SPACES OF p–INTEGRABLE FUNCTIONS WITH RESPECT
TO A VECTOR MEASURE DEFINED ON A δ–RING

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Abstract. In this paper we study the lattice properties of the Banach lattices $L^p(v)$ and $L^p_w(v)$ of p-integrable real-valued functions and weakly p-integrable real-valued functions with respect to a vector measure v defined on a δ-ring. The relation between these two spaces, the study of the continuity and some kind of compactness properties of certain multiplication operators between different spaces L^p and/or L^q_w and the representation theorems of general Banach lattices via these spaces play a fundamental role.

1. Introduction

Integration with respect to vector measures defined on δ-rings is the natural vector valued generalization of the case of integration with respect to positive σ-finite measures. In terms of the corresponding spaces of integrable functions, this consideration is also up to a point true. The spaces $L^1(v)$ of integrable functions and $L^1_w(v)$ of weakly integrable functions with respect to a vector measure v are broad classes of Banach lattices of measurable functions, and in fact represent a large family of Banach lattices. Regarding these representations, nowadays it is well-known that order continuous Banach lattices can also be written (isometrically and in order) as an $L^1(v)$-space of a certain vector measure v on a δ-ring, and a similar result holds for Banach lattices with the Fatou property and some additional requirement with the spaces $L^1_w(v)$ (see [11, Theorem 5 and Theorem 10]; see also [4, pp. 22–23]).

The case of finite positive scalar measures is generalized using vector measures on σ-algebras. Such measures provide spaces of integrable functions that can be used for representing any order continuous Banach lattice with a weak unit ($L^1(v)$) or any Banach lattice having the Fatou property with a weak unit belonging to the order continuous part of the space ($L^1_w(v)$) (see [5, Theorem 8] and [6, Theorem 2.5]). The corresponding representation theorems for p-convex Banach lattices having these additional lattice properties are also known; the spaces $L^p(v)$ and $L^p_w(v)$ are involved in this case (see [7, 20]). Although all the relevant (geometric, lattice, topological) properties of the spaces $L^p(v)$ of a vector measure v on a σ-algebra with $1 \leqslant p < \infty$ has been already studied (see [12, 21]), this is not the case for the δ-ring case. The difference

is important, since the case of vector measures on a σ-algebra only covers the cases of Banach function spaces (i.e. the reference measure space is σ-finite); for instance, Banach lattices as $c_0(I)$ or $\ell^\infty(I)$ for an uncountable set of indexes I can be written as spaces of p-integrable functions and weakly integrable functions with respect to a vector measure, respectively. In fact, these spaces represent (in the case of δ-rings) a big class of Banach lattices, as will be shown in this paper.

The aim of this work is to study the main properties of the spaces $L^p(\nu)$ and $L^p_w(\nu)$ of a vector measure ν on a δ-ring, the natural sets of multiplication operators and the inclusion relations with the spaces $L^\infty(\nu)$ and $L^1(\nu)$. After the preliminary Section 2, Section 3 is devoted to the study of the main Banach lattice properties of the spaces $L^p(\nu)$ and $L^p_w(\nu)$. The general case $0 < p < \infty$ is considered, although for $0 < p < 1$ these spaces are not necessarily Banach spaces; just consider the case when the vector measure is a scalar measure. However, completeness is proved also for this case but under a quasinorm. A general representation theorem for p-convex order continuous Banach lattices with $1 < p < \infty$ as $L^p(\nu)$ spaces is also given in Theorem 10 (the case $p = 1$ is already know, see [11, Theorem 5] and [4, pp. 22–23]). In Section 4 the spaces of multiplication operators between spaces of p-integrable functions and spaces of integrable functions with respect to the same vector measure are computed, and compactness type properties of these operators are studied, generalizing in this way what is known in the case of σ-algebras (see [8]). Finally, Section 5 is devoted to the analysis of the spaces $L^p(\nu)$ and $L^p_w(\nu)$ as intermediate spaces of $L^\infty(\nu) \cap L^1(\nu)$ and $L^\infty(\nu) + L^1(\nu)$, providing the vector measure version of the classical inclusions that hold for the Lebesgue spaces $L^p[0, \infty]$.

2. Preliminaries

2.1. Banach lattices

Let E be a Banach lattice with norm $\|\cdot\|$ and order $\leqslant$. Let B_E denotes the unit ball in E. A subspace F of E is an *ideal* of E if $y \in F$ whenever $y \in E$ with $|y| \leqslant |x|$ for some $x \in F$. An ideal F in E is said to be *order dense* in E if for every $0 \leqslant x \in E$ there exists an upwards directed system $0 \leqslant x_\tau \uparrow x$ such that $(x_\tau)_\tau \subset F$ and *super order dense* if this is the case by means of increasing sequences. An upwards directed system $(x_\tau)_\tau$ in E is said to be a *Cauchy system* if for any $\varepsilon > 0$ there exists τ_0 in $\{\tau\}$ such that $\|x_{\tau_1} - x_{\tau_2}\| < \varepsilon$ for all $x_{\tau_1} \geqslant x_{\tau_0}$ and $x_{\tau_2} \geqslant x_{\tau_0}$. A *weak unit* of E is an element $0 \leqslant e \in E$ such that $x \wedge e = 0$ implies $x = 0$.

The Banach lattice E is *order continuous* if for every downwards directed system in E, $(x_\tau)_\tau \downarrow 0$ it follows that $\|x_\tau\| \downarrow 0$. If $\|x_n\| \downarrow 0$ for any decreasing sequence $x_n \downarrow 0$ in E, then E is said to be σ-*order continuous*.

The Banach lattice E is said to be *Dedekind σ-complete* if every order bounded sequence has a supremum. We will say that E has the *weak Fatou property* if for every upwards directed system $0 \leqslant x_\tau \uparrow$ in E such that $\sup_\tau \|x_\tau\| < \infty$ it follows that there exists $x = \sup_\tau x_\tau$ in E. If moreover $\|x\| = \sup_\tau \|x_\tau\|$ then E will be said to has the *Fatou property*. We will say that E has the *weak σ-Fatou property* if for every increasing sequence $0 \leqslant x_n \uparrow$ in E such that $\sup_{n \geqslant 1} \|x_n\| < \infty$ there exists $x = \sup_{n \geqslant 1} x_n$

in E. If moreover $\|x\| = \sup_{n \geqslant 1} \|x_n\|$ we will say that E has the σ-*Fatou property*.

A Banach lattice E is said to be p-*convex* if there exists a constant $M > 0$ such that

$$\left\| \left(\sum_{j=1}^{n} |x_j|^p \right)^{\frac{1}{p}} \right\| \leqslant M \left(\sum_{j=1}^{n} \|x_j\|^p \right)^{\frac{1}{p}}$$

for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$. The smallest constant satisfying the previous inequality for all such $n \in \mathbb{N}$ and x_j's ($j = 1, \ldots, n$) is called the p-*convexity constant* of E and is denoted by $\mathbf{M}^{(p)}(E)$.

The definitions given above and the main results concerning Banach lattices that we use in this paper apply for the quasi-Banach lattice case, although they are not in general Banach spaces.

An operator $T \colon E \to F$ between Banach lattices is said to be an *order isomorphism* if it is one to one, onto and satisfies that $T(x \wedge y) = Tx \wedge Ty$ for all $x, y \in E$. In this case we will say that E and F are *order isomorphic*. If moreover $\|Tx\|_F = \|x\|_E$ for all $x \in E$, we will say that T is an *order isometry* and that E and F are *order isometric*. Every positive operator between Banach lattices is continuous (see [1, Theorem 12.3] or [15, page 2]).

The set consisting of all bounded linear maps from E into F will be denoted by $\mathscr{B}(E, F)$. A bounded linear operator $T \colon E \to F$ between Banach lattices is called $\mathscr{L}$-*weakly compact* if $T(B_E)$ is an $\mathscr{L}$-weakly compact subset of F, that is, if $\|x_n\| \to 0$ as $n \to \infty$ for every disjoint sequence $(x_n)_n$ contained in the solid hull of $T(B_E)$. We denote by $\mathscr{L}(E, F)$ this class of bounded operators and by $\mathscr{W}(E, F)$ the ideal of weakly compact operators. Note that $\mathscr{L}(E, F) \subset \mathscr{W}(E, F)$ by Proposition 3.6.12 in [19]. For these an other issues related to Banach lattices, see for instance [1], [15], [16], [19] and [24].

2.2. Integration with respect to vector measures on δ-rings

We recall here the integration theory of Lewis ([14]) and Masani and Niemi ([17], [18]). We refer also to [10]. Let $\mathscr{R}$ be a δ-*ring* of subsets of an abstract set Ω (i.e. a ring of sets closed under countable intersections) and consider $\mathscr{R}^{loc}$ the associated σ-algebra to $\mathscr{R}$ given by $\mathscr{R}^{loc} = \{ A \subset \Omega : A \cap B \in \mathscr{R}, \text{ for every } B \in \mathscr{R} \}$. Denote by $\mathscr{M}(\mathscr{R}^{loc})$ the space of measurable real functions on $(\Omega, \mathscr{R}^{loc})$ and by $\mathscr{S}(\mathscr{R}^{loc})$ and $\mathscr{S}(\mathscr{R})$ the space of simple functions with support in $\mathscr{R}^{loc}$ and $\mathscr{R}$ respectively.

Let $v \colon \mathscr{R} \to X$ be a set function with values in a real Banach space X such that $\sum_{n \geqslant 1} v(A_n)$ converges to $v(\cup_{n \geqslant 1} A_n)$ in X whenever $(A_n)_{n \geqslant 1}$ are pairwise disjoint sets in $\mathscr{R}$ with $\cup_{n \geqslant 1} A_n \in \mathscr{R}$. We will say that v is a *vector measure*. Denoting by X^* the dual space of X, the *semivariation* of v is given by $\|v\| \colon \mathscr{R}^{loc} \to [0, \infty]$ with $\|v\|(A) = \sup\{ |x^*v|(A) : x^* \in B_{X^*} \}$ for all $A \in \mathscr{R}^{loc}$ and where $|x^*v|$ is the variation of the measure $x^*v \colon \mathscr{R} \to \mathbb{R}$. A set $B \in \mathscr{R}^{loc}$ is v-*null* if $\|v\|(B) = 0$. A property holds v-*almost everywhere* (v-a.e.) if it holds except on a v-null set.

We will denote by $L_w^1(v)$ the space of functions in $\mathscr{M}(\mathscr{R}^{loc})$ which are integrable with respect to $|x^*v|$ for all $x^* \in X^*$ where functions which are equal v-a.e. are iden-

tified. The space $L_w^1(v)$ is a Banach space when endowed with the norm

$$\|f\|_v = \sup_{x^* \in B_{X^*}} \int_\Omega |f| \, d|x^* v|.$$

Moreover, it is a Banach lattice for the v-a.e. pointwise order and it is an ideal of measurable functions, that is, if $|f| \leqslant |g|$ v-a.e. with $f \in \mathcal{M}(\mathcal{R}^{loc})$ and $g \in L_w^1(v)$, then $f \in L_w^1(v)$. Even more, convergence in norm of a sequence implies v-a.e. convergence of some subsequence (see [18, Lemma 3.13]). A function $f \in L_w^1(v)$ is *integrable with respect to* v if for each $A \in \mathcal{R}^{loc}$ there exists a vector denoted by $\int_A f \, dv \in X$, such that

$$x^* \left(\int_A f \, dv \right) = \int_A f \, dx^* v \ \text{ for all } x^* \in X^*.$$

We denote by $L^1(v)$ the space of integrable functions with respect to v. It is an order continuous Banach lattice when endowed with the norm and the order structure of $L_w^1(v)$. Moreover, it is an ideal of measurable functions and so an ideal of $L_w^1(v)$. If $\varphi = \sum_{i=1}^n a_i \chi_{A_i} \in \mathcal{S}(\mathcal{R})$ then $\varphi \in L^1(v)$ with $\int_A \varphi \, dv = \sum_{i=1}^n a_i v(A_i \cap A)$ for all $A \in \mathcal{R}^{loc}$. In fact, the space $\mathcal{S}(\mathcal{R})$ is dense in $L^1(v)$. The integration operator $I_v \colon L^1(v) \to X$ given by $I_v(f) = \int_\Omega f \, dv$ is linear and continuous with $\|I_v(f)\| \leqslant \|f\|_v$.

A vector measure $v \colon \mathcal{R} \to E$ with values in a Banach lattice E is *positive* if $v(A) \geqslant 0$ for all $A \in \mathcal{R}$. In this case, the integration operator $I_v \colon L^1(v) \to E$ is positive (i.e. $I_v(f) \geqslant 0$ whenever $0 \leqslant f \in L^1(v)$) and it can be checked that $\|f\|_v = \|I_v(|f|)\|$ for all $f \in L^1(v)$ (see Lemma 3.13 in [20] with the obvious modifications in the case of δ-rings).

3. The spaces $L^p(v)$ and $L_w^p(v)$

As was said in the Introduction, the spaces $L^p(v)$ and $L_w^p(v)$ of p-integrable functions and weakly p-integrable functions are nowadays well-known when the vector measure v is defined on a σ-algebra. The scalar measure counterpart is given in this case by the finite measure spaces: every countably additive positive measure defined on a σ-algebra is bounded (finite). In the vector valued case, if this boundedness requirement is removed the measure must be defined on a δ-ring to make sense. In this section we introduce and study the main properties of the corresponding spaces of p-integrable functions. We extend the definition of $L^1(v)$ and $L_w^1(v)$ for a vector measure on a δ-ring given in [17, 18] and [14] to $L^p(v)$ and $L_w^p(v)$ as follows.

DEFINITION 1. Let $0 < p < \infty$ and let v be a vector measure defined on a δ-ring of subsets of an abstract set Ω. We say that a measurable real function $f \in \mathcal{M}(\mathcal{R}^{loc})$ is *weakly p-integrable with respect to* v if $|f|^p \in L_w^1(v)$, and *p-integrable with respect to* v if $|f|^p \in L^1(v)$. We denote by $L_w^p(v)$ the space of (equivalence classes of) weakly p-integrable functions with respect to v and by $L^p(v)$ the space of (equivalence classes of) p-integrable functions with respect to v.

It is clear that $L^p(v) \subset L^p_w(v)$. Moreover, they are ideals of the vector lattice (with the v-a.e. order) $\mathcal{M}(\mathcal{R}^{loc})$ with $\mathcal{S}(\mathcal{R}) \subset L^p(v)$. An homogeneous positive function can be defined over $L^p_w(v)$ by

$$\|f\|_{p,v} := \left\|\,|f|^p\,\right\|_v^{\frac{1}{p}} = \sup_{x^* \in B_{X^*}} \left(\int_\Omega |f|^p \, d|x^*v|\right)^{\frac{1}{p}}, \quad f \in L^p_w(v).$$

The following well-known inequalities involving positive real numbers will be necessary through the paper (see for instance [20, Section 2.2]).

LEMMA 2. *Let $a, b \in [0, +\infty)$. Then the following inequalities hold.*

$$(a+b)^r \leqslant a^r + b^r \text{ and } |a^r - b^r| \leqslant |a-b|^r, \text{ for } 0 < r \leqslant 1. \tag{1}$$

$$a^r + b^r \leqslant (a+b)^r \leqslant 2^{r-1}(a^r + b^r), \text{ for } r \geqslant 1. \tag{2}$$

$$|a^r - b^r| \leqslant r \cdot |a^{r-1} + b^{r-1}| \cdot |a-b|, \text{ for } r \geqslant 1. \tag{3}$$

Since $\|\cdot\|_v$ is a norm, straightforward calculations using the previous lemma show that $\|\cdot\|_{p,v}$ is in fact a quasi-norm. We also use the notations $\|\cdot\|_{L^p_w(v)}$ and $\|\cdot\|_{L^p(v)}$ when an explicit reference to the space is convenient. In what follows we prove some fundamental topological and lattice properties of the spaces $L^p(v)$ and $L^p_w(v)$. We write some of the proofs for the aim of completeness, since our arguments follow the lines of the ones that prove the corresponding results for the case of vector measures on σ-algebras (see [20, Ch.2, Ch.3] and [12, 21]). However, there are several technical things that makes the proofs slightly different. One of the reasons is that we are *not* working in the setting of the Banach function spaces. For instance, in our spaces convergence of a sequence in the norm still implies v-a.e. convergence of the sequence, but we have to use the result given in [18, Lemma 3.13] instead of [20, Proposition 2.2 (ii)], that is used in [20].

When $1 \leqslant p < \infty$, $\|\cdot\|_{p,v}$ is actually a lattice norm. To prove this result we need first the following lemma, that will be useful also in next sections.

LEMMA 3. *Let $q, r, s > 0$ such that $\frac{1}{q} = \frac{1}{r} + \frac{1}{s}$ and let $f \in L^r_w(v)$ and $g \in L^s_w(v)$. Then, $fg \in L^q_w(v)$ and $\|fg\|_{q,v} \leqslant \|f\|_{r,v} \|g\|_{s,v}$.*

Proof. Without loss of generality, it suffices to assume that $\|f\|_{r,v} = \|g\|_{s,v} = 1$. By using *Young's inequality*

$$\|fg\|_{q,v}^q = \left\|\,|f|^q |g|^q\,\right\|_v \leqslant \frac{q}{r}\left\|\,|f|^r\,\right\|_v + \frac{q}{s}\left\|\,|g|^s\,\right\|_v = \frac{q}{r}\|f\|_{r,v} + \frac{q}{s}\|g\|_{s,v} = \frac{q}{r} + \frac{q}{s} = 1. \quad \square$$

The same arguments prove that the result is also true in the case of $L^p(v)$.

Proposition 4. *Let* $0 < p < \infty$ *and* ν *a vector valued measure on a* δ-*ring. Then*

(1) *for* $1 \leqslant p < \infty$, $(L_w^p(\nu), \|\cdot\|_{p,\nu})$ *and* $(L^p(\nu), \|\cdot\|_{p,\nu})$ *are Banach lattices with the* ν-*a.e. order, and*

(2) *for* $0 < p < 1$, $(L_w^p(\nu), \|\cdot\|_{p,\nu})$ *and* $(L^p(\nu), \|\cdot\|_{p,\nu})$ *are quasi Banach lattices with the* ν-*a.e. order.*

Proof. We only prove the case of $L_w^p(\nu)$. Let $(f_n)_{n \geqslant 1}$ be a Cauchy sequence in $L_w^p(\nu)$. Due to the equality $|a - b| = |a^+ - b^+| + |a^- - b^-|$ for $a, b \in \mathbb{R}$ and the compatibility of the quasinorm $\|\cdot\|_{p,\nu}$ with the ν-a.e. pointwise order, we can assume that $f_n \in L_w^p(\nu)^+$ for all $n \in \mathbb{N}$.

Step 1. Completeness for $0 < p < 1$. Applying inequality (1) in Lemma 2 to f_n and f_m and taking norm $\|\cdot\|_\nu$ we have that

$$\big\||f_n^p - f_m^p|\big\|_\nu \leqslant \big\||f_n - f_m|^p\big\|_\nu.$$

Therefore, $(f_n^p)_{n \geqslant 1}$ is a Cauchy sequence in $L_w^1(\nu)$ and so it has a limit $f \in L_w^1(\nu)$. Note that $f \geqslant 0$ ν-a.e. as in $L_w^1(\nu)$ convergence in norm of a sequence implies ν-a.e. convergence of some subsequence. Fix $n \in \mathbb{N}$. Using inequality (3) in Lemma 2 with $r := \frac{1}{p}$, Lemma 3 with $q := p, r := \frac{p}{1-p}$ and $s := 1$ and again (1) and (2) in Lemma 2 but now with $r := \frac{1}{p} - 1$ we obtain

$$\big\||f_n - f^{\frac{1}{p}}|\big\|_{p,\nu} \leqslant \big\||(f_n^p)^{\frac{1}{p}} - f^{\frac{1}{p}}|\big\|_{p,\nu} \leqslant \frac{1}{p}\big\||(f_n^p)^{\frac{1}{p}-1} + f^{\frac{1}{p}-1}| \cdot |f_n^p - f|\big\|_{p,\nu}$$

$$\leqslant \frac{1}{p}\big\||(f_n^p)^{\frac{1}{p}-1} + f^{\frac{1}{p}-1}|\big\|_{\frac{p}{1-p},\nu}\big\||f_n^p - f|\big\|_\nu$$

$$\leqslant \frac{1}{p}\max\{2^{\frac{1-2p}{p}}, 1\}\big(\|f_n^p\|_\nu^{\frac{1-p}{p}} + \|f\|_\nu^{\frac{1-p}{p}}\big)\big\||f_n^p - f|\big\|_\nu$$

$$\leqslant \frac{1}{p}\max\{2^{\frac{1-2p}{p}}, 1\}\big(\sup_{m \in \mathbb{N}} \|f_m\|_{p,\nu}^{1-p} + \|f\|_\nu^{\frac{1-p}{p}}\big)\big\||f_n^p - f|\big\|_\nu,$$

hence $f_n \to f^{\frac{1}{p}}$ in $L_w^p(\nu)$ and consequently $L_w^p(\nu)$ is complete.

Step 2. $\|\cdot\|_{p,\nu}$ *is a lattice norm for* $1 \leqslant p < \infty$. Let $f, g \in L_w^p(\nu)$. Inequality (3) in Lemma 2 and Lemma 3 with $r := p, s := \frac{p}{p-1}$ and $q := 1$ yield that

$$\|f + g\|_{p,\nu}^p = \big\||f + g|^p\big\|_\nu = \big\||f + g| \cdot |f + g|^{p-1}\big\|_\nu$$

$$\leqslant \big\||f| \cdot |f + g|^{p-1}\big\|_\nu + \big\||g| \cdot |f + g|^{p-1}\big\|_\nu$$

$$\leqslant \|f\|_{p,\nu} \cdot \big\||f + g|^{p-1}\big\|_{\frac{p}{p-1},\nu} + \|g\|_{p,\nu} \cdot \big\||f + g|^{p-1}\big\|_{\frac{p}{p-1},\nu}$$

$$= \big\||f + g|^{p-1}\big\|_{\frac{p}{p-1},\nu}\big(\|f\|_{p,\nu} + \|g\|_{p,\nu}\big)$$

$$= \big\||f + g|^p\big\|_\nu^{\frac{p-1}{p}}\big(\|f\|_{p,\nu} + \|g\|_{p,\nu}\big)$$

$$= \|f + g\|_{p,\nu}^{p-1}\big(\|f\|_{p,\nu} + \|g\|_{p,\nu}\big),$$

hence $\|f+g\|_{p,v} \leqslant \|f\|_{p,v} + \|g\|_{p,v}$ and $\|\cdot\|_{p,v}$ is a norm. That it is also a lattice norm is direct since $\|\cdot\|_v$ so is. In fact this is also obviously true for the quasi-norm $\|\cdot\|_{p,v}$ if $0 < p < 1$.

Step 3. Completeness for $p \geqslant 1$. Fix $n, m \in \mathbb{N}$. As in the case $0 < p < 1$, use inequality (3) in Lemma 2 and Lemma 3 with $q := 1, r := \frac{p}{p-1}$ and $s := p$ to obtain

$$
\begin{aligned}
\|f_n^p - f_m^p\|_v &\leqslant p \left\| |f_n^{p-1} + f_m^{p-1}| \cdot |f_n - f_m| \right\|_v \\
&\leqslant p \left\| |f_n^{p-1} + f_m^{p-1}| \right\|_{\frac{p}{p-1},v} \|f_n - f_m\|_{p,v} \\
&\leqslant p \left(\left\| |f_n^{p-1}| \right\|_{\frac{p}{p-1},v} + \left\| |f_m^{p-1}| \right\|_{\frac{p}{p-1},v} \right) \|f_n - f_m\|_{p,v} \\
&= p \left(\left\| |f_n| \right\|_{p,v}^{p-1} + \left\| |f_m| \right\|_{p,v}^{p-1} \right) \|f_n - f_m\|_{p,v} \\
&\leqslant 2p \left(\sup_{k \in \mathbb{N}} \left\| |f_k| \right\|_{p,v}^{p-1} \right) \|f_n - f_m\|_{p,v}.
\end{aligned}
$$

Therefore $(f_n^p)_{n \geqslant 1}$ is a Cauchy sequence in $L^1_w(v)$. Hence there is a limit $f \in L^1_w(v)$. Again $f \geqslant 0$ v-a.e. by the same argument as the one used above. We will show that $f^{\frac{1}{p}}$ is the limit of $(f_n)_{n \geqslant 1}$ in $L^p_w(v)$. Indeed inequality (1) in Lemma 2 gives

$$
\|f_n - f^{\frac{1}{p}}\|_{p,v} = \left\| |f_n - f^{\frac{1}{p}}|^p \right\|_v^{\frac{1}{p}} \leqslant \|f_n^p - f\|_v^{\frac{1}{p}}.
$$

Hence $L^p_w(v)$ is complete. The proofs are similar for the space $L^p(v)$. $\quad\square$

REMARK 5. As it was already mentioned, $\mathscr{S}(\mathscr{R}) \subset L^p(v)$. Moreover $\mathscr{S}(\mathscr{R})$ is a dense set in $L^p(v)$. Indeed, let $f \in L^p(v)^+$, then $f^p \in L^1(v)$ and by the density of $\mathscr{S}(\mathscr{R})$ in $L^1(v)$ there exists an increasing sequence $0 \leqslant (\varphi_n)_{n \geqslant 1}$ converging to f^p v-a.e. and in the norm of $L^1(v)$. Clearly, $(\varphi_n^{\frac{1}{p}})_{n \geqslant 1} \in \mathscr{S}(\mathscr{R}) \subset L^p(v)$ and $\varphi_n^{\frac{1}{p}} \uparrow f$ v-a.e. The same corresponding inequalities used to prove the completeness of $L^p_w(v)$ can be used to take an inequality like $\|f - \varphi_n^{\frac{1}{p}}\|_{p,v} \leqslant K \|f^p - \varphi_n\|_v$ and conclude the result. The extension to the general case is routine. Consequently, $L^p(v)$ is a closed ideal in $L^p_w(v)$.

We study now the convexity behavior of our spaces.

PROPOSITION 6. *Let $0 < p < \infty$. The spaces $L^p_w(v)$ and $L^p(v)$ are p-convex with p-convexity constants $\mathbf{M}^{(p)}(L^p_w(v)) = \mathbf{M}^{(p)}(L^p(v)) = 1$. Moreover, for $0 < p < 1$, if $L^1_w(v)$ (resp. $L^1(v)$) is $\frac{1}{p}$-convex, then $L^p_w(v)$ (resp. $L^p(v)$) is a Banach lattice with the norm*

$$
\|\|f\|\|_p := \inf\{ \sum_{j=1}^n \|f_j\|_{p,v} : |f| \leqslant \sum_{j=1}^n |f_j|, \ f_j \in L^p_w(v), \ j = 1, \dots, n \},
$$

which is equivalent to the quasinorm $\|\cdot\|_{p,v}$. If moreover $\mathbf{M}^{(\frac{1}{p})}(L^1_w(v)) = 1$, the norm $\|\|\cdot\|\|_p$ coincides exactly with $\|\cdot\|_{p,v}$.

Proof. Fix $f_1, \ldots, f_n$, $n \in \mathbb{N}$, and compute

$$\Big\| \Big(\sum_{j=1}^{n} |f_j|^p \Big)^{\frac{1}{p}} \Big\|_{p,v} = \Big\| \sum_{j=1}^{n} |f_j|^p \Big\|_{v}^{\frac{1}{p}} \leqslant \Big(\sum_{j=1}^{n} \big\| |f_j|^p \big\|_{v} \Big)^{\frac{1}{p}} = \Big(\sum_{j=1}^{n} \|f_j\|_{p,v}^p \Big)^{\frac{1}{p}}.$$

We have clearly that $L_w^p(v)$ (and $L^p(v)$) is p-convex with p-convexity constant

$$\mathbf{M}^{(p)}(L_w^p(v)) = \mathbf{M}^{(p)}(L^p(v)) \leqslant 1.$$

Moreover, the p-convexity constant is $\mathbf{M}^{(p)}(L_w^p(v)) = \mathbf{M}^{(p)}(L^p(v)) = 1$. Indeed letting $n = 1$ in the inequality above $\big\| (|f|^p)^{\frac{1}{p}} \big\|_{p,v} = \|f\|_{p,v} = (\|f\|_{p,v}^p)^{\frac{1}{p}}$ and so $\mathbf{M}^{(p)}(L_w^p(v)) = \mathbf{M}^{(p)}(L^p(v)) \geqslant 1$.

It is direct to check that $\|\|\cdot\|\|_p$ is a lattice norm. From the definition of $\|\|\cdot\|\|_p$ it is clear that $\|\|f\|\|_p \leqslant \|f\|_{p,v}$ just taking $f_1 = f \in L_w^p(v)$. On the other hand, let $f \in L_w^p(v)$ and $\varepsilon > 0$ and choose $n \in \mathbb{N}$ and $f_1, \ldots f_n \in L_w^p(v)$ such that $|f| \leqslant \sum_{j=1}^n |f_j|$ and $\sum_{j=1}^n \|f_j\|_{p,v} \leqslant \|\|f\|\|_p + \varepsilon$. Since $L_w^1(v)$ is $\frac{1}{p}$-convex with $\frac{1}{p}$-convexity constant $\mathbf{M}^{(\frac{1}{p})}(L_w^1(v))$, we have

$$\|f\|_{p,v} = \big\| |f|^p \big\|_{v}^{\frac{1}{p}} \leqslant \Big\| \Big(\sum_{j=1}^{n} |f_j| \Big)^p \Big\|_{v}^{\frac{1}{p}} = \Big\| \Big(\sum_{j=1}^{n} (|f_j|^p)^{\frac{1}{p}} \Big)^p \Big\|_{v}^{\frac{1}{p}}$$

$$\leqslant \Big(\mathbf{M}^{(\frac{1}{p})}(L_w^1(v)) \Big(\sum_{j=1}^{n} \big\| |f_j|^p \big\|_{v}^{\frac{1}{p}} \Big)^p \Big)^{\frac{1}{p}} = \mathbf{M}^{(\frac{1}{p})}(L_w^1(v))^{\frac{1}{p}} \sum_{j=1}^{n} \|f_j\|_{p,v}$$

$$\leqslant \mathbf{M}^{(\frac{1}{p})}(L_w^1(v))^{\frac{1}{p}} (\|\|f\|\|_p + \varepsilon).$$

As ε is arbitrary, we obtain that $\|\|\cdot\|\|_p$ and $\|\cdot\|_{p,v}$ are equivalent. Note that $\|\|f\|\|_p = \|f\|_{p,v}$ whenever $\mathbf{M}^{(\frac{1}{p})}(L_w^1(v)) = 1$. $\square$

PROPOSITION 7. *Let $0 < p < \infty$. Then the space $L_w^p(v)$ has the σ-Fatou property.*

Proof. First, remark that $f \in L_w^p(v)$ if and only if $\big\| |f|^p \big\|_v < \infty$ since $f \in L_w^1(v)$ if and only if $\|f\|_v < \infty$. Now, let $0 \leqslant (f_n)_{n \geqslant 1}$ be an increasing sequence in $L_w^p(v)$ such that $\sup_n \|f_n\|_{p,v} < \infty$. By the measurability of f_n there exists $f := \sup_n f_n \in \mathcal{M}(\mathcal{R}^{loc})$. Using the same argument as the one in the proof of the completeness of $L_w^p(v)$, we have that $f \geqslant 0$. Let us show that $\|f\|_{p,v} < \infty$. Consider the increasing sequence $0 \leqslant (f_n^p)_{n \geqslant 1} \in L_w^1(v)$ and fix $x^* \in B_{X^*}$. Then for every $n \in \mathbb{N}$, $f_n^p \in L^1(|x^*v|)$ and by the *Monotone Convergence Theorem*

$$\int_{\Omega} |f|^p \, d|x^*v| = \lim_{n \to \infty} \int_{\Omega} |f_n|^p \, d|x^*v| \leqslant \lim_{n \to \infty} \big\| |f_n|^p \big\|_v = \sup_{n \in \mathbb{N}} \big\| |f_n|^p \big\|_v < \infty.$$

Hence $|f|^p \in L_w^1(v)$ and $f \in L_w^p(v)$. Consequently, $L_w^p(v)$ has the weak σ-Fatou property. Moreover, $\big\| |f|^p \big\|_v \leqslant \sup_n \big\| |f_n|^p \big\|_v$, hence $\|f\|_{p,v} \leqslant \sup_n \|\|f_n\|\|_{p,v}$. On the

other hand, $0 \leqslant f_n \leqslant f$ for every $n \in \mathbb{N}$, then $\|f_n\|_{p,v} \leqslant \|f\|_{p,v}$ for every $n \in \mathbb{N}$ and $\sup_n \|f_n\|_{p,v} \leqslant \|f\|_{p,v}$. Taking into account both inequalities we have $\sup_n \|f_n\|_{p,v} = \|f\|_{p,v}$, so $L^p_w(v)$ has the σ-Fatou property. $\quad\square$

PROPOSITION 8. *Let $0 < p < \infty$. Then the space $L^p(v)$ is order continuous.*

Proof. First, show that $L^p(v)$ is σ-order continuous. To this aim, take $(f_n)_{n \geqslant 1}$ a decreasing sequence in $L^p(v)^+$ with $\inf f_n = 0$, that is, such that $f_n \downarrow 0$ v-a.e. Then $(f_n^p)_{n \geqslant 1}$ is a decreasing sequence in $L^1(v)^+$ such that $f_n^p \downarrow 0$ v-a.e. The σ-order continuity of $L^1(v)$ yields $\lim_{n \to \infty} \|f_n^p\|_v = 0$, so $\lim_{n \to \infty} \|f_n^p\|_v^{\frac{1}{p}} = \lim_{n \to \infty} \|f_n\|_{p,v} = 0$ and $L^p(v)$ is σ-order continuous. Now, since $L^p_w(v)$ has the σ-Fatou property, it is Dedekind σ-complete ([24, Theorem 113.1]) and as $L^p(v)$ is a closed ideal in it, it is also Dedekind σ-complete ([16, Theorem 25.2]). Proposition 1.a.8 in [15] yields that $L^p(v)$ is then order continuous. $\quad\square$

EXAMPLE 9. It is easy to find examples of $L^p(v)$ spaces which have not the σ-Fatou property and $L^p_w(v)$ spaces which are not order continuous. In fact, under certain requirements having these properties implies the coincidence of $L^p(v)$ and $L^p_w(v)$ (for instance, when the δ-ring is a σ-algebra, see the comments at the end of this section). For example, take the δ-ring $\mathscr{R}$ of all the finite subsets of $\mathbb{N}$ and the vector measure $\eta : \mathscr{R} \to c_0$ given by $\eta(\{n\}) := e_n$, where $(e_n)_n$ is the canonical basis of c_0. It is known that in this case $L^1(\eta) = c_0$ and $L^1_w(\eta) = \ell^\infty$ (see [10, Example 2.2]). Just looking at the definition makes clear that for all $0 < p < \infty$, $L^p(\eta) = c_0$ and $L^p_w(\eta) = \ell^\infty$. The first one do not have the σ-Fatou property, and the second one is not order continuous. If we define the same vector measure but having values in the space ℓ^q, $1 \leqslant q \leqslant \infty$, instead of in c_0, it is shown in [10, Example 2.2] that $L^1(\eta) = L^1_w(\eta) = \ell^q$; for $1 \leqslant p < \infty$, then $L^p(\eta) = L^p_w(\eta) = \ell^{pq}$ that is order continuous and has the Fatou property.

The following result gives a representation theorem for abstract order continuous and p-convex Banach lattices. It generalizes the one in [12, Proposition 2.4] (see also [20, Proposition 3.30] for a complex version).

THEOREM 10. *Let $1 < p < \infty$ and let E be a p-convex order continuous Banach lattice. Then there exists a positive vector measure v defined on a δ-ring and with values in E such that $L^p(v)$ and E are order isomorphic.*

Proof. Since E is an order continuous Banach lattice, it can be renormed in order to have a p-convexity constant equal to 1 (see Proposition 1.d.8 in [15]) and there exists a vector measure v_1 defined on a δ-ring and with values in E, such that the space $L^1(v_1)$ of integrable functions with respect to v_1 is order isometric to E with the new norm. More precisely, the integration operator $I_{v_1} : L^1(v_1) \to E$ is an order isometry ([11, Theorem 5]; see also [4, pp. 22–23]). Consequently $L^1(v_1)$ is a p-convex and order continuous Banach lattice with p-convexity constant equal to 1 (as

above). Consider $L^{\frac{1}{p}}(v_1)$, by Proposition 6 we have that $\|\cdot\|_{\frac{1}{p}}$ is actually a lattice norm and then $L^{\frac{1}{p}}(v)$ is a Banach lattice. Define the set function $v_2 : \mathscr{R} \to L^{\frac{1}{p}}(v_1)$ by $A \mapsto \chi_A$. Clearly, v_2 is additive. Moreover, if $(A_i)_{i \geqslant 1} \subset \mathscr{R}$ is a pairwise disjoint sequence such that $\bigcup_{i \geqslant 1} A_i \in \mathscr{R}$, then by the order continuity of $L^1(v_1)$ we have that

$$\left\| v_2(\bigcup_{i \geqslant 1} A_i) - \sum_{i=1}^{n} v_2(A_i) \right\|_{\frac{1}{p}, v_1} = \left\| v_2(\bigcup_{i \geqslant 1} A_i) - v_2(\bigcup_{i \geqslant 1} A_i) \right\|_{\frac{1}{p}, v_1} = \left\| v_2(\bigcup_{i \geqslant n} A_i) \right\|_{\frac{1}{p}, v_1}$$

$$= \left\| |\chi_{\cup_{i \geqslant n} A_i}|^{\frac{1}{p}} \right\|_{v_1}^{p} = \left\| \chi_{\cup_{i \geqslant n} A_i} \right\|_{v_1} \to 0,$$

as $n \to \infty$. Hence, v_2 is a countably additive vector measure. Consider now the integration operator $I_{v_2} : L^1(v_2) \to L^{\frac{1}{p}}(v_1)$ which is linear and continuous and take $\varphi = \sum_{j=1}^{n} a_j \chi_{A_j} \in \mathscr{S}(\mathscr{R})$ (we assume that the sets A_j are pairwise disjoint). Then, since $L^{\frac{1}{p}}(v_1)$ is a Banach lattice and the vector measure v_2 is positive,

$$\|\varphi\|_{v_2} = \left\| \int_{\Omega} |\varphi| \, dv_2 \right\|_{\frac{1}{p}, v_1} = \left\| \sum_{j=1}^{n} |a_j| v_2(A_j) \right\|_{\frac{1}{p}, v_1} = \left\| \sum_{j=1}^{n} |a_j| \chi_{A_j} \right\|_{\frac{1}{p}, v_1} = \|\varphi\|_{\frac{1}{p}, v_1}.$$

On the other hand,

$$\|I_{v_2}(\varphi)\|_{\frac{1}{p}, v_1} = \left\| \int_{\Omega} \varphi \, dv_2 \right\|_{\frac{1}{p}, v_1} = \|\varphi\|_{\frac{1}{p}, v_1}.$$

Consequently, $\|I_{v_2}(\varphi)\|_{\frac{1}{p}, v_1} = \|\varphi\|_{\frac{1}{p}, v_1} = \|\varphi\|_{v_2}$. Moreover, the integration operator I_{v_2} over $\mathscr{S}(\mathscr{R})$ is the identity map. Extending now by density, we obtain that $L^1(v_2) = L^{\frac{1}{p}}(v_1)$ with equal lattice norms. Therefore, again the identity map will be an order isometry between $L^p(v_2)$ and $L^1(v_1)$ and $L^p(v_2) = L^1(v_1)$ with equal lattice norms. Hence, E and $L^p(v_2)$ are order isometric. $\square$

The properties of a vector measure v defined on a δ-ring $\mathscr{R}$ influence the spaces of integrable and weakly integrable functions $L^1(v)$ and $L^1_w(v)$ (see [10] and [3]). We explain here the corresponding consequences on the spaces $L^p_w(v)$ and $L^p(v)$ with $p > 1$.

In the general case it is not true that a measure v defined on a δ-ring is bounded, that is $\|v\|(\Omega) = \|\chi_\Omega\|_v < \infty$ which is equivalent to the fact that $\chi_\Omega \in L^1_w(v)$ (see [10, Example 2.1]). Due to the ideal property of $L^1(v)$ in $\mathscr{R}^{loc}$, the space of measurable bounded functions $L^\infty(v)$ is contained in $L^1(v)$ if and only if $\chi_\Omega \in L^1(v)$ (see the comments after [10, Example 2.1]). Consequently, if v is not bounded, this premise fails to hold. It is clear that the same conclusion holds for $L^p(v)$, $p > 1$. Moreover, recall that a measure v is said to be *strongly additive* if $(v(A_n))_{n \geqslant 1}$ converges to zero whenever $(A_n)_{n \geqslant 1}$ is a sequence of disjoint subsets of $\mathscr{R}$. Corollary 3.2 in [10] assures that v is strongly additive if and only if $\chi_\Omega \in L^1(v)$. Therefore, $L^\infty(v) \subset L^1(v)$ if and only if v is strongly additive. Again, the same result holds for $L^p(v)$, $p \geqslant 1$.

A measure v is said to be σ-*finite* if there exists a sequence $(A_n)_{n \geqslant 1}$ in $\mathscr{R}$ and a v-null set $N \in \mathscr{R}^{loc}$ such that $\Omega = (\cup_{n \geqslant 1} A_n) \cup N$. Theorem 3.3 in [10] ensures that the

σ-finiteness of v is equivalent to the existence of a weak unit in $L^1(v)$. Clearly, this is equivalent to the existence of a weak unit in $L^p(v)$ for any/some $p > 1$.

New requirements on the measure v (introduced and study in [3]) influence the structure of the spaces $L_w^1(v)$ strongly. Locally σ-finiteness plays an important and very special role as it gives the condition to $L^1(v)$ to be super order dense in $L_w^1(v)$. More concretely, recall that a measure v is *locally σ-finite* if given $B \in \mathscr{R}^{loc}$ with $\|v\|(B) < \infty$, B can be written as $B = (\cup_{n \geqslant 1} A_n) \cup N$, with $A_n \in \mathscr{R}$ and $N \in \mathscr{R}^{loc}$ a v-null set. Theorem 4.8 in [3] proves that v is locally σ-finite if and only if for every $0 \leqslant f \in L_w^1(v)$, there exists a sequence $(\varphi_n) \subset \mathscr{S}(R)$ such that $0 \leqslant \varphi_n \uparrow f$ v-a.e. This characterization allows us to use the same arguments as the ones in Proposition 3.9 and Corollary 3.10 in [12], as well as certain well-known theorems on general Banach lattices to prove the following proposition. We recall that a Banach lattice E is a KB-space if every monotone sequence in B_E is convergent.

PROPOSITION 11. *Let v be a locally σ-finite vector measure on a δ-ring $\mathscr{R}$. For $p > 1$, the following conditions are equivalent:*

1. *$L_w^p(v)$ is order continuous.*

2. *$L_w^p(v)$ is a KB-space.*

3. *$L_w^p(v)$ is weakly sequentially complete.*

4. *$L_w^p(v)$ does not contain a (lattice) copy of c_0.*

5. *$L_w^p(v)$ is reflexive.*

6. *$L^p(v)$ is reflexive.*

7. *$L^p(v)$ does not contain a (lattice) copy of c_0.*

8. *$L^p(v)$ is weakly sequentially complete.*

9. *$L^p(v)$ is a KB-space.*

10. *$L_w^p(v) = L^p(v)$.*

11. *$L_w^1(v) = L^1(v)$.*

Remark that $L_w^1(v) = L^1(v)$ holds if the space X where the vector measure takes its values does not contain a copy of c_0 ([14, Theorem 5.1]). This always happens if X is a weakly sequentially complete Banach space (see [1, page 226]) and the converse is also true for Banach lattices by Theorem 14.12 in [1]. This provides a quite large list of examples which guarantees the equality $L_w^1(v) = L^1(v)$.

Finally, let us note that there is a decomposition property for v which implies that the space $L_w^1(v)$ has the Fatou property. It is the so called $\mathscr{R}$-decomposability of v (see Definition 17). For a measure satisfying such property, Theorem 5.8 in [3] assures that $L_w^1(v)$ has the Fatou property and $L^1(v)$ is an order dense ideal in it. Thus, the order density of $L^p(v)$ in $L_w^p(v)$ is an obvious consequence; clearly, we also get that under this requirement for v the space $L_w^p(v)$ has the Fatou property for $p > 1$.

4. Multiplication operators

Let $1 < p < \infty$ and consider a vector measure v defined on a σ-algebra. It is well-known that in this case $L^p(v) \subset L^p_w(v) \subset L^1_w(v)$ and $L^p(v) \subset L^1(v) \subset L^1_w(v)$. Moreover, by Proposition 3.1 and Corollary 3.2 in [12] a further inclusion can be established: for $p > 1$, $L^p_w(v) \subset L^1(v)$. Actually, Proposition 3.3 in [12] establishes that this inclusion is an L-weakly compact operator (and so a weakly compact operator). However, for vector measures on δ-rings these inclusions are not necessarily true (for instance, $L^p[0,\infty]$ is not included in $L^1[0,\infty]$), but the inclusions between the products of the corresponding spaces are still preserved; for example the equality $L^p[0,\infty] \cdot L^{p'}[0,\infty] = L^1[0,\infty]$ remains true. In this section we analyze these inclusion relations and the compactness properties of the multiplication operators that appear in a natural way. Let us start with two simple examples related to Example 9.

EXAMPLE 12. (a) Let Γ be an uncountable abstract set and $\mathscr{R}$ the δ-ring of finite subsets of Γ. Clearly, $\mathscr{R}^{loc} = 2^\Gamma$. Consider the vector measure $v : \mathscr{R} \to \ell^1(\Gamma)$ defined by $v(A) := \sum_{\gamma \in A} e_\gamma$, where e_γ is the characteristic function of the point $\gamma \in \Gamma$. It is obvious that the only v-null set is the empty set. Since $\ell^1(\Gamma)$ does not contain a copy of c_0, we have that $L^1_w(v) = L^1(v)$ (see the explanation at the end of Section 3). Moreover $L^1_w(v) = L^1(v) = \ell^1(\Gamma)$ (see [10, Example 2.2] and Example 9). Take $1 < p < \infty$, then $L^p_w(v) = L^p(v) = \ell^p(\Gamma)$ and $\ell^p(\Gamma) \not\subset \ell^1(\Gamma)$. Therefore, $L^p(v) = L^p_w(v) \not\subset L^1_w(v) = L^1(v)$ and some of the inclusions above fail to be true.

(b) Let $1 < p < q$, and consider again the previous example. Then $L^p(v) = L^p_w(v) = \ell^p(\Gamma)$ and $L^q(v) = L^q_w(v) = \ell^q(\Gamma)$. Since $\ell^p(\Gamma) \subset \ell^q(\Gamma)$, we have $L^p(v) \subset L^q(v)$ which is just the opposite inclusion of the one that holds in the case of σ-algebras.

The multiplication operators between $L^p(v)$ spaces have been studied recently in a series of papers for the case of vector measures v on σ-algebras (see [20, Ch.3], [8], [9], [13] and [2]). In particular, the equality $L^p_w(v) \cdot L^{p'}(v) = L^1(v)$ and the compactness properties of the multiplication operators are nowadays well-known in this case. In what follows we will study multiplication operators and some of their properties in the context of vector measures defined on a δ-ring. We begin by proving some inclusions without further requirements on the measure.

LEMMA 13. *Let $p, p' > 1$ be conjugated exponents. Then*

1. $L^p_w(v) \cdot L^{p'}_w(v) = L^1_w(v)$, *and*

2. $L^p(v) \cdot L^{p'}(v) = L^p_w(v) \cdot L^{p'}(v) = L^p(v) \cdot L^{p'}_w(v) = L^1(v)$.

Proof. (1) Taking into account Lemma 3 we have that $L^p_w(v) \cdot L^{p'}_w(v) \subset L^1_w(v)$. Now let $f \in L^1_w(v)$. Then we can write $f = \text{sign}(f)|f| = (\text{sign}(f)|f|^{\frac{1}{p}}) \cdot |f|^{\frac{1}{p'}} \in L^p_w(v) \cdot L^{p'}_w(v)$ and check the converse inclusion.

(2) Note that the same proof of (1) yields $L^p(v) \cdot L^{p'}(v) = L^1(v)$. We will prove that

$L_w^p(v) \cdot L^{p'}(v) = L^1(v)$. For this aim, let $f \in L_w^p(v)$ and $g \in L^{p'}(v)$. We can suppose without loss of generality that $f, g \geqslant 0$ v-a.e. Since $f \in L_w^p(v)$, there exists a sequence $(\psi_n)_{n \geqslant 1}$ in $\mathscr{S}(\mathscr{R}^{loc})$ such that $0 \leqslant \psi_n \uparrow f$ v-a.e. and since $g \in L^{p'}(v)$, there exists a sequence $(\varphi_n)_{n \geqslant 1}$ in $\mathscr{S}(\mathscr{R})$ such that $0 \leqslant \varphi_n \uparrow g$ v-a.e. and in the norm of $L^{p'}(v)$. Note that for every $n \in \mathbb{N}$, $\psi_n \varphi_n \in \mathscr{S}(\mathscr{R})$ and that $fg \in L_w^p(v) \cdot L_w^{p'}(v) = L_w^1(v)$, so it suffices to prove that $\|\psi_n \varphi_n - fg\|_v \to 0$ as $n \to \infty$ as $L^1(v)$ is closed in $L_w^1(v)$. Indeed

$$\|\psi_n \varphi_n - fg\|_v = \left\| f\chi_{\mathrm{supp}(f)} \left(\frac{\psi_n}{f\chi_{\mathrm{supp}(f)}} \varphi_n - g \right) \right\|_v \leqslant \|f\|_{L_w^p(v)} \cdot \left\| \frac{\psi_n}{f\chi_{\mathrm{supp}(f)}} \varphi_n - g \right\|_{L^{p'}(v)},$$

where the last computation has been made taking into account that $L^\infty(v) \cdot L^{p'}(v) \subset L^{p'}(v)$ due to the ideal property of $L^{p'}(v)$. Since $0 \leqslant \frac{\psi_n}{f\chi_{\mathrm{supp}(f)}} \varphi_n \uparrow g$ v-a.e., the order continuity of $L^{p'}(v)$ yields $\left\| \frac{\psi_n}{f\chi_{\mathrm{supp}(f)}} \varphi_n - f \right\|_{L^{p'}(v)} \to 0$. Hence $\|\psi_n \varphi_n - fg\|_v \to 0$ and $fg \in L^1(v)$. Finally, since $L^1(v) = L^p(v) \cdot L^{p'}(v) \subset L_w^p(v) \cdot L^{p'}(v)$, the equality holds. Symmetry on the exponents p and p' gives the final result. $\square$

REMARK 14. Note that in general $L_w^p(v) \cdot L_w^{p'}(v) \not\subset L^1(v)$. To see this, just consider a vector measure v such that $L^1(v) \neq L_w^1(v)$ and take a function $f \in L_w^1(v) \setminus L^1(v)$. Then f can be written as $f = \mathrm{sign}(f)|f| = (\mathrm{sign}(f)|f|^{\frac{1}{p}}) \cdot |f|^{\frac{1}{p'}}$, but $f \notin L^1(v)$.

In fact, these spaces can be of a completely different size. Let us show an example. Take a family of disjoint probability spaces $(\Omega_\gamma, \Sigma_\gamma, \mu_\gamma)_{\gamma \in \Gamma}$ for an uncountable set of indexes Γ, the δ-ring $\mathscr{R}$ defined by finite unions $B = \cup_{i=1}^n A_{\gamma_i}$, $A_{\gamma_i} \in \Sigma_{\gamma_i}$ and the vector measure $\kappa : \mathscr{R} \to c_0(\Gamma)$ given by $\kappa(B) = \sum_{i=1}^n \mu_{\gamma_i}(A_{\gamma_i}) \chi_{\{\gamma_i\}}$. Then a direct extension of the arguments that are used in Example 9 ([10, Example 2.2]) gives that the space $L^1(\kappa)$ is the direct sum $\bigoplus_{c_0(\Gamma)} L^1(\mu_\gamma)$. In particular, the support of each elements of this space is contained in a countable union of sets Ω_γ, $\gamma \in \Gamma$. However, $L_w^1(\kappa) = \bigoplus_{\ell^\infty(\Gamma)} L^1(\mu_\gamma)$ and the functions of this space can be even strictly positive in all points of $\bigcup_{\gamma \in \Gamma} \Omega_\gamma$. (Notice that the notations $\bigoplus_{c_0(\Gamma)}$ and $\bigoplus_{\ell^\infty(\Gamma)}$ indicate that the support of each function in the first space is included in a countable subset of indexes γ which do not happen in the case of the second space).

Given $g \in \mathscr{M}(\mathscr{R}^{loc})$ we denote by $M_g : \mathscr{M}(\mathscr{R}^{loc}) \to \mathscr{M}(\mathscr{R}^{loc})$ the multiplication operator by g.

LEMMA 15. *Let $p, p' > 1$ be conjugated exponents and $g \in L^{p'}(v)$. Then*

1. $M_g \in \mathscr{B}(L^p(v), L^1(v))$, *and*

2. $M_g \in \mathscr{B}(L_w^p(v), L^1(v))$.

In both cases $\|M_g\|$ coincides with $\|g\|_{L^{p'}(v)}$.

Proof. It is a consequence of Lemma 13. Indeed, M_g is well defined and so it is automatically continuous since $M_g = M_{g^+} - M_{g^-}$, that is, the difference of two positive

operators between Banach lattices. Moreover $\|M_g(f)\|_{L^1(v)} = \|gf\|_{L^1(v)} \leqslant \|g\|_{L^{p'}(v)} \cdot \|f\|_{L^p_w(v)}$ for all $f \in L^p_w(v)$, thus $\|M_g\| \leqslant \|g\|_{L^{p'}(v)}$. For the other inequality, just take the function

$$f_0 = \|g\|_{L^{p'}(v)}^{-\frac{p'}{p}} |g|^{p'-1} \in B_{L^p(v)}. \qquad \square$$

The arguments used in the previous proof prove also the next lemma.

LEMMA 16. *Let* $p, p' > 1$ *be conjugated exponents and* $g \in L^{p'}_w(v)$. *Then*

1. $M_g \in \mathscr{B}(L^p(v), L^1(v))$ *with* $\|M_g\| \leqslant \|g\|_{L^{p'}_w(v)}$, *and*

2. $M_g \in \mathscr{B}(L^p_w(v), L^1_w(v))$ *with* $\|M_g\| = \|g\|_{L^{p'}_w(v)}$.

In what follows we need further requirements on the measure space $(\Omega, \mathscr{R}, v)$. We will assume that v is $\mathscr{R}$-decomposable. This is a vector measure extension of a well-known decomposition property for scalar measure spaces that is called to be decomposable (or strictly localizable) (see [22, Definition 46]). Let us consider a δ-ring $\mathscr{R}$ of subsets of Ω and a vector measure v on it. Then *Zorn's Lemma* gives a class of pairwise disjoint sets $\{A_i : i \in I\} \subseteq \mathscr{R}$ and a disjoint v-null subset $N \subseteq \Omega$ such that $A = \cup_{i \in I} A_i \cup N$.

DEFINITION 17. A vector measure v over a δ-ring $\mathscr{R}$ of subsets of Ω is said to be *$\mathscr{R}$-decomposable* if there exists a maximal decomposition of Ω as before given by $(\Omega_\alpha)_{\alpha \in \Delta}$ in $\mathscr{R}$ and a v-null N such that

1. for every arbitrary family $(A_\alpha)_{\alpha \in \Delta}$ of elements or $\mathscr{R}$ such that $A_\alpha \subset \Omega_\alpha$ for all $\alpha \in \Delta$, the union $\cup_{\alpha \in \Delta} A_\alpha$ is in $\mathscr{R}^{loc}$, and

2. for each $x^* \in X^*$ and every arbitrary family of $|x^* v|$-null sets $(Z_\alpha)_{\alpha \in \Delta}$ in $\mathscr{R}$ such that $Z_\alpha \subset \Omega_\alpha$ for all $\alpha \in \Delta$, the union $\cup_{\alpha \in \Delta} Z_\alpha$ is $|x^* v|$-null.

For an $\mathscr{R}$-decomposable vector measure, Theorem 5.8 in [3] assures that

THEOREM 18. *Let* $\mathscr{R}$ *be a* δ-ring of subsets of Ω, X *a Banach space and* $v : \mathscr{R} \to X$ *an* $\mathscr{R}$-decomposable vector measure. Then $L^1_w(v)$ has the Fatou property and $L^1(v)$ is an order dense ideal in it.

Consequently, in such case for every $p > 1$, $L^p_w(v)$ has also the Fatou property and $L^p(v)$ is an order dense ideal. We will use this result in the sequel.

THEOREM 19. *Let* $p, p' > 1$ *be conjugated exponents and let* $g \in \mathscr{M}(\mathscr{R}^{loc})$. *If* v *is* $\mathscr{R}$-decomposable, then the following statements are equivalent:

1. $g \in L^{p'}_w(v)$.

2. $M_g \in \mathscr{B}(L^p(v), L^1(v))$.

3. $M_g \in \mathscr{B}(L^p(\nu), L_w^1(\nu))$.

4. $M_g \in \mathscr{B}(L_w^p(\nu), L_w^1(\nu))$.

Proof. By Lemma 16 we have that $(1) \Rightarrow (2)$. Let us see the converse. Assume that $M_g \in \mathscr{B}(L^p(\nu), L^1(\nu))$ and so $\|M_g\| < \infty$. Consider

$$A := \{f \in L_w^{p'}(\nu) : 0 \leqslant f \leqslant |g|\}.$$

By Lemma 16 for $f \in L_w^{p'}(\nu)$, $M_f \in \mathscr{B}(L^p(\nu), L^1(\nu))$ and $\|M_f\| = \|f\|_{L_w^{p'}(\nu)}$. In particular, this is the case for $f \in A$. We order A to see it as an upwards directed system where $f_1, f_2 \in A$ are majorized by $f_1 \vee f_2 \in L_w^{p'}(\nu) \cap A$. We use $(A, \vee)$ to denote it. Then, we have an upwards directed system such that

$$\sup_{f \in (A, \vee)} \|f\|_{L_w^{p'}(\nu)} = \sup_{f \in (A, \vee)} \|M_f\| \leqslant \|M_{|g|}\| = \|M_g\| < \infty.$$

The Fatou property of $L_w^{p'}(\nu)$ ensures that there exists $f_0 := \sup_{f \in (A, \vee)} f \in L_w^{p'}(\nu)$ and that

$$\|f_0\|_{L_w^{p'}(\nu)} = \sup_{f \in (A, \vee)} \|f\|_{L_w^{p'}(\nu)} = \sup_{f \in (A, \vee)} \|M_f\|.$$

We claim that $f_0 = |g|$ ν-a.e. Suppose that this is not the case. Then the set $B := \{\omega \in \Omega : f_0(\omega) \neq |g|(\omega)\} \subset \mathscr{R}^{loc}$ satisfies that $\|\nu\|(B) > 0$. By Lemma 3.4 in [18], $\|\nu\|(B) = \sup_{D \in \mathscr{R} \cap 2^B} \|\nu\|(D)$. Thus, there exists $D \in \mathscr{R}, D \subset B$ such that $0 < \|\nu\|(D) < \infty$. Note that $D \subset \text{supp}(|g| - f_0)$, then $D \cap \text{supp}(|g| - f_0) \neq \emptyset$ ν-a.e. We know that $0 \leqslant |g| - f_0 \in \mathscr{M}(\mathscr{R}^{loc})$, therefore, there exists a sequence $(\varphi_n)_{n \geqslant 1}$ in $\mathscr{S}(\mathscr{R}^{loc})$ such that $\varphi_n \uparrow |g| - f_0$ ν-a.e. This implies that $\varphi_n \chi_D \uparrow (|g| - f_0) \chi_D$ ν-a.e. and so that $\varphi_n \chi_D + f_0 \chi_D \uparrow |g| \chi_D$ ν-a.e.

On the other hand, there exists $n \in \mathbb{N}$ such that $\varphi_n \chi_D \neq 0$ ν-a.e. In other case, if for all $n \in \mathbb{N}$ $\varphi_n \chi_D = 0$ ν-a.e., then $\text{supp}(\varphi_n \cap D) = \emptyset$ ν-a.e. for all $n \in \mathbb{N}$, thus $\left(\bigcup_{n \geqslant 1} \text{supp}(\varphi_n)\right) \cap D = \emptyset$ ν-a.e., a contradiction. Let $k \in \mathbb{N}$ such a number. We have that $\varphi_k \chi_D + f_0 \chi_D \leqslant f_0 \chi_D = (\sup_{f \in (A, \vee)} |f|) \chi_D$ where $\varphi_k \chi_D + f_0 \chi_D \in L_w^{p'}(\nu)$, which contradicts the definition of the supremum. Consequently, $f_0 = |g|$ ν-a.e. and $g \in L_w^q(\nu)$.

The proof of $(1) \Longleftrightarrow (3)$ is analogous. $(4) \Rightarrow (3)$ is evident so let us show now that $(3) \Rightarrow (4)$. For this aim we consider for every $I \subset \Delta$ finite the set $\Omega_I = \cup_{\alpha \in I} \Omega_\alpha$. Consider $0 \leqslant f \in L_w^p(\nu)$ and choose $(\varphi_n)_{n \geqslant 1} \subset \mathscr{S}(\mathscr{R}^{loc})$ such that $0 \leqslant \varphi_n \uparrow f$. For each $n \geqslant 1$ and $I \subset \Delta$ finite, we define $\xi_{(n,I)} = \varphi_n \chi_{\Omega_I} \in \mathscr{S}(\mathscr{R})$. Then $(\xi_{(n,I)})_{(n,I)} \subset L^p(\nu)$ is an upwards directed system $0 \leqslant \xi_{(n,I)} \uparrow f$. Moreover $\sup_{(n,I)} \xi_{(n,I)} = f$. By (3) we have that $0 \leqslant |g|\xi_{(n,I)} = \text{sign}(g)g\xi_{(n,I)} \in L_w^1(\nu)$. Moreover, it is clear that $|g|\xi_{(n,I)} \uparrow |g|f \in \mathscr{M}(\mathscr{R}^{loc})$ and that for every $n \in \mathbb{N}$ and $I \subset \Delta$ finite,

$$\||g|\xi_{(n,I)}\|_{L_w^1(\nu)} = \|M_g(\xi_{(n,I)})\|_{L_w^1(\nu)} \leqslant \|M_g\| \cdot \|\xi_{(n,I)}\|_{L^p(\nu)} \leqslant \|M_g\| \cdot \|f\|_{L_w^p(\nu)}.$$

The Fatou property of $L_w^1(\nu)$ yields that $|g|f \in L_w^1(\nu)$ and so that $gf \in L_w^1(\nu)$. The extension to the general case is routine. Therefore $M_g : L_w^p(\nu) \to L_w^1(\nu)$ is well defined

and the continuity is guaranteed since it is a difference of positive operators between Banach lattices.   □

THEOREM 20. *Let $p, p' > 1$ be conjugate exponents and let $g \in \mathcal{M}(\mathcal{R}^{loc})$. If v is $\mathcal{R}$-decomposable then the following conditions are equivalent:*

1. $g \in L^{p'}(v)$.

2. $M_g \in \mathcal{B}(L_w^p(v), L^1(v))$.

Proof. By Lemma 15 we have that $(1) \Rightarrow (2)$. Let us see $(2) \Rightarrow (1)$. Suppose that $M_g \in \mathcal{B}(L_w^p(v), L^1(v))$. Then also $M_g \in \mathcal{B}(L^p(v), L^1(v))$ so $g \in L_w^{p'}(v)$ by Theorem 19. That is, $|g|^{p'} \in L_w^1(v)$ which clearly implies that $|g|^{p'-1} \in L_w^p(v)$. Therefore,

$$|g|^{p'} = |g| \cdot |g|^{p'-1} \in M_{|g|}(L_w^p(v)) = M_g(L_w^p(v)) \subset L^1(v).$$

Consequently, $g \in L^{p'}(v)$.   □

We finish this section by analyzing the compactness properties of the multiplication operators.

THEOREM 21. *Let $p, p' > 1$ conjugate exponents and let $g \in \mathcal{M}(\mathcal{R}^{loc})$. If v is $\mathcal{R}$-decomposable then the following statements are equivalent:*

1. $g \in L^{p'}(v)$.

2. $M_g \in \mathcal{B}(L_w^p(v), L^1(v))$.

3. $M_g \in \mathcal{L}(L_w^p(v), L^1(v))$.

4. $M_g \in \mathcal{L}(L^p(v), L^1(v))$.

5. $M_g \in \mathcal{L}(L_w^p(v), L_w^1(v))$.

6. $M_g \in \mathcal{L}(L^p(v), L_w^1(v))$.

7. $M_g \in \mathcal{W}(L_w^p(v), L^1(v))$.

8. $M_g \in \mathcal{W}(L^p(v), L^1(v))$.

9. $M_g \in \mathcal{W}(L_w^p(v), L_w^1(v))$.

10. $M_g \in \mathcal{W}(L^p(v), L_w^1(v))$.

Proof. The equivalence $(1) \Longleftrightarrow (2)$ is precisely Theorem 20. Let us see $(1) \Rightarrow (3)$. We already have that $M_g \in \mathcal{B}(L_w^p(v), L^1(v))$. We want to see that $M_g(B_{L_w^p(v)})$ is an $\mathcal{L}$-weakly compact set in $L^1(v)$, that is, $M_g(B_{L_w^p(v)})$ is norm-bounded and such that $\|h_n\|_{L^1(v)} \to 0$ as $n \to \infty$ for every disjoint sequence $(h_n)_{n \geqslant 1}$ contained in the solid hull of $M_g(B_{L_w^p(v)})$. Note that $M_g(B_{L_w^p(v)})$ is clearly norm-bounded by the continuity

of M_g. Moreover, the solid hull of $M_g(B_{L_w^p(v)})$ is itself, since $M_g(B_{L_w^p(v)})$ is solid in $L^1(v)$. In fact, let $|h| \leqslant |\widetilde{h}|$, with $h \in L^1(v)$ and $\widetilde{h} \in M_g(B_{L_w^p(v)})$. We have that $\widetilde{h} = gf$ with $f \in B_{L_w^p(v)}$ and then $|h| \leqslant |gf|$. Thus,

$$\frac{|h|}{|g|}\chi_{\text{supp}(g)} \leqslant |f|\chi_{\text{supp}(g)} \leqslant |f|.$$

The ideal property of $L_w^p(v)$ yields that $\frac{h}{g}\chi_{\text{supp}(g)} \in L_w^p(v)$ and so $\|\frac{h}{g}\chi_{\text{supp}(g)}\|_{L_w^p(v)} \leqslant \|f\|_{L_w^p(v)} \leqslant 1$. Then $h = g\frac{h}{g}\chi_{\text{supp}(g)} \in M_g(B_{L_w^p(v)})$. Finally let $(h_n)_n$ be such a sequence and consider for each $n \in \mathbb{N}$ the set $\text{supp}(h_n) \in \mathscr{R}^{loc}$. As $(h_n)_n$ is a disjoint sequence $(\text{supp}(h_n))_n$ is a disjoint family in $\mathscr{R}^{loc}$. On the other hand, for every $n \in \mathbb{N}$ there exists $f_n \in B_{L_w^p(v)}$ such that $h_n = M_g(f_n) = gf_n = g\chi_{\text{supp}(h_n)}f_n$. By *Hölder's Inequality*

$$\|M_g(f_n)\|_{L^1(v)} = \|h_n\|_{L^1(v)} \leqslant \|f_n\|_{L_w^p(v)} \cdot \|g\chi_{\text{supp}(h_n)}\|_{L^{p'}(v)} \leqslant \|g\chi_{\text{supp}(h_n)}\|_{L^{p'}(v)},$$

but $\|g\chi_{\text{supp}(h_n)}\|_{L^{p'}(v)} \to 0$ since $(g\chi_{\text{supp}(h_n)})_n$ is an order bounded disjoint sequence in the order continuous space $L^{p'}(v)$. The implication $(3) \Rightarrow (2)$ is evident and so we close the chain $(1) \Longleftrightarrow (2) \Longleftrightarrow (3)$.

The implication $(3) \Rightarrow (4)$ is clear because $L^p(v)$ is continuously contained in $L_w^p(v)$ and the composition of a continuous operator (to the right) with an $\mathscr{L}$-weakly compact is an $\mathscr{L}$-weakly compact operator. Let us show now $(4) \Rightarrow (1)$ and close the equivalences from (1) to (4). Assume that $M_g \in \mathscr{L}(L^p(v), L^1(v))$. In particular, $M_g \in \mathscr{B}(L^p(v), L^1(v))$ and Theorem 19 yields that $g \in L_w^{p'}(v)$.

In order to show that $g \in L^{p'}(v)$, we consider, for every $I \subset \Delta$ finite, the set $\Omega_I = \cup_{\alpha \in I}\Omega_\alpha$ and the σ-algebra $\Sigma_I = \{\cup_{\alpha \in I}A_\alpha : A_\alpha \in \Sigma_\alpha \text{ for all } \alpha \in I\}$ of Ω_I where $\Sigma_\alpha = \mathscr{R} \cap \Omega_\alpha$. Note that $\Omega_I \subset \Omega$ and $\Sigma_I \subset \mathscr{R}$. Denote by $v_I : \Sigma_I \to X$ the restriction of v to Σ_I. For each $f \in \mathscr{M}(\mathscr{R}^{loc})$, denote by f^I the function resulting from the restriction of f to Ω_I. For every $f \in L_w^1(v)$ we have that $f\chi_{\Omega_I} \in L_w^1(v)$ and $f^I \in L_w^1(v_I)$ with $\|f^I\|_{v_I} = \|f\chi_{\Omega_I}\|_v$ (see the proof of Theorem 5.8 in [3]). Moreover, for every $f \in L^1(v)$ we have that $f\chi_{\Omega_I} \in L^1(v)$ and $f^I \in L^1(v_I)$ (see [11]). If Z is a v-null set then $Z \cap \Omega_I$ is v_I-null. Conversely, each function in $L^1(v_I)$ (respectively in $L_w^1(v_I)$) can be considered as a function in $L^1(v)$ (respectively $L_w^1(v)$) with the same corresponding relationships.

Define now $B_k := \{\omega \in \Omega : 0 \leqslant |g(\omega)| < k\}$, for $k \in \mathbb{N}$, and consider $(|g|\chi_{B_k})_{(k,I)}^I \in L^\infty(v_I) \subset L^{p'}(v_I)$. Then $|g|\chi_{B_k}\chi_{\Omega_I} \in L^{p'}(v)$. Clearly, $|g|\chi_{B_k}\chi_{\Omega_I} \uparrow |g|$ v-a.e.

We claim that the upwards directed system $(|g|\chi_{B_k}\chi_{\Omega_I})_{(k,I)}$ is a Cauchy system in $L^{p'}(v)$; in this case it is also convergent in norm to the suprema, that is convergent to g (see Theorem 100.8 in [24]) and then $g \in L^{p'}(v)$. Otherwise, there would exist a number $\varepsilon > 0$ and an increasing sequence $(|g|\chi_{B_k}\chi_{\Omega_{I_k}})_k$ in $(|g|\chi_{B_k}\chi_{\Omega_I})_{(k,I)}$ such that $\left\||g|\chi_{B_{k+1}}\chi_{\Omega_{I_{k+1}}} - |g|\chi_{B_k}\chi_{\Omega_{I_k}}\right\|_{L^{p'}(v)} > \varepsilon$ for all $k \in \mathbb{N}$, i.e. such that $\left\||g|\chi_{C_k}\right\|_{L^{p'}(v)} > \varepsilon$ where $C_k := (B_{k+1} \cap \Omega_{I_{k+1}}) \setminus (B_k \cap \Omega_{I_k})$ (note that $C_k \neq \emptyset$). Let $f_k := \|g\|_{L_w^{p'}(v)}^{-\frac{p'}{p}}|g|^{p'-1}\chi_{C_k}$

$\in B_{L^p(v)}$. Then $\|M_g(f_k)\|_{L^1(v)} \to 0$ as $k \to \infty$ due to the $\mathscr{L}$-weakly compactness of M_g, but

$$M_g(f_k) = g|g|^{p'-1}\|g\|_{L_w^{p'}(v)}^{-\frac{p'}{p}}\chi_{C_k} = \text{sign}(g)|g|^{p'}\|g\|_{L_w^{p'}(v)}^{-\frac{p'}{p}}\chi_{C_k},$$

and hence $\|M_g(f_k)\|_{L^1(v)} = \||g|^{p'}\chi_{C_k}\|_{L^1(v)}\|g\|_{L_w^{p'}(v)}^{-\frac{p'}{p}}$. Therefore

$$\||g|\chi_{C_k}\|_{L^{p'}(v)} = \|M_g(f_k)\|_{L^1(v)} \cdot \|g\|_{L_w^{p'}(v)}^{\frac{p'}{p}} \to 0$$

as $k \to \infty$, that gives a contradiction.

Clearly, $(3) \Rightarrow (5)$ since $L^1(v)$ is continuously included in $L_w^1(v)$ and the implication $(5) \Rightarrow (6)$ follows by the same argument as the one used to prove $(3) \Rightarrow (4)$. We will show now that $(6) \Rightarrow (4)$. Assume that $M_g \in \mathscr{L}(L^p(v), L_w^1(v))$. In particular, $M_g \in \mathscr{B}(L^p(v), L_w^1(v))$. Theorem 19 yields that $M_g \in \mathscr{B}(L^p(v), L^1(v))$ and then $M_g(L^p(v)) \subset L^1(v)$ which gives $M_g \in \mathscr{L}(L^p(v), L^1(v))$. We already have the equivalences (1) to (6).

Since every $\mathscr{L}$-weakly compact operator is weakly compact, $(3) \Rightarrow (7)$. Again, since $L^1(v) \subset L_w^1(v)$ with equal norms, $(7) \Rightarrow (9)$. The same argument that proves $(3) \Rightarrow (4)$ gives $(9) \Rightarrow (10)$. $(10) \Rightarrow (8)$ can be proved in the same way that $(6) \Rightarrow (4)$. Only $(8) \Rightarrow (1)$ is needed to close the chain. Let us see this. Suppose $M_g \in \mathscr{W}(L^p(v), L^1(v))$, then $M_g \in \mathscr{B}(L^p(v), L^1(v))$ and $g \in L_w^{p'}(v)$. Let $A_k := \{\omega \in \Omega : k-1 \leqslant |g(\omega)|^{p'} < k\}$, for $k \in \mathbb{N}$, and consider $(|g|^{p'}\chi_{A_k})_k^I \in L^\infty(v_I) \subset L^1(v_I)$ (we follow the notation in the proof of $(4) \Rightarrow (1)$). Then $|g|^{p'}\chi_{A_k}\chi_{\Omega_I} \in L^1(v)$. Define

$$S_{(n,I)} := \sum_{k=1}^{n} \int_{\Omega} |g|^{p'}\chi_{A_k}\chi_{\Omega_I} \, dv = \int_{\Omega} g(\text{sign}(g)) \sum_{k=1}^{n} |g|^{p'-1}\chi_{A_k}\chi_{\Omega_I} \, dv.$$

If we write $f_{(n,I)} := \text{sign}(g)\sum_{k=1}^{n} |g|^{p'-1}\chi_{A_k}\chi_{\Omega_I}$ (note that $f_{(n,I)} \in L^p(v)$), we have $S_{(n,I)} = \int_{\Omega} gf_{(n,I)} \, dv = I_v \circ M_g(f_{(n,I)})$. The ideal property of the weakly compact operators gives that $S_{(n,I)} \in \mathscr{W}(L^p(v), X)$. Since $|f_{(n,I)}|^p \leqslant |g|^{p'}$, we have that $\|f_{(n,I)}\|_{L^p(v)} \leqslant \|g\|_{L_w^{p'}(v)}^{\frac{p'}{p}}$, and hence $(f_{(n,I)})_{(n,I)} \subset \|g\|_{L_w^{p'}(v)} \cdot B_{L^p(v)}$; that is $(f_{(n,I)})_{(n,I)}$ is included in a multiple of $B_{L^p(v)}$. Therefore, $(S_{(n,I)})_{(n,I)}$ is contained in a relatively weakly compact subset of X. Consequently, there exists $(\widetilde{S}_{(n,I)})_{(n,I)} \subset (S_{(n,I)})_{(n,I)}$ weakly convergent to some $x_0 \in X$. On the other hand, recall that each weakly v-integrable function has an integral belonging to X^{**} (this fact can be easily proved following the same arguments as in Corollary 3 and definitions in page 224 in [23]). So there is an element $x_0'' \in X^{**}$ such that for every $x^* \in X^*$

$$x^*(S_{(n,I)}) = \int_{\Omega} gf_{(n,I)} \, dx^*v \to \int_{\Omega} |g|^{p'} \, dx^*v = x^*(x_0'')$$

due to the order continuity of $L^1(|x^*v|)$. Hence $(S_{(n,I)})_{(n,I)}$ converges in the weak* topology of X^{**} to x_0''. Since the weak* topology of X^{**} coincides in X with the weak

topology of X, we can take $x_0 := x_0'' \in X$. This assures the existence of $x_0 \in X$ such that $x^*(x_0) = \int_\Omega |g|^{p'} dx^* v$. So the second condition in the definition of $L^1(v)$ is verified and we conclude the result. $\quad\square$

REMARK 22. Following the results in [8], the previous theorem can be extended to the corresponding cases of semi-compact and $\mathcal{M}$-weakly compact operators. For the definitions we refer to [19, Definition 3.6.9] and for the proof check Theorem 7 in [8].

5. L^p and L_w^p as intermediate spaces

It is well-known that in the case of σ-finite measures, the inclusions $L^1(\mu) \cap L^\infty(\mu) \subset L^p(\mu) \subset L^1(\mu) + L^\infty(\mu)$ substitute for many purposes the inclusions $L^\infty(\mu) \subset L^p(\mu) \subset L^1(\mu)$ that hold for finite measures. To finish the paper, in this section we analyze the inclusion between the spaces $L^1(v) \cap L^\infty(v)$, $L^p(v)$ and $L^1(v) + L^\infty(v)$, and also for the corresponding weak spaces.

PROPOSITION 23. *Let* $1 < p \leqslant \infty$. *Then the following (continuous) inclusions hold.*

1. $L_w^1(v) \cap L^\infty(v) \subset L_w^p(v) \subset L_w^1(v) + L^\infty(v)$.

2. $L^1(v) \cap L^\infty(v) \subset L^p(v) \subset L^1(v) + L^\infty(v)$.

Proof. (1) Consider the Banach lattices $L_w^1(v) \cap L^\infty(v)$ and $L_w^1(v) + L^\infty(v)$ with the v-a.e. order and the usual lattice norms

$$\|f\|_{L_w^1(v) \cap L^\infty(v)} = \max\{\|f\|_{L_w^1(v)}, \|f\|_\infty\}, \text{ and}$$

$$\|h\|_{L_w^1(v) + L^\infty(v)} = \inf\{\|f\|_{L_w^1(v)} + \|g\|_\infty : h = f + g, f \in L_w^1(v), g \in L^\infty(v)\},$$

respectively.

For every $f \in L_w^1(v) \cap L^\infty(v)$ we have that $|f(\omega)| \leqslant \|f\|_\infty \leqslant \|f\|_{L_w^1(v) \cap L^\infty(v)}$ for v-almost all $\omega \in \Omega$, so $\|f\|_{L_w^1(v) \cap L^\infty(v)}^{-1} |f(\omega)| \leqslant 1$, v-a.e and then

$$\|f\|_{L_w^1(v) \cap L^\infty(v)}^{-p} |f(\omega)|^p \leqslant \|f\|_{L_w^1(v) \cap L^\infty(v)}^{-1} |f(\omega)|, \quad v-\text{a.e.}$$

Hence, $|f(\omega)|^p \leqslant \|f\|_{L_w^1(v) \cap L^\infty(v)}^{p-1} \cdot |f(\omega)|$, v-a.e. The ideal property of $L_w^1(v)$ yields that $|f|^p \in L_w^1(v)$, and therefore $f \in L_w^p(v)$.

On the other hand, let $0 \leqslant f \in L_w^p(v)$. For an arbitrary $\varepsilon > 0$ define the measurable set $A_\varepsilon := \{\omega \in \Omega : f(\omega) > \varepsilon\}$. Note that if $\omega \in A_\varepsilon$, then $f(\omega)^p > \varepsilon^p \chi_{A_\varepsilon}$. Therefore,

$$\infty > \sup_{x^* \in B_{X^*}} \int_\Omega |f|^p d|x^* v| \geqslant \varepsilon^p \sup_{x^* \in B_{X^*}} \int_\Omega \chi_{A_\varepsilon} d|x^* v| = \varepsilon^p \|\chi_{A_\varepsilon}\|_{L_w^1(v)} = \varepsilon^p \|v\|(A_\varepsilon),$$

that is $\|v\|(A_\varepsilon) < \infty$. In particular, $\chi_{A_\varepsilon} \in L_w^{p'}(v)$. Write $f = f\chi_{A_\varepsilon} + f\chi_{\Omega \setminus A_\varepsilon}$. Clearly, $f\chi_{\Omega \setminus A_\varepsilon} \in L^\infty(v)$. Moreover, by Lemma 13, $L_w^p(v) \cdot L_w^{p'}(v) = L_w^1(v)$ with p, p' conjugate exponents. Hence, $f\chi_{A_\varepsilon} \in L_w^1(v)$ and we conclude that $f \in L_w^1(v) + L^\infty(v)$. The extension to the general case is routine. Finally, the continuity of the inclusions is clear, since it is a positive operator between Banach lattices. The inclusions $L^1(v) \cap L^\infty(v) \subset L^p(v) \subset L^1(v) + L^\infty(v)$ in (2) follow by the same arguments, using in this case the identification $L^p(v) \cdot L_w^{p'}(v) = L^1(v)$ for proving the second one. $\square$

In general these relations cannot be improved by changing spaces of weakly integrable functions by spaces of integrable functions. The inclusions $L_w^1(v) \cap L^\infty(v) \subset L^p(v)$ and $L_w^p(v) \subset L^1(v) + L^\infty(v)$ fail sometimes, as the following examples show.

EXAMPLE 24. Let $\Gamma = (0, \infty)$ and consider the δ-ring $\mathscr{R}$ of the finite subsets of Γ. Let $1 < p < \infty$ and $v : \mathscr{R} \to c_0(\Gamma)$ be the vector measure given by $v(A) := \sum_{\gamma \in A} \chi_{\{\gamma\}}$. The corresponding spaces of integrable functions can be calculated easily and are $L^1(v) = L^p(v) = c_0(\Gamma)$ and $L_w^1(v) = \ell^\infty(\Gamma)$ (see Example 9, Example 12 and [10, Example 2.2]). Note also that $L^\infty(v) = \ell^\infty(\Gamma)$. Therefore $L_w^1(v) \cap L^\infty(v) = \ell^\infty(\Gamma) \not\subset c_0(\Gamma) = L^p(v)$.

EXAMPLE 25. Consider an uncountable index set I and a family of disjoint non atomic probability spaces $(\Omega_i, \Sigma_i, \mu_i)_{i \in I}$. Consider the vector measure $v : \mathscr{R} \to c_0(I)$ constructed as the one in Remark 14. Let $1 < p < \infty$. The spaces $L^1(v)$ and $L_w^p(v)$ can be identified with the spaces $\bigoplus_{c_0(I)} L^1(\mu_i)$ and $\bigoplus_{\ell^\infty(I)} L^p(\mu_i)$, respectively, and the space $L^\infty(v)$ is $\bigoplus_{\ell^\infty(I)} L^\infty(\mu_i)$. Take an element of $L_w^p(v)$ defined by a set of functions $(f_i)_{i \in I}$, each f_i with support in Ω_i, with $0 < f_i \in L^p(\mu_i) \setminus L^\infty(\mu_i)$ for all $i \in I$ and $\sup_{i \in I} \|f_i\|_{L^p(\mu_i)} < \infty$. Then it cannot be written as a sum of elements of $L^1(v)$ and $L^\infty(v)$ since the elements of $L^1(v)$ are 0 in each Ω_i except in a countable subset of indexes of I. Moreover, the functions in $\{f_i : i \in I\}$ are not essentially bounded, so f_i does not belong to $L^\infty(\mu_i)$ for any i. Consequently, $L_w^p(v) \not\subset L^1(v) + L^\infty(v)$.

However, an improvement is still possible in the right hand side inclusion by defining a new space. For this aim, denote

$$L_{w,0}^1(v) := \overline{L_w^1(v) \cap L^\infty(v)}^{L_w^1(v)} \subset L_w^1(v).$$

Remark that $L^1(v) \subset L_{w,0}^1(v)$ since $\mathscr{S}(\mathscr{R}) \in L_w^1(v) \cap L^\infty(v)$ and $\mathscr{S}(\mathscr{R})$ is a dense set in $L^1(v)$. We claim that,

THEOREM 26. Let $1 < p < \infty$. Then the (continuous) inclusion $L_w^p(v) \subset L_{w,0}^1(v) + L^\infty(v)$ holds.

Proof. Let $0 \leqslant f \in L_w^p(v)$ and fix $\varepsilon > 0$. Consider again the sets $A_\varepsilon := \{\omega \in \Omega : f(\omega) > \varepsilon\}$ of the previous proof, and recall that $\|v\|(A_\varepsilon) < \infty$ and $f = f\chi_{A_\varepsilon} + f\chi_{\Omega \setminus A_\varepsilon} \in L_w^1(v) + L^\infty(v)$. Define now for every $n \in \mathbb{N}$, $B_{\varepsilon,n} := \{\omega \in A : f(\omega) \leqslant n\} \in$

$\mathscr{R}^{loc}$ and $f_n := f\chi_{B_{\varepsilon,n}}$. Note that $\|v\|(B_{\varepsilon,n}) < \infty$. Then $f_n \in L_w^1(v) \cap L^\infty(v)$. We claim that $\|f\chi_{A_\varepsilon} - f_n\|_{L_w^1(v)} \to 0$ as $n \to \infty$. To see this, use *Hölder's Inequality*

$$\|f\chi_{A_\varepsilon} - f_n\|_{L_w^1(v)} = \|f\chi_{A_\varepsilon}(\chi_{A_\varepsilon} - \chi_{B_{\varepsilon,n}})\|_{L_w^1(v)} \leqslant \|f\|_{L_w^p(v)} \cdot \|\chi_{A_\varepsilon} - \chi_{B_{\varepsilon,n}}\|_{L_w^{p'}(v)}.$$

Let $C_n := A_\varepsilon \setminus B_{\varepsilon,n}$, $n \in \mathbb{N}$. Let us see that $(C_n)_n \downarrow \bigcap_n C_n$ and $\|\chi_{C_n}\|_{L_w^{p'}(v)} \to 0$ as $n \to \infty$. Otherwise, there would exist a number $\delta > 0$ such that $\|v\|(C_n) > \delta$ for an infinite subset M of $\mathbb{N}$. Since $\|v\|(C_n) = \sup_{C \in \mathscr{R} \cap 2^{C_n}} \|v\|(C)$ (see Lemma 3.4 in [18]) there would exist also $C_{n,\delta} \subset C_n$ with $C_{n,\delta} \in \mathscr{R}$ such that $\|v\|(C_{n,\delta}) > \delta$, $n \in M$. But $n\chi_{C_{n,\delta}} \leqslant f\chi_{C_{n,\delta}}$ for all $n \in \mathbb{N}$, thus $n\delta^{\frac{1}{p}} < n\|\chi_{C_{n,\delta}}\|_{L_w^p(v)} \leqslant \|f\|_{L_w^p(v)} < \infty$, which is a contradiction.

Consequently, $\|f\chi_{A_\varepsilon} - f_n\|_{L_w^1(v)} \to 0$ as $n \to \infty$, hence $f\chi_{A_\varepsilon} \in L_{w,0}^1(v)$. The extension to non positive functions is clear. $\square$

REMARK 27. Consider the case where the vector measure is defined on a σ-algebra. Then, $L_w^1(v) \cap L^\infty(v) = L^\infty(v) \subset L^1(v)$ which is closed in $L_w^1(v)$. Hence, $L_{w,0}^1(v) = L^1(v)$ and the inclusion in the theorem gives $L_w^p(v) \subset L^1(v)$. Therefore, Theorem 26 is a generalization of Proposition 3.1 in [8].

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