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FREDHOLM STABILITY RESULTS FOR LINEAR COMBINATIONS OF m–POTENT OPERATORS

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Abstract. We investigate the stability of the nullity, defect and index of linear combinations $uA+vB$ of generalized quadratic operators, extending in several directions the recent results of Gau, Wang and Wong (*Oper. Matrices* **2** (2008), 193–199), and of commuting scalar m-potent operators. For the latter case we prove that the nullspaces of $uA+vB$ themselves are also stable.

1. Introduction

Stability referred to in the title means that the nullity, defect and index of a linear combination $uA+vB$ of two linear operators are independent of the choice of u and v, or more generally that $uA+vB$ belongs to a certain class of operators, under suitable restrictions on the operators and coefficients. This topic has recently received considerable attention in a number of papers in various settings, ranging from matrices and operators to elements of algebras. Most often considered classes of operators are idempotents [1, 4, 7, 9, 10, 11], while square-zero, nilpotent, k-potent and quadratic operators are studied in [6], and commuting algebraic operators in [2].

Our focus in the present paper is on two special classes of algebraic operators. We consider generalized quadratic operators with the aim of extending the results of Gau, Wang and Wong [6]. Then we discuss commuting scalar m-potent operators and derive a new stability result for the nullspaces of $aA+bB$, showing that for this class of algebraic operators it is possible to avoid the machinery of multidimensional spectral theory which is used by Chalendar, Fricain and Timotin [2].

X denotes a complex Banach space, $B(X)$ the set of all bounded linear operators on X. The identity operator is denoted by I. For $A\in B(X)$, $N(A)$ and $R(A)$ denote the nullspace and range of A, respectively, the nullity and defect of A are de-

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noted by $\mathrm{nul}A = \dim N(A)$ and $\mathrm{def}A = \dim(X/R(A))$, respectively. An operator is *upper semi-Fredholm* if $R(A)$ is closed and $\mathrm{nul}A < \infty$, and *lower semi-Fredholm* if $\mathrm{def}A < \infty$; the classes of upper and lower semi-Fredholm operators will be denoted by $F_+(X)$ and $F_-(X)$, respectively. An operator A is Fredholm if it belongs to the class $F(X) = F_+(X) \cap F_-(X)$, and its index is defined by $\mathrm{ind}A = \mathrm{nul}A - \mathrm{def}A$. The set of all invertible operators on X is denoted by $\mathrm{Inv}(X)$. An operator $A \in B(X)$ is *idempotent* if $A^2 = A$. We use the symbol $\mathbb{C}$ for the set of all complex numbers, and $\mathbb{Z}_+$ for the set of all positive integers.

To prove the stability results about $F_+(X)$, $F_-(X)$, $F(X)$ and $\mathrm{Inv}(X)$ for m-potent operators, we need two results summarized in the following proposition. The concept of the essential enlargement $\widetilde{X}$ of the Banach space X originated with Sadovskii [12].

PROPOSITION 1.1. (i) ([8, Theorem 6.9.2]) *For any Banach space X there exists the so-called* essential enlargement $(\widetilde{X}, \omega)$ of X *which consists of a Banach space $\widetilde{X}$ and a continuous algebra homomorphism $\omega \colon T \mapsto \widetilde{T}$ with the property that*

$$T \in F_+(X) \iff N(\widetilde{T}) = \{0\}. \tag{1.1}$$

(ii) ([8, Theorem 6.12.1]) *Let $A \in B(X)$ and let A' be its Banach space adjoint. Then*

$$A \in F_-(X) \iff A' \in F_+(X), \quad \mathrm{def}A = \mathrm{nul}A'.$$

Several authors [1, 10, 13] studied the linear combinations of two idempotent matrices A, B, and showed that the invertibility and the rank of $aA + bB$ are independent of the choice of a, b provided $(a, b) \in \Gamma$, where

$$\Gamma = \{(a,b) \in \mathbb{C} \times \mathbb{C} : ab(a+b) \neq 0\}. \tag{1.2}$$

A generalization to operators was given by Koliha and Rakočević in [10], later extended by Du, Deng, Mbehta and Müller in [4].

PROPOSITION 1.2. [4] *Let $A, B \in B(X)$ be idempotents. If $A + B$ is in one of the classes of injective, surjective, bounded below, Fredholm, upper Fredholm, lower Fredholm, left invertible, right invertible, left essentially invertible, right essentially invertible, inner invertible operators, then $aA + bB$ is in the same class independently of the choice of $(a, b) \in \Gamma$. The nullity, defect and index of $aA + bB$ are constant for all $(a, b) \in \Gamma$.*

We study a subclass of algebraic operators defined as follows.

DEFINITION 1.3. Let $m \in \mathbb{Z}_+$, $m \geqslant 2$. An operator $A \in B(X)$ is a *generalized scalar m-potent operator* if it belongs to the class $\Omega_m = \bigcup_{c,\lambda \in \mathbb{C}} \Omega_m(c, \lambda)$, where

$$\Omega_m(c,\lambda) = \{A \in B(X) : (A + cI)^m = \lambda (A + cI)\}. \tag{1.3}$$

$\Omega_2(0,1)$ is the set of all *idempotent operators*, $\Omega_m(0,0)$ the *nilpotent operators*, $\Omega_2(0,0)$ the *square-zero operators*, $\Omega_m(0,\lambda)$ the *scalar m-potent operators*. Further, Ω_2 coincides with the class of all *quadratic operators*, that is, operators with a quadratic annihilating polynomial: $A \in \Omega_m(c,\lambda)$ has annihilating polynomial $f(z) = (z+c)((z+c)^{m-1} - \lambda)$ which is reduced to $(z+c)(z+c-\lambda)$ in the case $m = 2$. It is easy to see that the annihilating polynomial of any quadratic operator can be written in this form.

Chalendar, Fricain and Timotin [2] complemented the results of [4] by studying the stability of algebraic operators. The main result of their paper is that this stability is not true in general. In fact, $p(z) = z^2 - z$ is essentially the only polynomial such that the spectral stability holds for all algebraic operators with the minimal polynomial p. The authors applied multidimensional spectral theory to extend Proposition 1.2 to commuting algebraic operators.

Our aim is to give an elementary proof of the Fredholm stability for scalar m-potent operators without the use of multidimensional spectral theory. We need to restrict the set of scalars admissible in the linear combinations $aA + bB$; for this purpose we follow [14] and define

$$\Gamma_m(\lambda,\mu) = \{(a,b) \in \mathbb{C} \times \mathbb{C} : ab(\lambda a^{m-1} + (-1)^m \mu b^{m-1}) \neq 0\}. \tag{1.4}$$

We note that $\Gamma = \Gamma_2(1,1)$. In Section 3 we will prove the Fredholm stability of $aA+bB$ when A,B are commuting scalar m-potent operators with $A^m = \lambda A$, $B^m = \mu B$, and $(a,b) \in \Gamma_m(\lambda,\mu)$. Unlike the case of quadratic operators, the case of m-potent operators with $m \geqslant 3$ cannot be reduced to the idempotent case, so instead we establish the equality of nullspaces of $uA + vB$ for different choices of $(u,v) \in \Gamma_m(\lambda,\mu)$ following elementary methods of [14].

2. Generalized quadratic operators

Quadratic operators are important in stochastic theory, differential equations, quantum mechanics and many other areas of mathematics and physics. This class of operators was recently generalized [3, 5]. We prove stability results for—possibly non-commuting—generalized quadratic operators.

DEFINITION 2.1. An operator $A \in B(X)$ is a generalized quadratic operator if it satisfies the equation

$$(A - \lambda P)(A - \mu P) = 0,$$

where λ,μ are complex scalars and $P \in B(X)$ is such that

$$P^2 = P, \quad AP = PA = A.$$

If $P = I$, this definition reduces to that of a quadratic operator. Gau, Wang and Wong [6] showed that for two quadratic operators $A,B \in B(X)$ sharing the same annihilating polynomial f, the linear combinations $aA + bB - 2\lambda I$, where λ is a zero of f, exhibit a stability with respect to being Fredholm independently of the choice of (a,b).

LEMMA 2.2. *Let $A \in B(X)$ be a generalized quadratic operator satisfying $(A-\lambda P)(A-\mu P)=0$ with $\lambda \neq \mu$. Then the operator $U=(A-\lambda P)/(\mu-\lambda)$ is idempotent, that is, $U^2=U$.*

Proof. Write $c=(\mu-\lambda)^{-1}$; then $U=c(A-\lambda P)$ and

$$\begin{aligned} U^2 &= c^2(A-\lambda P)[(A-\mu P)+(\mu-\lambda)P] = c^2(\mu-\lambda)(AP-\lambda P^2) \\ &= c(A-\lambda P) = U. \end{aligned}$$ □

In order to extend the spectral stability properties to generalized quadratic operators, we need to constrain the set of admissible coefficients in linear combinations of the operators. We observe that (a,b) belongs to $\Gamma_2(\lambda,\mu)$ if and only if $ab(a\lambda+b\mu)\neq 0$.

THEOREM 2.3. *Let $A,B \in B(X)$ be generalized quadratic operators satisfying $(A-\lambda P)(A-\mu P)=0$ and $(B-\rho Q)(B-\sigma Q)=0$ with $\lambda\neq\mu$ and $\rho\neq\sigma$. Then for any two pairs $(a,b),(u,v)\in\Gamma_2(\mu-\lambda,\sigma-\rho)$,*

$$aA+bB-(a\lambda P+b\rho Q)\in\Theta \iff uA+vB-(u\lambda P+v\rho Q)\in\Theta, \tag{2.1}$$

where Θ is one of the classes of injective, surjective, bounded below, Fredholm, upper Fredholm, lower Fredholm, left invertible, right invertible, left essentially invertible, right essentially invertible, inner invertible, operators. Further, the operators in (2.1) *have the same nullity, defect and index.*

Proof. Define $U=(\mu-\lambda)^{-1}(A-\lambda P)$ and $V=(\sigma-\rho)^{-1}(B-\rho Q)$. By Lemma 2.2, U and V are idempotent. Since (a,b), $(u,v)\in\Gamma_2(\mu-\lambda,\sigma-\rho)$, the pairs $(a(\mu-\lambda),b(\sigma-\rho))$ and $(u(\mu-\lambda),v(\sigma-\rho))$ belong to Γ and we can apply Proposition 1.2 to conclude that

$$a(\mu-\lambda)U+b(\sigma-\rho)V=a(A-\lambda P)+b(B-\rho Q)$$

belongs to the same class as

$$u(\mu-\lambda)U+v(\sigma-\rho)V=u(A-\lambda P)+v(B-\rho Q)$$

and that the nullity, defect and index are independent of a and b.

REMARK 2.4. Specialising $P=Q=I$ in the preceding theorem, we obtain a result for quadratic operators. In the case when the annihilating polynomials of the two operators in the preceding theorem are the same, we recover Theorem 2.3 in [6].

3. Scalar m-potent operators

Throughout this section, m is a positive integer, $m \geqslant 2$. Gau, Wang and Wong [6] gave examples of m-potent matrices A,B for which the invertibility of $aA+bB$ depends on the choice of $(a,b) \in \Gamma$; this means that the coefficients have to be further restricted. They also gave examples of matrices where the stability fails even after a further restriction of coefficients, because the matrices do not commute. Consequently, for a general result we consider only commuting operators.

For the class of scalar m-potent operators we give a direct proof of the Fredholm stability without resorting to multidimensional spectral theory used in [2] by first proving a stronger result that the nullspaces of $aA+bB$ are stable independently of the choice of coefficients.

THEOREM 3.1. *Let $m \in \mathbb{Z}_+$, $m \geqslant 2$, and let $A,B \in B(X)$ be commuting scalar m-potent operators satisfying $A^m = \lambda A$ and $B^m = \mu B$, respectively. Then*

$$N(aA+bB) = N(A) \cap N(B) \tag{3.1}$$

for any $(a,b) \in \Gamma_m(\lambda,\mu)$.

Proof. The inclusion $N(aA+bB) \supset N(A) \cap N(B)$ is clear. Conversely assume that $(aA+bB)x = 0$, $x \neq 0$. Then $aAx = -bBx$, and $a^m A^m x = (-1)^m b^m B^m x$ by commutativity of A,B. So

$$a^m \lambda Ax = (-1)^m b^m \mu Bx = -(-1)^m b^{m-1} \mu(-bBx) = -(-1)^m b^{m-1} \mu(aAx),$$

and $(\lambda a^{m-1} + (-1)^m \mu b^{m-1})Ax = 0$. Since $(a,b) \in \Gamma_m(\lambda,\mu)$, we conclude that $Ax = 0$. Also $Bx = -ab^{-1}Ax = 0$. Thus $N(A) \cap N(B) \supset N(aA+bB)$, and (3.1) is proved.

THEOREM 3.2. *Let the assumptions of Theorem* 3.1 *hold and let, for some $(u,v) \in \Gamma_m(\lambda,\mu)$, $uA+vB \in \Theta$ where $\Theta \in \{F_+(X), F_-(X), F(X), \mathrm{Inv}(X)\}$. Then, for any $(a,b) \in \Gamma_m(\lambda,\mu)$, $aA+bB \in \Theta$, and the nullity, defect, and index, respectively, of the two linear combinations are equal.*

Proof. Let $\widetilde{X}$ be the essential enlargement of X as outlined in Proposition 1.1 (i), and let $\widetilde{T} \in B(\widetilde{X})$ be the image of $T \in B(X)$ under the continuous algebra homomorphism ω. We can apply (3.1) to operators $\widetilde{A},\widetilde{B}$ in place of A,B since $A^m = \lambda A$ implies $\widetilde{A}^m = \lambda\widetilde{A}$; similarly $\widetilde{B}^m = \lambda\widetilde{B}$. According to (3.1) and Proposition 1.1 (i),

$$aA+bB \in F_+(X) \iff uA+vB \in F_+(X).$$

The statement about $F_-(X)$ follows from Proposition 1.1 (ii).

Specializing the preceding theorem to matrices and taking $\Theta = \mathrm{Inv}(X)$, we obtain the main result of [14].

COROLLARY 3.3. *Let the assumptions of Theorem* 3.1 *hold, let, in addition,* $\lambda\mu \neq 0$*, and let* γ, δ *be any solution to*

$$\gamma^{m-1} = \lambda^{-1}, \qquad \delta^{m-1} = (-1)^m \mu^{-1}.$$

If $\Theta \in \{F_+(X), F_-(X), F(X), \mathrm{Inv}(X)\}$ *and* $\gamma A + \delta B \in \Theta$*, then* $uA + vB \in \Theta$ *for all* $(u,v) \in \Gamma_m(\lambda,\mu)$*, and* $\theta(uA+vB) = \theta(\gamma A + \delta B)$*, where* θ *is the nullity, defect or index, as appropriate.*

Proof. This follows from Theorem 3.2 when we observe that (γ, δ) belongs to $\Gamma_m(\lambda, \mu)$:

$$\lambda\gamma^{m-1} + (-1)^m \mu\delta^{m-1} = \lambda\lambda^{-1} + (-1)^{2m}\mu\mu^{-1} = 2 \neq 0. \qquad \square$$

The preceding stability results extend in an obvious way to generalized scalar operators.

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