} a \\ b \end{bmatrix}^{\dagger} \begin{bmatrix} a \\ b \end{bmatrix} = a^{\dagger}a + v^{\dagger}v, \tag{4}$$

where $\xi = (a^*a + b^*b)^{\dagger}$ and $v = b(1 - a^{\dagger}a)$. We may re-express the former element as

$$\xi = t\mu^{-1}t^* + (v^*v)^{\dagger}, \tag{5}$$

in which

$$t = (1 - v^{\dagger}b)a^{\dagger}, \ \ x = (1 - vv^{\dagger})ba^{\dagger} = bt, \ \ \mu = 1 + x^*x. \tag{6}$$

Indeed, from [4, Theorem 1],

$$\xi = (a^*a + b^*b)^{\dagger} = t\ell t^* + v^{\dagger}(v^*)^{\dagger},$$

where

$$\ell = 1 - ((1 - vv^{\dagger})ba^{\dagger})^* k(ba^{\dagger})$$

and

$$k = (1 + (1 - vv^{\dagger})ba^{\dagger}((1 - vv^{\dagger})ba^{\dagger})^*)^{-1} = (1 + xx^*)^{-1}.$$

Since $(1 - vv^{\dagger})k = k(1 - vv^{\dagger}) = (1 - vv^{\dagger})k(1 - vv^{\dagger})$,

$$\begin{aligned} \ell &= 1 - (ba^{\dagger})^*(1 - vv^{\dagger})k(1 - vv^{\dagger})ba^{\dagger} \\ &= 1 - ((1 - vv^{\dagger})ba^{\dagger})^* k(1 - vv^{\dagger})ba^{\dagger} \\ &= 1 - x^*(1 + xx^*)^{-1}x \\ &= (1 + x^*x)^{-1} = \mu^{-1} \end{aligned}$$

Lastly, $v^{\dagger}(v^*)^{\dagger} = (v^*v)^{\dagger}$ by [5, Lemma 5], or simply by checking the Penrose equations (2).

2.2. The lower triangular case

Consider the 2×2 triangular matrix $M = \begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$. We may split M as

$$M = \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} + \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} = \mathscr{A} + \mathscr{C},$$

where $\mathscr{C}\mathscr{A}^* = 0$. In order to apply (3) to this semi-orthogonal splitting, we need to show that $I + A^*A$ is invertible for any matrix $A \in R_{2\times 2}$. This we now undertake.

The key fact is the following factorization. If α is a unit then

$$\begin{bmatrix} \alpha & \beta^* \\ \beta & \delta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \beta\alpha^{-1} & 1 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} 1 & \alpha^{-1}\beta^* \\ 0 & 1 \end{bmatrix}, \tag{7}$$

where z is the Schur complement $z=\delta-\beta\alpha^{-1}\beta^*$. Now consider the matrix $A=[\mathbf{a},\mathbf{b}]=\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$. Then

$$I+A^*A=\begin{bmatrix} 1+\mathbf{a}^*\mathbf{a} & \mathbf{a}^*\mathbf{b} \\ \mathbf{b}^*\mathbf{a} & 1+\mathbf{b}^*\mathbf{b} \end{bmatrix}. \tag{8}$$

and its Schur complement becomes

$$\begin{aligned} z &= 1+\mathbf{b}^*\mathbf{b}-(\mathbf{b}^*\mathbf{a})(1+\mathbf{a}^*\mathbf{a})^{-1}\mathbf{a}^*\mathbf{b} \\ &= 1+\mathbf{b}^*[I_2-\mathbf{a}(1+\mathbf{a}^*\mathbf{a})^{-1}\mathbf{a}^*]\mathbf{b} \\ &= 1+\mathbf{b}^*[I_2+\mathbf{a}\mathbf{a}^*]^{-1}\mathbf{b}, \end{aligned}$$

since $(I_2+\mathbf{a}\mathbf{a}^*)^{-1}=I_2-\mathbf{a}(1+\mathbf{a}^*\mathbf{a})^{-1}\mathbf{a}^*$.

We now turn to the matrix

$$\begin{aligned} G &= I+\mathbf{a}\mathbf{a}^* \\ &= \begin{bmatrix} 1+a_1a_1^* & a_1a_2^* \\ a_2a_1^* & 1+a_2a_2^* \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ \beta\alpha^{-1} & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & \zeta \end{bmatrix}\begin{bmatrix} 1 & \alpha^{-1}\beta^* \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

where $\alpha=1+a_1a_1^*$ is a unit, $\beta=a_2a_1^*$ and the Schur complement ζ takes the form

$$\begin{aligned} \zeta &= 1+a_2a_2^*-a_2a_1^*(1+a_1a_1^*)^{-1}a_1a_2^* \\ &= 1+a_2(1-a_1^*(1+a_1a_1^*)^{-1}a_1)a_2^* \\ &= 1+a_2(1+a_1^*a_1)^{-1}a_2^*, \end{aligned}$$

since $(1+a_1^*a_1)^{-1}=1-a_1^*(1+a_1a_1^*)^{-1}a_1$.

By using the square root axiom, we may set $1+a_1a_1^*=ee^*$ and therefore e is a unit. Consequentely, there exists f such that $(1+a_1a_1^*)^{-1}=ff^*$ and hence $\zeta=1+(a_2f)(a_2f)^*$. Again ζ is a unit, and by the square root axiom, $\zeta=hh^*$, which leads to $\zeta^{-1}=gg^*$, for some g.

Substituting into z now gives

$$\begin{aligned} z &= 1+\mathbf{b}^*(I+\mathbf{a}\mathbf{a}^*)^{-1}\mathbf{b} \\ &= 1+\mathbf{b}^*\begin{bmatrix} 1 & -\alpha^{-1}\beta^* \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \alpha^{-1} & 0 \\ 0 & \zeta^{-1} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\beta\alpha^{-1} & 1 \end{bmatrix}\mathbf{b} \\ &= 1+\begin{bmatrix} b_1^* & w^* \end{bmatrix}\begin{bmatrix} ff^* & 0 \\ 0 & gg^* \end{bmatrix}\begin{bmatrix} b_1 \\ w \end{bmatrix} \\ &= 1+b_1^*ff^*b_1+w^*gg^*w, \end{aligned}$$

where $w=b_2-\beta\alpha^{-1}b_1$, and therefore z is a unit. Thus $R_{2\times 2}$ is again symmetric.

We now may apply (3) to our matrix M, giving

$$M^\dagger = \mathscr{U}^\dagger + (I+\mathscr{Y}^*)(I+\mathscr{Y}\mathscr{Y}^*)^{-1}S, \tag{9}$$

where

$$\mathscr{U} = (I-\mathscr{A}\mathscr{A}^\dagger)\mathscr{C} = \begin{bmatrix} a & 0 \\ B & 0 \end{bmatrix},$$

with $B = (1-dd^\dagger)b$, and

$$\mathscr{Y} = \mathscr{A}^\dagger\mathscr{C}(I-\mathscr{U}^\dagger\mathscr{U}).$$

We next compute $\mathscr{U}^\dagger = \begin{bmatrix} \xi a^* & \xi B^* \\ 0 & 0 \end{bmatrix}$ in which

$$\begin{aligned}
\xi &= (a^*a+B^*B)^\dagger = t\mu^{-1}t^* + (v^*v)^\dagger, \\
v &= B(1-a^\dagger a),\ t = (1-v^\dagger B)a^\dagger, \\
\mu &= I + x^*x, \quad \text{and} \\
x &= (I-vv^\dagger)Ba^\dagger = Bt.
\end{aligned}$$

By combining these, and by using the equalities in (4), we arrive at

$$\mathscr{U}^\dagger\mathscr{U} = \begin{bmatrix} a^\dagger a + v^\dagger v & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\mathscr{Y} = \begin{bmatrix} 0 & 0 \\ 0 & d^\dagger \end{bmatrix}\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}\begin{bmatrix} 1-a^\dagger a - v^\dagger v & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ f & 0 \end{bmatrix},$$

where

$$f = d^\dagger b(1-a^\dagger a - v^\dagger v).$$

Likewise,

$$\begin{aligned}
\mathscr{S} &= \mathscr{A}^\dagger - \mathscr{A}^\dagger\mathscr{C}\mathscr{U}^\dagger \\
&= \begin{bmatrix} 0 & 0 \\ 0 & d^\dagger \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & d^\dagger \end{bmatrix}\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}\begin{bmatrix} \xi a^* & \xi B^* \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ -d^\dagger b\xi a^* & d^\dagger - d^\dagger b\xi B^* \end{bmatrix}.
\end{aligned}$$

We then compute

$$(I+\mathscr{Y}^*)(I+\mathscr{Y}\mathscr{Y}^*)^{-1} = \begin{bmatrix} 1 & f^*(1+ff^*)^{-1} \\ 0 & (1+ff^*)^{-1} \end{bmatrix},$$

followed by

$$(I+\mathscr{Y}^*)(I+\mathscr{Y}\mathscr{Y}^*)^{-1}\mathscr{S} = \begin{bmatrix} 1 & f^*(1+ff^*)^{-1} \\ 0 & (1+ff^*)^{-1} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -d^\dagger b\xi a^* & d^\dagger - d^\dagger b\xi B^* \end{bmatrix}.$$

This then gives, using equation (9),

$$\begin{bmatrix} a & 0 \\ b & d \end{bmatrix}^{\dagger} = (I+\mathscr{Y}^*)(I+\mathscr{Y}\mathscr{Y}^*)^{-1}\mathscr{S} + \mathscr{U}^{\dagger} = \begin{bmatrix} p & q \\ s & r \end{bmatrix}, \tag{10}$$

where

$$p = \xi a^* - (1+f^*f)^{-1} f^* d^{\dagger} b \xi a^* \tag{11}$$

$$s = -(1+ff^*)^{-1} d^{\dagger} b \xi a^* \tag{12}$$

$$q = \xi b^*(1-dd^{\dagger}) + (1+f^*f)^{-1} f^* d^{\dagger}[1 - b\xi b^*(1-dd^{\dagger})] \tag{13}$$

$$r = (1+ff^*)^{-1} d^{\dagger}[1 - b\xi^* b^*(1-dd^{\dagger})] \tag{14}$$

in which

$$\xi = [a^*a + b^*(1-dd^{\dagger})b]^{\dagger} = \xi^* = t(1+x^*x)^{-1}t^* + (v^*v)^{\dagger} \tag{15}$$

$$x = (1-vv^{\dagger})(1-dd^{\dagger})ba^{\dagger} \tag{16}$$

$$t = [1 - v^{\dagger}(1-dd^{\dagger})b]a^{\dagger} \tag{17}$$

$$f = d^{\dagger} b(1 - a^{\dagger}a - v^{\dagger}v) \text{ and} \tag{18}$$

$$v = (1-dd^{\dagger})b(1-a^{\dagger}a) \text{ (corner stone).} \tag{19}$$

We have presented an alternative expression to main theorem of [7] for the Moore-Penrose inverse of a 2×2 lower triangular matrix.

For later use, we observe that

(a) $va^* = 0,\ (v^*v)^{\dagger}a^* = 0$.

(b) $\xi a^* = t\mu^{-1}t^*a^*$,where $\mu = 1 + x^*x$.

(c) $t = [1 - v^{\dagger}(1-dd^{\dagger})b]a^{\dagger} = (1 - v^{\dagger}b)a^{\dagger} = a^{\dagger} - v^{\dagger}ba^{\dagger}$.

(d) $taa^{\dagger} = t$ and $at = aa^{\dagger} - av^{\dagger}ba^{\dagger} = aa^{\dagger} = (at)^*$

(e) $xaa^{\dagger} = x$ and so $aa^{\dagger}x^* = x^*$, and

(f) $\mu aa^{\dagger} = aa^{\dagger}\mu$ and $\mu^{-1}aa^{\dagger} = aa^{\dagger}\mu^{-1}$.

From the above,

$$\xi a^* = t\mu^{-1}(at)^* = t\mu^{-1}aa^{\dagger} = taa^{\dagger}\mu^{-1} = t\mu^{-1}.$$

The equality

$$\xi a^* = t\mu^{-1} \tag{20}$$

will be used later in this document.

3. The four "faces" of $M^\dagger$

We now examine the four cases where the block lower triangular matrix $M = \begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$ has a Moore-Penrose inverse of the form:

(i) $M^\dagger = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}$ the (1,2,0) case (unflipped),

(ii) $M^\dagger = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$ the (2,1,0) case (flipped),

(iii) $M^\dagger = \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}$ the (1,1,0) case,

(iv) $M^\dagger = \begin{bmatrix} * & * \\ * & 0 \end{bmatrix}$ the (2,2,0) case.

3.1. The (1,2,0) case (unflipped)

The Moore-Penrose inverse of the block lower triangular matrix $M = \begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$ is again of (1,2,0) type if and only if $b = dd^\dagger b = ba^\dagger a$ (see [12]).

We may also use the general triangular case (10) to rederive this consistency. Indeed this occurs precisely when

$$0 = q = \xi b^*(1 - dd^\dagger) + (1 + f^* f)^{-1} f^* d^\dagger [1 - b\xi b^*(1 - dd^\dagger)].$$

By post-multiplying by $dd^\dagger$ gives $(1 + f^* f)^{-1} f^* d^\dagger = 0$ which reduces to $df = 0$. By substituting this back into q, then shows that also $\xi b^*(1 - dd^\dagger) = 0$. Thus $M^\dagger$ has the desired lower triangular form if and only if

$$df = 0 \text{ and } \xi b^*(1 - dd^\dagger) = 0. \tag{21}$$

Now recall that if $B = (1 - dd^\dagger)b$ then $\xi = (a^* a + B^* B)^\dagger$. Hence the second consistency condition becomes $(a^* a + B^* B)^\dagger B^* = 0$, which is equivalent to $(a^* a + B^* B)B^* = 0$. This implies that $B(a^* a + B^* B)B^* = 0$ and hence by star-cancellation, $BB^* = 0$ and thus $B = 0$. This says that b = $dd^\dagger b$ and hence $v = 0$.

By substituting in $0 = df = dd^\dagger b[1 - a^\dagger a - v^\dagger v]$ then yields $0 = b(1 - a^\dagger a)$, and we recover the necessary condition $b = dd^\dagger b a^\dagger a$, which is also sufficient. We have proved

THEOREM 3.1. *Given* $M = \begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$*, the following conditions are equivalent:*

1. $M^\dagger$ *is of* $(1,2,0)$ *type.*
2. $b \in dRa$.
3. $b = dd^\dagger ba^\dagger a$.
4. $dd^\dagger b = b = ba^\dagger a$.

In this case, $M^\dagger = \begin{bmatrix} a^\dagger & 0 \\ -d^\dagger ba^\dagger & d^\dagger \end{bmatrix}$.

This can be extended to the $n \times n$ case (as in [6]).

3.2. The (2,1,0) case (flipped)

Next we examine the case here the Moore-Penrose inverse of the lower triangular matrix M "flips" and takes the form $M^\dagger = \begin{bmatrix} p & q \\ 0 & r \end{bmatrix}$ for some p, q, and r. We will give necessary and sufficient conditions for this to happen, in terms of the blocks a, b and d.

From (12) we see that a necessary and sufficient condition for $M^\dagger$ to have the flipped form $\begin{bmatrix} p & q \\ 0 & r \end{bmatrix}$ is that $d^\dagger b\xi a^* = 0$.

We now observe from Equation (20), that the consistency condition collapses to $0 = d^*bt = d^*b(1-v^\dagger b)a^\dagger$, which yields

$$d^*ba^* = d^*bv^\dagger ba^*, \tag{22}$$

or equivalently

$$dd^\dagger b(b^\dagger - v^\dagger)ba^\dagger a = 0.$$

We thus have

THEOREM 3.2. *Given* $M = \begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$*, then* $M^\dagger$ *is of* $(2,1,0)$ *type if and only if*

$$dd^\dagger b(b^\dagger - v^\dagger)ba^\dagger a = 0,$$

in which case

$$M^\dagger = \begin{bmatrix} \xi a^* & \xi b^*(1-dd^\dagger) + (1+f^*f)^{-1}f^*d^\dagger[1-b\xi b^*(1-dd^\dagger)] \\ 0 & (1+ff^*)^{-1}d^\dagger[1-b\xi^* b^*(1-dd^\dagger)] \end{bmatrix},$$

where ξ, f *are as above.*

If we set $e = a^\dagger a$ and $f = dd^\dagger$, then the consistency condition can be written as

$$\zeta = fbe - fb[(1-f)b(1-e)]^\dagger be = 0,$$

which is the (2,2) Schur complement in $\begin{bmatrix} fbe & be \\ fb & (1-f)b(1-e) \end{bmatrix}$. It only involves b, e and f. It is not clear how to simplify this condition. All we have is that $vv^\dagger = (1-dd^\dagger)bv^\dagger$.

3.3. The (2,2,0) case

From (10) we see that $M^\dagger$ is of (2,2,0) type if and only if $r = 0$, which is equivalent to

$$d^\dagger = d^\dagger b\xi^* b^*(1-dd^\dagger).$$

Right multiplication by $dd^\dagger$ shows that necessarily $d^\dagger = 0$, that is, $d = 0$. The sufficiency is clear. We may thus state the following result:

THEOREM 3.3. *Given* $M = \begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$, $M^\dagger$ *is of* $(2,2,0)$ *type if and only if* $d = 0$, *in which case* $\begin{bmatrix} a & 0 \\ b & d \end{bmatrix}^\dagger = \begin{bmatrix} (a^*a+b^*b)^\dagger a^* & (a^*a+b^*b)^\dagger b^* \\ 0 & 0 \end{bmatrix}$.

3.4. The (1,1,0) case

Lastly, we analyze the case where $M^\dagger$ is of (1,1,0) type. This corresponds to

$$p = \xi a^* - (1+f^*f)^{-1} f^* d^\dagger b\xi a^* = 0,$$

with $\xi = (a^*a + B^*B)^\dagger$, $B = (1-dd^\dagger)b$, $f = d^\dagger b(1-a^\dagger a - v^\dagger v)$ and $v = (1-dd^\dagger)b(1-a^\dagger a)$.

Now recall, from equation (20), that $\xi a^* = t\mu^{-1}$, where $\mu = 1 + x^*x = (1-vv^\dagger)Ba^\dagger$ and $t = a^\dagger - v^\dagger ba^\dagger$. Thus $p = 0$ is equivalent to $t\mu^{-1} = (1+f^*f)^{-1} f^* d^\dagger bt\mu^{-1}$, i.e. to

$$(1+f^*f)t = f^* d^\dagger bt. \tag{23}$$

Since $va^\dagger = 0 = av^\dagger$ we know that $fa^\dagger = 0 = fv^\dagger$ and consequently $ft = 0$.

The equality (23) now reduces to $t = f^* d^\dagger bt$. Lastly, left multiplication by $a^\dagger a$ shows that necessarily $a^\dagger = 0$, that is, $a = 0$. This is, trivially, sufficient for $p = 0$. We may thus conclude that

THEOREM 3.4. *Given* $M = \begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$, $M^\dagger$ *is of* $(1,1,0)$ *type if and only if* $a = 0$, *in which case* $\begin{bmatrix} 0 & 0 \\ b & d \end{bmatrix}^\dagger = \begin{bmatrix} 0 & b^*(bb^*+dd^*)^\dagger \\ 0 & d^*(bb^*+dd^*)^\dagger \end{bmatrix}$.

It is easily seen that these reduce to Cline's result.

4. Questions and remarks

1. The consistency condition for the Moore-Penrose inverse to flip involves the corner matrix $v = (1-dd^\dagger)b(1-a^\dagger a)$. Its Moore-Penrose inverse is a perturbation of $b^\dagger$.

2. Can we use the theory of Schur complements or partial orders, to simplify the consistency condition $fbe = fb[(1-f)b(1-e)]^\dagger be$?

3. No further simplification of the Condition (22) seems possible.

4. The unflipped case can be, inductively, generalized to the $n\times n$ case. What can be said for the flipped case for $n\times n$ matrices?

5. To ensure the symmetry of $R_{2\times 2}$ with R regular and symmetric, we may replace the square-root axiom on R by the condition SC_4.

6. $SC_2(R)$ does not imply $SC_n(R)$, nor implies the square root property, as remarked in Examples 2 and 3 in [13, page 215].

7. We have not used *any* of the other conditions that relate p, q, and r to a, b and d.

Acknowledgement

The authors wish to thank both referees for their valuable comments and corrections, and for pointing out the results presented in [7], [2] and [13].

REFERENCES

[1] A. BEN ISRAEL AND T. N. E. GREVILLE, *Generalized Inverses, Theory and Applications*, 2nd Edition, Springer, New York 2003.
[2] S. K. BERBERIAN, *Baer rings and Baer *-rings*, available `www.ma.utexas.edu/mp_arc/c/03/03-181.pdf`
[3] R. CLINE, *Representations for the generalized inverse of a partitioned matrix*, J. Soc. Indust. Appl. Math. **12** (1964), 588–600.
[4] R. CLINE, *Representations for the generalized inverse of sums of matrices*, J. SIAM Numer. Anal Ser B **2**, 1 (1965), 99–114.
[5] R. HARTWIG, *Block generalized inverses*, Arch. Rational Mech. Anal. **61**, 3 (1976), 197–251.
[6] R. HARTWIG AND P. PATRICIO, *G-inverses and path products*, in preparation.
[7] C.-H. HUNG AND T. L. MARKHAM, *The Moore-Penrose inverse of a partitioned matrix $M = \left(\begin{smallmatrix} A & 0 \\ B & C \end{smallmatrix}\right)$*, Czechoslovak Mathematical Journal **25 (100)**, 3 (1975), 354–361.
[8] I. KAPLANSKY, *Elementary divisors and modules*, Trans. Amer. Math. Soc. **66** (1949), 464–491.
[9] I. KAPLANSKY, *Any orthocomplemented complete modular lattice is a continuous geometry*, Ann. of Math. (3) **61** (1955), 524–541.
[10] I. KAPLANSKY, *Rings of operators*, W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[11] J. J. KOLIHA, D. DJORDJEVIĆ AND D. CVETKOVIĆ, *Moore-Penrose inverse in rings with involution*, Linear Algebra Appl. **426**, 2–3 (2007), 371–381.
[12] C. MEYER, *Generalized inverses of block triangular matrices*, SIAM J. Appl. Math. **19** (1970), 741–750.
[13] N. PRIJATELJ AND I. VIDAV, *On special *-regular rings*, Michigan Math. J. **18** (1971), 213–221.

(Received May 4, 2010)

R.E. Hartwig
Mathematics Department, N.C.S.U.
Raleigh, NC 27695-8205
U.S.A.
e-mail: hartwig@unity.ncsu.edu

P. Patrício
Departamento de Matemática e Aplicações
Universidade do Minho
4710-057 Braga
Portugal
e-mail: pedro@math.uminho.pt