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M–IDEALS AND THE BISHOP–PHELPS THEOREM

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Abstract. We give new proofs for the known important approximative properties of M-ideals, using only the definition of an M-ideal and the Bishop-Phelps theorem. Unlike the known proofs, these proofs do not use the 3-ball intersection property of M-ideals.

1. Introduction

In [8], a new proof of proximinality for M-ideals was given, using only the definition of an M-ideal and the Bishop- Phelps theorem. We refer the reader to [6] and observe that the known proofs of proximinality and other approximative properties of M-ideals use either the 3 or the 2-ball intersection property of the M-ideals. In this paper, we further pursue the simple technique used in [8] and derive some of the well known approximative properties of M-ideals, using mainly the definition of an M-ideal and the Bishop- Phelps theorem.

We now describe the notation we use in this paper. Throughout X denotes a real Banach space and X^*, the dual of X. If x is in X and $r>0$, the open (closed) ball with x as center and r as radius is denoted by $B(x,r)$ $(B[x,r])$. The unit sphere $\{x \in X : \|x\| = 1\}$, of X is denoted by S_X. The class of all norm attaining functionals on X is denoted by $NA(X)$ and $NA_1(X)$ denotes the set $NA(X) \cap S_X$. If $A \subseteq X$ then $sp(A)$ denotes the span of A, $\bar{A}$ the norm closure of A and $bd(A)$, the boundary of A. Throughout this paper, we identify an element of a Banach space X with its image under the canonical embedding of X into X^{**}.

For any x in X and a subspace Y of X, we denote by $d(x,Y)$, the distance of x from Y. Let

$$P_Y(x) = \{y \in Y : \|x-y\| = d(x,Y)\}.$$

If $P_Y(x)$ is a non-empty set for each x in X, then the subspace Y is said to be proximinal in X. For a subspace Y of X, let $Y^\perp$ denote the annihilator of Y in X, that is

$$Y^\perp = \{f \in X^* : f \equiv 0 \text{ on } Y\}.$$

For f in X^*, the restriction of f to Y is denoted by $f_{|Y}$. The following definitions can be found in [6].

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DEFINITION 1.1. [6] A subspace Y of a Banach space X is called an L-summand of X if there exists a subspace Z of X such that $X = Y \oplus_1 Z$, the l_1-direct sum of Y and Z. This implies that if $x = y + z$ for x in X, y in Y and z in Z, then $\|x\| = \|y\| + \|z\|$.

DEFINITION 1.2. [6] A subspace Y of a Banach space X is called an M-ideal in X if $Y^{\perp}$ is an L-summand in X^*.

If Y is an M-ideal in X then there exists a subspace Z of X^* such that

$$X^* = Y^{\perp} \oplus_1 Z. \tag{1.1}$$

Using the canonical isometric isomorphism $Y^* \simeq X^*/Y^{\perp} \simeq Z$, one can conclude (see Proposition 1.12, [6]) that

$$Z = \{f \in X^* : \|f\| = \|f_{|Y}\|\}. \tag{1.2}$$

Throughout this paper we use the following notation. If X is a Banach space, Y an M-ideal in X then Z would be given by (1.2).

Finally, If A and B are bounded, nonempty subsets of a Banach space, we denote by $d_H(A,B)$ the Hausdorff metric distance between A and B, given by

$$d_H(A,B) = \max\{\sup_{a \in A} d(a,B), \sup_{b \in B} d(b,A)\}.$$

2. The new proofs

In this section, we give new proofs for the following known results. These results, from various sources (See [1], [4], [6], [7], [9] and [11]), are all given in [6] (See Propositions II.1.3, II.1.8 and Corollaries II.1.5 and II.1.7 of [6]).

Let X be a Banach space and Y be an M-ideal in X. If x is in X with $d(x,Y) = 1$ then
i) The set $P_Y(x) - P_Y(x)$ contains the open ball $B_Y(0,2)$
ii) The weak * closure of $P_Y(x)$ in $Y^{\perp\perp}$ is a closed ball of radius one
and
iii) if z_i is in X for $i = 1,2$, then

$$d_H(P_Y(z_1), P_Y(z_2)) \leqslant 2\|z_1 - z_2\|.$$

We begin with a Fact, that is essentially needed in the proofs that follow. We first have a remark, recalling an observation from [8], that is needed here.

REMARK 2.1. Let Y be an M-ideal and a hyperplane in a Banach space X. Let $x \in X \backslash Y$. If $f \in S_{X^*}$ and $y \in Y$ satisfy

$$\|f\| > \|f_{|Y}\| \text{ and } f(x+y) = \pm\|x+y\|,$$

then it follows from the proof of the Proposition 1 of [8], that $-y$ is in $P_Y(x)$.

FACT 2.1. Let X be a Banach space and Y be an M-ideal and a hyperplane in X. Assume x is in X and $d(x,Y)=1$. Then given f in S_Z and $\varepsilon>0$, there exist f_i in $NA_1(X)$ and $-y_i$ in $P_Y(x)$ for $i=1,2$ such that

$$\|f-f_i\|<\varepsilon,\ \|f_i\|>\|f_{i|Y}\| \text{ for } i=1,2 \tag{2.1}$$

and

$$f_i(x+y_i)=(-1)^{i+1}\|x+y_i\| \text{ for } i=1,2. \tag{2.2}$$

Proof. Assume w.l.o.g $0<\varepsilon<2$. Since Y is a hyperplane in X, $X=Y\oplus sp(x)$. We have $\|f\|=1=\|f_{|Y}\|$. Let g in $Y^{\perp}$ satisfy

$$1=\|g\|=g(x)=d(x,Y)$$

and set

$$h_i=(-1)^{i+1}\frac{\varepsilon g}{2}+\left(1-\frac{\varepsilon}{2}\right)f, \text{ for } i=1,2.$$

Then $\|h_i\|=1$ and $\|h_{i|Y}\|=1-\frac{\varepsilon}{2}$ for $i=1,2$. If h_i is in $NA(X)$, take $f_i=h_i$. Otherwise, using the Bishop Phelps theorem get f_i in $NA_1(X)$ such that $\|h_i-f_i\|<\frac{\varepsilon}{4}$. Then $\|f_{i|Y}\|<\|f_i\|=1$, for $i=1,2$. In fact,

$$\|f_{i|Y}\|\leqslant\|h_{i|Y}\|+\frac{\varepsilon}{4}=1-\frac{\varepsilon}{2}+\frac{\varepsilon}{4}=1-\frac{\varepsilon}{4}.$$

Further

$$\|f-f_i\|\leqslant\|f-h_i\|+\|h_i-f_i\|\leqslant\frac{\varepsilon}{2}+\frac{\varepsilon}{4}<\varepsilon.$$

This proves (2.1).

Now, f_i attains its norm on X at some $\alpha_i x+z_i$, where α_i is a non-zero scalar and z_i is in Y, for $i=1,2$. Hence

$$\alpha_i f_i\left(x+\frac{z_i}{\alpha_i}\right)=f_i(\alpha_i x+z_i)=\|\alpha_i x+z_i\|=|\alpha_i|\left\|x+\frac{z_i}{\alpha_i}\right\|$$

for $i=1,2$. Taking $y_i=\frac{z_i}{\alpha_i}$ for $i=1,2$, we get

$$f_i(x+y_i)=(sgn\ \alpha_i)\|x+y_i\|, \text{ for } i=1,2$$

where $sgn\ \alpha_i$ is the sign of α_i. Now by Remark 2.1, $-y_i$ is in $P_Y(x)$ and so $\|x+y_i\|=1=d(x,Y)$ for $i=1,2$.

We now claim that $\alpha_1\alpha_2<0$. Suppose not. We assume $\alpha_i>0$ for $i=1,2$, the proof being similar for the case $\alpha_i<0$ for $i=1,2$,. Since $\alpha_i>0$ for $i=1,2$, we have

$$h_i(x+y_i)>\|x+y_i\|-\frac{\varepsilon}{4}=1-\frac{\varepsilon}{4}$$

for $i=1,2$. Hence

$$1-\frac{\varepsilon}{4}\leqslant h_i(x+y_i)=(-1)^{i+1}\frac{\varepsilon g(x+y_i)}{2}+\left(1-\frac{\varepsilon}{2}\right)f(x+y_i) \text{ for } i=1,2.$$

Since $\|f\| = 1 = \|x + y_i\|$ for $i = 1,2$ and g is in $Y^{\perp}$ with $g(x) = 1$, this implies

$$1 - \frac{\varepsilon}{4} \leqslant (-1)^{i+1}\frac{\varepsilon}{2} + 1 - \frac{\varepsilon}{2}$$

for $i = 1,2$. Thus

$$(-1)^{i+1}\frac{\varepsilon}{2} \geqslant \frac{\varepsilon}{4}, \text{ for } i = 1,2$$

which is a contradiction. This proves our claim that $\alpha_1\alpha_2 < 0$ and completes the proof of the Fact. □

The notion of a psuedoball as given in [2] is as follows (See Definition 1.2 of [6]).

DEFINITION 2.1. A closed, convex bounded subset B of a Banach space X is called a pseudoball of radius r if its diameter $2r > 0$ and if for each finite collection $x_1, x_2, \cdots, x_n$ of points with $\|x_i\| < r$ there is x in B such that

$$x + x_i \in B, \text{ for } i = 1,2,\cdots,n.$$

It was shown in [2] that B is a pseudoball of radius r in a Banach space X if and only if the weak* closure of B in X^{**} is a ball of radius r. We refer the reader to Theorem II.1.6 of [6] for the details. The proof in [6], of this implication is simple and uses only a separation theorem, due to Tukey. We are now in a position to prove the results mentioned earlier. In the rest of the paper, if A is a subset of Y, we denote the weak* closure of A in $Y^{\perp\perp}$ by $\overline{A}^{w*}$.

THEOREM 2.1. *Let X be a Banach space and Y be an M-ideal in X. Assume x is in X and $d(x,Y) = 1$. Then*

1. *The open ball $B_Y(0,2)$ is contained in the set $P_Y(x) - P_Y(x)$.*
2. *The weak* closure of the set $P_Y(x)$ in $Y^{\perp\perp}$, is a ball of radius one.*
3. *The set $P_Y(x)$ is a psuedoball of radius one in Y.*

Proof. In view of the remark just above the theorem, we prove only first two statements.

Using Remark 2 of [8], we assume without loss of generality that $X = Y \oplus sp(x)$. We first show that for any f in S_Z,

$$\sup\{f(w) : w \in [P_Y(x) - P_Y(x)]\} = 2. \tag{2.3}$$

Select any f in S_Z and $0 < \varepsilon < 1$. Obtain f_i in $NA_1(X)$ and y_i in $P_Y(x)$, $i = 1,2$, as in the above Fact. Then

$$f(x - y_1) \geqslant 1 - \varepsilon \text{ and } f(x - y_2) < -1 + \varepsilon.$$

Hence

$$f(y_2-y_1)=f(x-y_1)-f(x-y_2)>2-2\varepsilon.$$

Since $\varepsilon>0$ was chosen arbitrarily, this implies (2.3).

Select any y in Y, with $\|y\|<2$ and choose $\varepsilon>0$ such that $\|y\|+\varepsilon<2$. Let $A=P_Y(x)-P_Y(x)$. Then A is a symmetric, convex subset of Y and we claim that y is in A. Suppose not. We discuss two cases. First we assume that y is in $\bar{A}\backslash A$. Then y is in $bd(A)$ and by using the Bishop- Phelps support point theorem [3] (See Theorem 2.11.9 in [10]), we can get z in $bd(A)$, such that z is a support point to A and $\|y-z\|<\varepsilon$. If f in S_Z is the support functional to A at z, then

$$\sup f(A)=\sup\{f(w): w\in A\}\leqslant f(z)<f(y)+\varepsilon\leqslant\|y\|+\varepsilon<2.$$

But this contradicts (2.3).

We now assume that y is not in $\bar{A}$. We again arrive at a contradiction using the separation theorem and similar steps as above. This completes the proof of 1.

To prove 2, we first recall that $Y^{\perp\perp}\cong Y^{**}$. Define ϕ in X^{**} by

$$\phi(f)=h(x),\ \text{for all } f\in X^*,$$

where $f=g+h$, with g in $Y^{\perp}$ and h in Z. Clearly, ϕ is in $Y^{\perp\perp}$ and

$$\phi(f)=f(x),\ \text{for all } f\in Z. \tag{2.4}$$

We will now show that the weak*closure of the set $P_Y(x)$ in $Y^{\perp\perp}$ is the closed ball $B_{Y^{\perp\perp}}[\phi,1]$.

We first show that the weak*closure of the set $P_Y(x)$ in $Y^{\perp\perp}$ is contained in the closed ball $B_{Y^{\perp\perp}}[\phi,1]$. Let $\psi\in\overline{P_Y(x)}^{w^*}$. Then for any $f\in Z\simeq Y^*$ with norm 1, we have $f(\phi-\psi)=f(x-\psi)$. Given $\varepsilon>0$, pick $y\in P_Y(x)$ such that $|f(\psi-y)|<\varepsilon$. Then

$$|f(x-\psi)|=|f(x-y)|+|f(y-\psi)|<1+\varepsilon,$$

as $\|x-y\|=d(x,Y)=1$. So

$$|f(\phi-\psi)|=|f(x-\psi)|<1+\varepsilon$$

for every $\varepsilon>0$ and hence

$$|f(\phi-\psi)|\leqslant 1 \text{ for any } f\in Z.$$

This implies $\|\phi-\psi\|\leqslant 1$ and $\psi\in B_{Y^{\perp\perp}}[\phi,1]$.

We now claim $\overline{P_Y(x)}^{w^*}=B_{Y^{\perp\perp}}[\phi,1]$. Suppose not. Then by the Separation theorem, there is ψ in $Y^{\perp\perp}$ with $\|\phi-\psi\|\leqslant 1$ and ψ is not in the weak* closure of $P_Y(x)$ in $Y^{\perp\perp}$. This with (2.4) implies that there exists h in S_Z and $\varepsilon>0$ satisfying

$$0<\varepsilon<(\phi-\psi)(h)-\sup\{h(x-y): y\in P_Y(x)\}. \tag{2.5}$$

Now by Fact 2.1, we can get f in $NA_1(X)$ and y_1 in $-(P_Y(x))$ such that

$$\|h-f\|<\varepsilon \text{ and } f(x+y_1)=\|x+y_1\|=d(x,Y)=1.$$

Hence

$$h(x+y_1)>1-\varepsilon$$

while

$$(\phi-\psi)(h)\leqslant 1, \text{ as } \|\phi-\psi\|\leqslant 1 \text{ and } \|h\|=1.$$

This contradicts (2.5) and completes the proof of 2. □

We need the following result, which must be known, in proving the Lipschitz continuity of the metric projection onto an M-ideal.

PROPOSITION 2.2. *Let C and D be non-empty, bounded , closed convex sets in a Banach space. Then*

$$d_H(C,D)=d_H(\bar{C}^{w^*},\bar{D}^{w^*}).$$

Proof. We only need to prove the two facts

$$d(x,C)=d(x,\bar{C}^{w^*}), \text{ for any } x\in X \tag{2.6}$$

and

$$\sup_{x\in D} d(x,\bar{C}^{w^*})=\sup_{x^{**}\in\bar{D}^{w^*}} d(x^{**},\bar{C}^{w^*}). \tag{2.7}$$

If

$$d(x,C)>r>d(x,\bar{C}^{w^*}) \tag{2.8}$$

then

$$B_{X^{**}}[x,r]\cap\bar{C}^{w^*}\neq\emptyset.$$

As $B_{X^{**}}[x,r]=\overline{B_X[x,r]}^{w^*}$, we have

$$\overline{B_X[x,r]}^{w^*}\cap\bar{C}^{w^*}\neq\emptyset$$

and so 0 is in the weak closure in X of the convex set $B_X[x,r]-C$ and thus by Mazur's theorem 0 is in the norm closure of $B_X[x,r]-C$. Hence $\inf_{y\in C}\|x-y\|=r$ which contradicts (2.8). So (2.6) holds.

To see the second assertion note that for any $r>0$, the set $\bar{C}^{w^*}+rB_{X^{**}}$ is weak* closed. Thus $D\subseteq\bar{C}^{w^*}+rB_{X^{**}}$ implies $\bar{D}^{w^*}\subseteq\bar{C}^{w^*}+rB_{X^{**}}$. It is now clear that 2.7 holds. □

We also need a simple observation and a definition.

DEFINITION 2.2. Let X be a Banach space and X_i be closed subspaces of X for $i=1,2$ such that $X=X_1\oplus X_2$. The bounded linear projection Q from X onto X_1 along X_2, is called an M-projection if for any x in X,

$$\|x\| = \max\{\|Q(x)\|, \|x-Q(x)\|\}.$$

The following remark is easy to verify.

REMARK 2.2. Let X be a Banach space and X_i be closed subspaces of X for $i=1,2$ such that $X=X_1\oplus X_2$. Assume that the projection Q from X onto X_1 is an M- projection. Then X_1 is proximinal and for any x in X,

$$P_{X_1}(x) = B_{X_1}[Qx, d_x]$$

where $d_x = d(x, X_1)$.

PROPOSITION 2.3. *Let Y be an M-ideal in a Banach space X. Then for any x_i in X for $i=1,2$,*

$$d_H(P_Y(x_1), P_Y(x_2)) \leqslant 2\|x_1-x_2\|.$$

Proof. We have

$$X^* = Y^\perp \oplus_1 Z,$$

since Y is an M-ideal in X and so the bounded linear projection Q from X^{**} onto $Y^{\perp\perp}$ along $Z^\perp$, is an M-projection. Using 2 of Theorem 2.1 and the above Remark (See also Corollary 1.7 in [6]), we see that

$$P_{Y^{\perp\perp}}(x) = \overline{P_Y(x)}^{w*} = B_{Y^{\perp\perp}}[Qx, d_x],$$

where $d_x = d(x,Y)$.

Let $d_i = d(x_i, Y)$ for $i=1,2$. We have by Proposition 2.2,

$$\begin{aligned} d_H(P_Y(x_1), P_Y(x_2)) &= d_H(\overline{P_Y(x_1)}^{w^*}, \overline{P_Y(x_2)}^{w^*}) \\ &= d_H(P_{Y^{\perp\perp}}(x_1), P_{Y^{\perp\perp}}(x_2)) \\ &= d_H(B_{Y^{\perp\perp}}[Qx_1, d_1], B_{Y^{\perp\perp}}[Qx_2, d_2]) \\ &\leqslant \|Qx_1 - Qx_2\| + |d_1 - d_2| \\ &\leqslant \|x_1 - x_2\|, \end{aligned}$$

and this completes the proof for the Lipschitz continuity. □

REMARK 2.3. It follows from Remark 1 of [8] and the nature of the results of Theorem 2.1 and Proposition 2.3 that, it suffices to consider real Banach spaces for our purpose and that these results hold for the complex case also.

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