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QUASILINEAR MAPPINGS, M–IDEALS AND POLYHEDRA

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Abstract. We survey the connection between two results from rather different areas: failure of the 3-space property for local convexity (and other properties) within the category of quasi-Banach spaces, and the irreducibility (in the sense of Minkowski difference) of large families of finite dimensional polytopes.

1. Introduction

We all know that every closed subspace of a Hilbert space is *complemented*, i.e. the range of a continuous linear projection. Curiously perhaps, the proof of this is non-linear. The projection is the closest point mapping, which in most Banach spaces is non-linear (and often ill-defined). In Hilbert spaces, some work is needed to establish its linearity, although the nonlinear identity $\|x\|^2 = \|Px\|^2 + \|x - Px\|^2$ is fairly obvious.

Our first result, due to Enflo, Lindenstrauss and Pisier [5], was therefore surprising when it was published in 1975.

THEOREM 1. *There is a Banach space X with an uncomplemented subspace H such that both H and X/H are isomorphic to Hilbert spaces.*

Finite dimensional subspaces of Banach spaces are always complemented, so this example is very much an infinite-dimensional phenomenon. On the other hand, the next result [22, Theorem 11] is very finite-dimensional.

THEOREM 2. *If P is an n-dimensional polytope without a centre of symmetry, then the difference set $P - P$ has at least $4n$ vertices, or is a hexagon.*

These two results are actually related, and this paper tries to explain their connection. It gives the history of how some results from the isomorphic theory of Banach and quasi-Banach spaces led to some new results in convex geometry.

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2. Quasilinear mappings

In 1978, Kalton [10, Theorem 4.6] and Ribe [19] independently published the following.

THEOREM 3. *There is a topological vector space X containing an uncomplemented one-dimensional subspace L such that $X/L \cong \ell_1$.*

Although X cannot be locally convex, it is completely metrizable, in fact a quasi-Banach space. Its topology is induced by a quasi-norm, i.e. a positively homogeneous functional satisfying this weak version of the triangle inequality

$$\|x+y\| \leqslant K(\|x\|+\|y\|)$$

where K is a constant. In this example, K can be chosen arbitrarily close to 1.

So, a common property of a subspace and a quotient space need not be shared by the whole space. In contemporary language, being locally convex is not a 3-space property in the category of quasi-Banach spaces. Theorem 1 says that being isomorphic to a Hilbert space is not a 3-space property in the category of Banach spaces. The study of 3-space problems in functional analysis is now vast; we refer to [4] for a comprehensive introduction.

We shall see that quasi-Banach spaces turn out to be a more natural category to work in than Banach spaces, when studying twisted sums. We recall that a twisted sum of two quasi-Banach spaces Y and Z is any quasi-Banach space X containing a subspace isomorphic to Y, with $X/Y \cong Z$. Functional analysts sometimes call this an extension of Y by Z, although the longer established convention in homological algebra [8] would be to call it an extension of Z by Y.

A map $\Omega : Z \to Y$ between two quasi-Banach spaces is called quasilinear if it is homogeneous, and satisfies the inequality

$$\|\Omega(x)+\Omega(y)-\Omega(x+y)\| \leqslant K(\|x\|+\|y\|)$$

for some constant K. Such mappings have applications to several areas of mathematics, in particular partial differential equations and interpolation spaces, but we will only consider them in relation to twisted sums.

A quasinorm can be defined on the algebraic direct sum $Y \oplus Z$ by

$$\|(y,z)\|_\Omega = \|z\| + \|y-\Omega(z)\|.$$

It is routine to verify that $X = Y \oplus_\Omega Z$ is a quasi-Banach space, $Y \oplus \{0\}$ is a closed subspace isometric to Y, and $X/Y \cong Z$. It turns out that Y is complemented if and only if Ω is "close" to a linear map, i.e. if and only if $\sup_{\|z\|\leqslant 1} \|\Omega(z)-L(z)\|$ is finite for some linear map $L : Z \to Y$. The converse is less obvious, but it is true and useful to know that any twisted sum of Y and Z is isomorphic to $Y \oplus_\Omega Z$ for a suitable Ω. We refer to [2, Chapter 16] for a detailed explanation of these facts, and to [3] for an interesting discussion about different constructions.

Such mappings from $\ell_2(n) \to \ell_2(n^2)$ were in fact the basis of the construction in [5]. It was shown there that any projection from the resulting twisted sum onto the copy of $\ell_2(n^2)$ has norm at least some constant times $\sqrt{\log n}$. Piecing these twisted sums together gives Theorem 1, whose proof is ultimately finite dimensional.

Quasilinear maps from ℓ_1 to $\mathbb{R}$ were also the basis of the proof of Theorem 3. Kalton and Peck [12] first identified the special role that quasilinear maps were playing in the development of twisted sums, and elaborated the construction above.

From now on, we only consider the case when Y and Z are Banach spaces. Even then, X need not be. We have seen that imposing the most stringent conditions on the subspace does not help us in this regard. However, imposing some conditions on the quotient space does: X will be locally convex, i.e. isomorphic to a Banach space, if Z has type (in particular if Z is superreflexive) [11], or if $Z \cong c_0$ (or more generally any quotient of a $\mathscr{L}_\infty$ space) [13].

What about imposing conditions on Ω? Kalton [10] showed that X is locally convex (i.e. isomorphic to a Banach space) if and only if there is a constant K so that

$$\left\|\Omega(\sum_{i=1}^{n} z_i) - \sum_{i=1}^{n} \Omega(z_i)\right\| \leqslant K \sum_{i=1}^{n} \|z_i\|$$

for all finite collections $(z_i)_{i=1}^{n} \in Z$. (We remark that it is possible to define a sequence of different properties here, one for each value of n. The relationship between these properties does not seem to have been investigated.)

But why should we have to renorm? It is also reasonable to ask about conditions under which X is already a Banach space, i.e. $\|\cdot\|_\Omega$ is already a norm. Following a strong hint from Kalton, Lima and Yost [16] introduced the following definition.

A map $\Omega : Z \to Y$ between two Banach spaces is called *pseudolinear* if it is homogeneous, and satisfies the inequality

$$\|\Omega(x) + \Omega(y) - \Omega(x+y)\| \leqslant \|x\| + \|y\| - \|x+y\|$$

for all $x, y \in Z$.

The motivation for this definition is the straightforward fact that $\|\cdot\|_\Omega$ is a norm if and only if Ω is pseudolinear. In this case, we will call $X = Y \oplus_\Omega Z$ a semi-L-sum of Y and Z. The reason for this name is the intimate connection with the concept of semi-L-summands first defined by Lima [14]. Before defining them, we note that twisted sums and semi-L-sums behave very differently, despite the similarities in their construction. In particular, we are not aware of any uncomplemented semi-L-sums, although we see no reason why they should not exist. The following result [16, Proposition 10] reformulates this as a problem about pseudolinear mappings.

THEOREM 4. *Let $\Omega : Z \to Y$ be pseudolinear, and X the corresponding semi-L-sum. Then Y is complemented in X if, and only if, Ω can be decomposed in the form $\Omega = T + A$ where $T : Z \to Y$ is a linear map and $A : Z \to Y$ is continuous. This always holds if Y is complemented in Y^{**}, in particular if Y is reflexive.*

Recall that a Chebyshev subspace of a Banach space is one whose metric projection is single-valued. This means that for each point x in the larger space, there is a unique point $Px = P_Y x$ in the subspace Y which minimizes $\|x-y\|$ over all $y \in Y$. A Chebyshev subspace Y of X is called a semi-L-summand when the metric projection satisfies the identity $\|x\| = \|Px\| + \|x-Px\|$. Obviously every L-summand is a semi-L-summand, and in $L_1(\mu)$ spaces there are no other examples [14, Theorem 5.5]. However the subspace of constant functions in a real $C(K)$ space is a natural example of a semi-L-summand whose metric projection is not linear.

The next result [16, Theorem 7] shows that, as with twisted sums, the existence of a non-trivial semi-L-sum depends only on the quotient space.

THEOREM 5. *Given real Banach spaces Z and Y, the following are equivalent.*
(i) There is a proper semi-L-sum of $\mathbb{R}$ and Z.
(ii) There is a proper pseudolinear map $\Omega : Z \to \mathbb{R}$.
(iii) There is a proper pseudolinear map $\Omega : Z \to Y$.
(iv) There is a proper semi-L-sum of Y and Z.
(v) The unit ball of Z^* is weak reducible, i.e. there is an asymmetric, weak* compact, convex set $S \subset Z^*$ such that $S - S$ is the unit ball.*

The proofs of (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv) are fairly straightforward. To show that (v)$\Rightarrow$(i), consider the sup-normed space of weak* continuous affine functions on S.

However the proof that (iv)$\Rightarrow$(v) is more complicated, depending on the theory of M-ideals. We briefly summarize this topic in the next section, cheerfully mentioning that our original interest in reducibility arose from studying M-ideals in Banach spaces.

3. M-ideals

The theory of M-ideals goes back to Alfsen and Effros [1]. For further details about them, particularly the following results, see also [14], [7, Chapter 1] or [21].

A subspace Y is said to have the n-ball property in X if, whenever $B_1, \ldots, B_n$ are open balls in X, with $\bigcap_{i=1}^n B_i \neq \emptyset$, and $Y \cap B_i \neq \emptyset$ for each i, then we also have $Y \cap \bigcap_{i=1}^n B_i \neq \emptyset$. It is well known now that the 3-ball property implies the n-ball property for all n, and this happens if and only if Y° is an L-summand in X^*; such subspaces are called M-ideals. Examples of M-ideals include any ideal in a C^*-algebra; many ideals in uniform algebras; and the compact operators in $B(\ell_p)$, for $1 < p < \infty$.

Likewise Y has the 2-ball property in X if and only if Y° is a semi-L-summand in X^*; such subspaces are called semi-M-ideals.

The duality is complete: Y is an L-summand in X if and only if Y° is an M-ideal in X^*, and Y is a semi-L-summand in X if and only Y° is a semi-M-ideal in X^*.

Using intersection properties of balls, Lima [15, Theorems 1.2 and 1.3] showed that if Y is a semi-M-ideal in a Banach space X, and the unit ball of Y is irreducible, then Y is actually an M-summand in X. Combined with the duality results, this yields the implication (iv)$\Rightarrow$(v) in Theorem 5, which was the original motivation for our interest in all the topics discussed here.

4. Polytopes and polyhedra

By facial cone in a Banach space, we mean as usual the set of all positive multiples of some proper face of the unit ball. The norm is clearly additive on any facial cone, and hence so is any pseudolinear function. So on any facial cone, any pseudolinear function will agree with the restriction of a linear function. This means that the equivalence of (ii) and (v) in Theorem 5 tells us something interesting about the reducibility of finite dimensional balls, in particular of polytopes. Let's think this through.

We smuggled a definition of reducibility into the statement of Theorem 5, and used the concept again in §3. Due originally to Grünbaum [6], it is useful to repeat it here for finite dimensional sets (not necessarily polytopes). A compact convex set P, symmetric about the origin, is *reducible*, if there is a nonsymmetric closed convex set Q for which $P = Q - Q$. The latter term denotes the set of all differences, $\{x - y : x \in Q, y \in Q\}$. If the identity $P = Q - Q$ is only possible when Q is centrally symmetric, then P is said to be *irreducible*.

Suppose we are given a polytope P, with vertex set V and edge set E, sitting in some ambient vector space X. If P is symmetric about the origin and full dimensional, we may choose to interpret it as the unit ball of some norm on X. Let us define $C_v = \{f \in X^* : f(v) = \|f\|\}$, for each $v \in V$. It is easy to see that each C_v is a cone in X^*, with nonempty interior.

If P is reducible, Theorem 5 then furnishes a real-valued, homogeneous but nonlinear mapping $\Omega : X^* \to \mathbb{R}$ which satisfies the inequality

$$|\Omega(f) + \Omega(g) - \Omega(f+g)| \leqslant \|f\| + \|g\| - \|f+g\|,$$

for all $f, g \in X^*$. The previous remarks imply that $\Omega|C_v$ has a unique linear extension to the whole vector space X^*, which we will denote by $\rho(v) \in X^{**} = X$. So we have a well defined map $\rho : V \to X$, with some interesting properties, which essentially proves one direction of the following result.

THEOREM 6. *The polytope P is reducible if, and only if, there is a nonconstant function $\rho : V \to X$ such that*

(i) $\rho(v) = \rho(-v)$ for all $v \in V$, and

(ii) $\rho(v) - \rho(w)$ is a scalar multiple of $v - w$, whenever $[v, w]$ is an edge of P.

Again, we will only sketch the proof. If P is reducible, property (i) of the function ρ just constructed follows from the homogeneity of pseudolinear mappings, while (ii) is essentially a consistency condition. For if $[v, w]$ is an edge of P, then the corresponding faces in the dual polytope have non-empty intersection, so $\rho(v)$ and $\rho(w)$ must both agree with Ω on $C_v \cap C_w$.

For the converse, suppose a nonconstant "reducing function" ρ is given. We define Ω by $\Omega(f) = f(\rho(v))$ for each $f \in C_v$. The property (ii) of reducing functions ensures that Ω is well defined and continuous. Homogeneity and nonlinearity are clear. Finally, note that the two functions on $X^* \times X^*$ defined by $(f,g) \mapsto \Omega(f) + \Omega(g) - \Omega(f+g)$ and $(f,g) \mapsto \|f\| + \|g\| - \|f+g\|$ are positive homogeneous, continuous, and that the former vanishes everywhere the latter does. A routine compactness argument then shows that $|\Omega(f)+\Omega(g)-\Omega(f+g)| \leqslant K(\|f\|+\|g\|-\|f+g\|)$ for all $f,g \in X^*$ and a suitable constant K.

Once stated, Theorem 6 can be given a simple geometric proof [22, Theorem 1], but its inadvertent discovery required a lot of functional analysis. It is most unlikely that we would even have formulated this result without going through the convoluted process just presented.

The condition in Theorem 6 may be expressed in the form: does a certain finite family of linear equations have a nontrivial solution? Thus, to determine the reducibility of a given polytope, it suffices to find the rank of some rather large matrix. But we won't.

The rigidity of triangles implies that a symmetric polytope is irreducible if "many" of its 2-dimensional faces are triangles. This was known long ago [20]. However, it is instructive to check that if u,v,w are three vertices of a polytope, each two of which are adjacent, then any function ρ, satisfying the conditions in our theorem, must coincide on $\{u,v,w\}$ with the restriction of a homothety. (In fact, u,v,w need not form a triangular face here.) So the existence of sufficiently many triangles in the graph (1-skeleton) of P, together with the condition $\rho(v)=\rho(-v)$, forces ρ to be constant, from which irreducibility follows.

It is not hard to check that a parallelotope of any dimension is irreducible. Although it is still surprising today to learn that a 2-dimensional euclidean disc is reducible, this fact was apparently known to Euler; the "reducing set" is the well known Reuleaux triangle. In fact, any 2-dimensional convex body other than a parallelogram is reducible, and euclidean balls of any dimension (other than one) are reducible. This more or less summarizes what was known about this topic before 1960.

Using Theorem 6, we [22] were able to describe some large families of irreducible polytopes. We just list the main ones. Some were proved much earlier with different techniques by Shephard [20].

- A symmetric polytope is irreducible if every 2-dimensional face is a parallelogram.
- A symmetric polytope is irreducible if it is the direct sum of two irreducible polytopes.
- A symmetric polytope is irreducible if it is the convex hull of two (possibly reducible) polytopes lying in complementary subspaces.
- A symmetric polytope is irreducible if it is the convex hull of each pair of opposite maximal faces.
- A symmetric polytope is irreducible if it is the convex hull of a maximal face, with no pair of its edges parallel, and the opposite face.
- As mentioned at the beginning, every n-dimensional symmetric polytope with $4n-2$ or fewer vertices is irreducible (unless $n=2$).
- Combined with Baire category, these arguments can also be used to establish the existence of irreducible, smooth, strictly convex bodies.

A related, and much more studied, concept is the following: a finite dimensional compact convex set A (not necessarily symmetric) is said to be *decomposable* if it can be expressed as a sum $A=B+C$, where B and C are compact convex sets not homothetic to A; otherwise A is indecomposable. It is clear that a (symmetric) reducible set is decomposable, but the converse is false. For example, any parallelogram is decomposable, but not reducible. The only 2-dimensional indecomposable bodies are triangles. In three and higher dimensions, it is much harder to decide which sets are decomposable. Not surprisingly, the existence of large families of triangles guarantees indecomposability.

More recently we have realized that similar methods can be used to study decomposability of polytopes. This is the object of current research work with K. Przesławski [17, 18]. In particular, functions from the vertex set into the ambient vector space play a vital role. We thought that the use of such maps in [22] to study irreducibility (where Theorem 6 was first proved) was a new idea. However such functions were implicitly used by Kallay in [9], albeit in the context of decomposability.

An interesting feature of our work is that 4-cycles which are not coplanar play a role as important as triangles. Using them, we have now classified (as decomposable or indecomposable) all polytopes with 15 or fewer edges [18].

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