Operators and Matrices
Volume 6, Number 2 (2012), 271–278 doi:10.7153/oam-06-19

GEOMETRIC THEORY OF WEAK KOROVKIN SETS

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Abstract. A set S of generators of an abstract W^*-algebra $\mathscr{A}$ is called weakly hyperrigid if for every faithful representation of $\mathscr{A}$ on a Hilbert space $\mathscr{A} \subseteq B(H)$ and every net $\{\Phi_\alpha\}_{\alpha\in I}$ of completely positive maps, of norm $\leqslant 1$, on $B(H)$,

$$\lim_\alpha \Phi_\alpha(s) = s \text{ weakly, for all } s \in S \Rightarrow \lim_\alpha \Phi_\alpha(a) = a \text{ weakly},$$

for all $a \in \mathscr{A}$. This is analogous to W. B. Arveson's [6] hyperrigid sets in C^* algebras. A characterisation of weakly hyperrigid sets and a noncommutative analogue of Y. Saskin's theorem [17] on geometric characterisation of Korovkin sets in $C(X)$ is proved.

1. Introduction

The classical theorem of P. P. Korovkin [8] in 1953 unified many approximation processes such as Bernstein polynomial approximation of continuous functions. Korovkin's celebrated theorem is as follows:

1.1. Theorem

Let $\{\Phi_n\}$ be a sequence of positive linear maps on $C[a,b]$ and for each of the functions $g_k(x) = x^k$, $x \in [a,b]$, $k = 0,1,2$, let

$$\lim_{n\to\infty} \Phi_n(g_k) = g_k \text{ uniformly on } [a,b],\ k = 0,1,2.$$

Then

$$\lim_{n\to\infty} \Phi_n(f) = f \text{ uniformly on } [a,b], \text{ for all } f \in C[a,b].$$

The above theorem leads to Korovkin-type theorems and Korovkin sets in various settings such as more general function spaces, Banach algebras, Banach lattices and operator algebras. A detailed survey of these developments can be found in the monograph of F. Altomare and M. Campiti [1],and the survey article of F. Altomare [2] which contains several new results also. Sets such as $\{g_0, g_1, g_2\}$ are called test sets or Korovkin sets. Another important discovery was the geometric formulation of Korovkin's theory by Y. Saskin in 1966 [17]. An excellent article by H. Berens and G. G. Lorentz in 1975 [7] contains these developments. Let us recall the definition of Choquet boundary for function systems, the main tool of Saskin's formulation.

Mathematics subject classification (2010): Primary 46L07; Secondary 46L52.
Keywords and phrases: Korovkin, W^*-algebras, completely positive maps.

Paper OaM-06-19

1.2. Definition

Let $S \subset C(X)$ containing the constant function 1, where X is a compact Hausdorff space. The Choquet boundary ∂S of S is defined as

$$\partial S = \{x \in X : \varepsilon_x|_S \text{ has a unique positive linear extension to } C(X), \text{ where } \varepsilon_x \text{ denotes the evaluation functional defined by } \varepsilon_x(f) = f(x),\ f \in C(X)\}.$$

1.3. Saskin's theorem

Let $S \subset C(X)$ be a subset of $C(X)$ that contains the constant function and separates points of X. Let

$$K(S) = \big\{f \in C(X) : \Phi_n(s) \to s, \text{ uniformly on } X, \forall s \in S \Longrightarrow \Phi_n(f) \to f \text{ uniformly on } X \text{ for every sequence } \{\Phi_n\} \text{ of positive linear maps on } C(X) \text{ such that } \|\Phi_n\| \leqslant 1,\ n = 1,2,\ldots\big\}.$$

Then $K(S) = C(X)$ if and only if the Choquet boundary $\partial S = X$.

Theorem 1.3 was extended for contractive linear maps on $C(X)$ by D. E. Wulbert [20]. The noncommutative analogue of the formulation was obtained by W. B. Arveson in 2009 [6] using his theory of noncommutative Choquet boundary for unital completely positive (UCP) linear maps on C^*-algebras. Now we recall the definition non commutative Choquet boundary:

Let $\mathscr{A}$ be a C^*-algebra with identity $1_{\mathscr{A}}$ and S a subset of $\mathscr{A}$ containing $1_{\mathscr{A}}$. An irreducible representation $\pi : \mathscr{A} \to B(H)$, H a complex Hilbert space, is called a *boundary representation* for S if $\pi|_{span(S)}$ has a unique *completely positive* linear extension to $\mathscr{A}$, namely π itself. The set of unitary equivalence classes of all boundary representations of S is called the *noncommutative Choquet boundary* ∂S of S [5].

For simplicity we assume that all C^* algebras considered here contains the identity, unless mentioned specifically. An *operator system* in a C^*-algebra $\mathscr{A}$ is a subset S of $\mathscr{A}$ containing $1_{\mathscr{A}}$ such that $C^*(S) = \mathscr{A}$, where $C^*(S)$ denotes the C^*-algebra generated by $\mathscr{A}$. Arveson called *Korovkin* sets in $\mathscr{A}$ for completely positive maps as *hyperrigid sets*. A finite or countably infinite set G of generators of a C^*-algebra $\mathscr{A}$ is said to be *hyperrigid* if every faithful representation of $\mathscr{A} \subseteq B(H)$, H a separable Hilbert space, and for every sequence of identity preserving completely (UCP) positive maps $\Phi_n : B(H) \to B(H), n = 1,2,3,\ldots,$

$$\lim_{n\to\infty} \|\Phi_n(g) - g\| = 0,\ \forall g \in G \Longrightarrow \lim_{n\to\infty} \|\Phi_n(a) - a\| = 0\ \forall a \in \mathscr{A}.$$

The purpose of this short paper is to introduce the notion of *weak hyperrigid* sets in W^*-algebras and to obtain an analogue of the characterization theorem due to Arveson.

1.4. Theorem (Arveson)

For every separable operator system S, that generates a C^*-algebra $\mathscr{A}$, the following are equivalent.

(i) S is hyperrigid.

(ii) For every nondegenerate representation $\pi : \mathscr{A} \to B(H)$ on a separable Hilbert space H and every sequence $\Phi_n : \mathscr{A} \to B(H)$ of UCP maps

$$\lim_{n\to\infty} \|\Phi_n(s) - \pi(s)\| = 0, \quad \forall s \in S \Rightarrow$$
$$\lim_{n\to\infty} \|\Phi_n(a) - \pi(a)\| = 0, \quad \forall a \in \mathscr{A}.$$

(iii) For every nondegenerate representation $\pi : \mathscr{A} \to B(H)$ on a separable Hilbert space H, $\pi|_S$ has unique UCP extension to $\mathscr{A}$, namely π itself.

(iv) For every unital C^*-algebra $\mathscr{B}$, every unital homomorphism $\theta : \mathscr{A} \to \mathscr{B}$ and every UCP map $\Phi : \mathscr{B} \to \mathscr{B}$,

$$\Phi(x) = x\ \forall x \in \theta(S) \Longrightarrow \Phi(x) = x \forall x \in \theta(\mathscr{A}).$$

2. Korovkin-type theorems for weak convergence and weak hyperrigid sets

2.1. Definition

A subset S of a W^*-algebra $\mathscr{A}$ containing identity $1_{\mathscr{A}}$ is called weakly hyperrigid if

(i) $\mathscr{A}$ equals the W^*-algebra $W^*(\mathscr{S})$ generated by S, and

(ii) for every faithful representation of $\mathscr{A} \subseteq B(H)$, H a separable Hilbert space and for every net of contractive completely postive map $\Phi_\alpha : B(H) \to \mathscr{B}(H)$,

$$\lim_\alpha \Phi_\alpha(s) = s \text{ weakly } \forall s \in S \Longrightarrow \lim_\alpha \Phi_\alpha(a) = a \text{ weakly } \forall a \in \mathscr{A}.$$

Before identifying examples of weakly hyperrigid sets, we prove the following characterisation of weak hyperrigidity. The main tool is Krein Millman theorem for compact convex sets. Here the compact convex set shall be the set of all contractive completely positive maps equipped with B-W topology [3]. It is known that a net $\{\Phi_\alpha\}_{\alpha\in I}$ of contractive completely positive maps converges to Φ in the B-W topology iff $\{\Phi_\alpha(T)\}$ converges in the weak operator topology for each operator T in the domain of Φ_α, $\alpha \in I$. This also implies that Φ is a contractive completely positive map. However the proof technique is more or less similar to that of Arveson [5] except for the use of Krein Millmann theorem.

2.2. Theorem

Let S be a $*$ closed operator system in a W^*-algebra $\mathscr{A}$. Then the following statements are equivalent.

(i) S is weakly hyperrigid.

(ii) For every nondegenerate representation $\pi : \mathscr{A} \to B(H)$, on a separable Hilbert space H and every net $\{\Phi_\alpha\}_{\alpha \in I}$ of contractive completely positive maps from $\mathscr{A}$ to $B(H)$

$$\lim_\alpha \Phi_\alpha(s) = \pi(s) \text{ weakly } \forall s \in S \Longrightarrow \lim_\alpha \Phi_\alpha(a) = \pi(a) \text{ weakly } \forall a \in \mathscr{A}.$$

(iii) For every non degenerate representation $\pi : \mathscr{A} \to \mathscr{B}(H)$ on a separable Hilbert space H, $\pi|_S$ has the unique extension property, i.e. $\pi|_S$ has unique completely positive extension to $\mathscr{A}$ namely π itself.

*(iv) For every W^* algebra $\mathscr{B}$, *homomorphism $\theta : \mathscr{A} \to \mathscr{B}$ such that $\theta(1_A) = 1_B$ and every contractive completely positive map $\Phi : \mathscr{B} \to \mathscr{B}$,*

$$\Phi(x) = x \quad \forall x \in \theta(S) \Rightarrow \Phi(x) = x \quad \forall x \in \theta(\mathscr{A}).$$

Proof. (i) $\Rightarrow$ (ii): Let $\pi : \mathscr{A} \to B(H)$ be a nondegenerate representation on a Hilbert space H and $\Phi_\alpha : \mathscr{A} \to B(H)$ be completely positive maps, $\|\Phi_\alpha\| \leqslant 1$ for all $\alpha \in I$ such that

$$\lim_\alpha \Phi_\alpha(s) = \pi(s) \text{ weakly, for all } s \in S.$$

Let $\sigma : \mathscr{A} \to B(K)$ be a faithful representation of $\mathscr{A}$ on a Hilbert space K. Then $\sigma \oplus \pi : \mathscr{A} \to B(K \oplus H)$ is a faithful representation. Let $\omega_\alpha : (\sigma \oplus \pi)(\mathscr{A}) \to B(K \oplus H)$ be defined as

$$\omega_\alpha(\sigma(a) \oplus \pi(a)) = \sigma(a) \oplus \Phi_\alpha(a), \; a \in \mathscr{A}.$$

Then ω_α is a completely positive map with $\|\omega_\alpha\| \leqslant 1$ for all $\alpha \in I$. Also

$$\lim_\alpha \omega_\alpha(\sigma(s) \oplus \pi(s)) = \sigma(s) \oplus \pi(s) \text{ weakly for all } s \in S.$$

Therefore, for all $x, y \in H$

$$\begin{aligned}
&\left|\langle \Phi_\alpha(a)x, y\rangle - \langle \pi(a)x, y\rangle\right| \\
&= \left|\langle \sigma(a) \oplus \Phi_\alpha(a)(0 \oplus x), 0 \oplus y\rangle - \langle \sigma(a) \oplus \pi(a)(0 \oplus x), 0 \oplus y\rangle\right| \\
&= \left|\langle \omega_\alpha(\sigma(a) \oplus \pi(a)(0 \oplus x), 0 \oplus y\rangle - \langle \sigma(a) \oplus \pi(a)(0 \oplus x), 0 \oplus y\rangle\right|.
\end{aligned}$$

Therefore $\lim_\alpha \Phi_\alpha(a) = \pi(a)$ weakly for every $a \in \mathscr{A}$.

Now (ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (iv): Exactly the same proof as in [6, Theorem 2.1]. So we omit it.

(iv) $\Rightarrow$ (i): Let $\{\Phi_\alpha\}_{\alpha \in I}$ be a net of completely positive maps from $B(H)$ such that

$$\lim_\alpha \Phi_\alpha(s) = s \text{ weakly for every } s \in S.$$

Let Φ be a limit point of $\{\Phi_\alpha\}_{\alpha \in I}$ in the B. W. topology. Therefore Φ is a completely positive map on $B(H)$ such that $\|\Phi\| \leqslant 1$ and $\Phi(s) = s$ for all $s \in S$. Hence $\Phi(a) = a$ for all $a \in \mathscr{A}$ by (iv).

Therefore

$$\lim_\alpha \Phi_\alpha(a) = a, weakly \quad \forall a \in \mathscr{A}. \qquad \square$$

2.3. Remarks

It is to be noted that to establish theorem 1.4 the separability of the C^*-algebra as well as the representing Hilbert space was crucial. However to deal with characterisation Theorem 2.2 for weak hyperrigidity such assumptions are not needed. Recall that the *spectrum of a C^*-algebra* is defined as the set of all *unitary equivalence classes of all irreducible representations* of $\mathscr{A}$. If $\mathscr{A}$ is $C(X)$ then every irreducible representation is one dimentional, where X is a compact Hausdorff space .Therefore the spectrum of $C(X)$ reduces to a singleton set.

For hyperrigid sets Arveson proved the following theorem.

2.4. Theorem

[6, Theorem 5.1]

Let S be a separable operator system whose generated C^*-algebra has countable spectrum, such that every irreducible representation of $\mathscr{A}$ is a boundary representation for S. Then S is hyperrigid.

Though separability condition is not needed the case of weak hyperrigidity, we present the following conjecture.

2.5. Conjecture

Let S be a separable operator system in a W^* algebra $\mathscr{A}$ whose generated C^* algebra has countable spectrum, such that every irreducible representation of $\mathscr{A}$ is a boundary representation for S. Then S is weakly hyperrigid. A partial answer to the above problem puts further assumptions on S.

2.6. Theorem

Let S be a separable operator system in a W^* algebra $\mathscr{A}$ such that $C^*(S)$ has countable spectrum and every irreducible representation of $\mathscr{A}$ is boundary representation. Also, assume that every irreducible representation of $C^*(S)$ has a unique extreme completely positive extention to $\mathscr{A}$. Then S is weakly hyperrigid.

Proof. It is enough to show that if π is a representation of $\mathscr{A}$ on $B(H)$ then $\pi|_S$ has a unique completely positive extension to $\mathscr{A}$. We have,as in the proof of [6, Theorem 5.1,p.13]

$$\pi|_{C^*(S)} = \oplus \pi_n,$$

where π_n's are representations of $C^*(S)$ on Hilbert space H_n, such that $H = \oplus H_n$. Also each π_n is unitary equivalent to a multiple of an irreducible representation ρ_n of $C^*(S)$ on H_n. Let $\tilde{\rho}_n$ be the unique extreme completely positive extension of ρ_n to A.

Let

$$\Omega = \{\Phi \in CP(\mathscr{A}, B(H)) : \Phi|_{C^*(S)} = \rho_n\}.$$

Here $CP(\mathscr{A}, B(H))$ denote the set of all completely positive maps from $\mathscr{A}$ to $B(H)$. Then Ω is a compact convex set in the B-W topology. We show that $\Omega = \{\tilde{\rho}_n\}$. Let Φ be an extreme point in Ω. We show that Φ is extreme in $CP(\mathscr{A}, B(H))$. Let Φ_1 and Φ_2 in $CP(\mathscr{A}, B(H))$ be such that

$$\alpha\Phi_1 + (1-\alpha)\Phi_2 = \Phi \text{ for some } \alpha \in [0,1].$$

Then

$$\alpha\Phi_1|_{C^*(S)} + (1-\alpha)\Phi_2|_{C^*(S)} = \Phi|_{C^*(S)} = \rho_n.$$

Since ρ_n is irreducible, it is extreme in $CP(\mathscr{A}, B(H))$. Hence

$$\Phi_1|_{C^*(S)} = \Phi_2|_{C^*(S)} = \rho_n \Rightarrow \Phi_i \in \Omega,$$

for $i = 1, 2$.

$$\Rightarrow \Phi_1 \equiv \Phi_2 \equiv \Phi$$

Therefore, Φ is extreme in $CP(\mathscr{A}, B(H))$.

Hence $\tilde{\rho}_n|_S$ is an irreducible boundary representation for S. Therefore, $\pi|_S$ has a unique completely positive extension to $\mathscr{A}$. Hence, by Theorem 2.2, S is weakly hyperrigid. $\square$

As an immediate consequence of Theorem 2.6,we present below a noncommutative analogue of Saskin's theorem for operator systems satisfying conditions mentioned in the above theorem.Being a straight implication ,the proof is omitted.

2.7. Corollary

Let S be an operator system in the W^* algebra $\mathscr{A}$ with identity $1_{\mathscr{A}}$ in S satisfying conditions of Theorem 2.6. Then S is weakly hyperrigid if and only if $\partial(S) = \hat{\mathscr{A}}$, where $\partial(S)$ is the noncommutative Choquet boundary of S and $\hat{\mathscr{A}}$ is the spectrum of $\mathscr{A}$.

3. Examples of weakly hyperrigid sets

The relation of boundary representation and weak Korovkin sets are already known in the paper of B. V. Limaye and M. N. N. Namboodiri, 1984 [10]. Let us recall the following theorem [10, 3.2, p.205].

3.1. Theorem

Let S be an irreducible set in $B(H)$ such that S contains the identity operator I and $C^*(S)$ contains a non zero compact operator. Then S is a weak Korovkin set in $B(H)$ if and only if $id|_S$ has a unique completely positive linear extension to $C^*(S)$ namely $id|_{C^*(S)}$.

The famous boundary theorem of Arveson [4] implies the following Corollary [10] of it.

3.2. Corollary

Let S be an irreducible set in $B(H)$ which contains the identity operator I. Suppose that there are $T \in$ span $\{S+S^*\}$ and a compact operator K in $B(H)$ such that $\|T-K\| < \|T\|$. Then S is a weak Korovkin set in $B(H)$.

The following example is given in [10].

3.3. Example

Let T be an irreducible operator which is almost normal (i.e., $T^*T - TT^*$ is a compact operator), but not normal (i. e. $T^*T - TT^* \neq 0$). Then the set $S = \{I, T, T^*T + TT^*\}$ is a weak Korovkin set.

3.4. Remark.

If H is separable, then one can show that S in example 3.3 is weakly hyperrigid. This is because in this case every faithful representation of $B(H)$ is continuous in the weak operator topology. However, the noncommutative Choquet boundary of S contains only the identity representation upto unitary equivalence whereas the spectrum of $B(H)$ contains more elements.

Acknowledgement

The author wish to thank Prof. B. V. Limaye for constant inspiration. The author also wish to thank the organizers of the ICM Satellite Conference held at ISI, Bangalore, Aug. 7–10, 2010.

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(Received December 31, 2010)

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