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BINARY SHIFTS OF HIGHER COMMUTANT INDEX

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For Robert T. Powers,
on the occasion of his seventieth birthday

Abstract. In a previous paper the author has shown that all binary shifts of commutant index 2 are cocycle conjugate. In this paper we show that there are only finitely many conjugacy classes of binary shifts of commutant index 3.

1. Introduction

We continue a study of the binary shifts on the hyperfinite II_1 factor R. A binary shift α is a unital $*$-endomorphism on R with the property that the subfactor index, $[R:\alpha(R)]$, is 2. The study of binary shifts was initiated by R. T. Powers. In his original paper Powers classified binary shifts up to conjugacy, [8][Theorem 3.6]. The cocycle conjugacy classification (Definition 1.1) is still an open problem, but partial results have been obtained previously by the author and others, see [2, 4, 9, 10, 11, 12]. In [10] the author has shown that all binary shifts of commutant index 2 are cocycle conjugate, and some results on binary shifts of higher commutant index were obtained in [11]. It follows from a result in [2] that there are at least 2^{k-2} distinct cocycle conjugacy classes of binary shifts of commutant index k, $k \geqslant 2$. Here we consider the binary shifts of commutant index 3. We show that there are at most 5 distinct cocycle conjugacy classes of these shifts.

In [10] the author carried out an analysis of the congruence classes of Toeplitz matrices over $GF(2)$ associated with binary shifts of commutant index 2 (see Definition 1.2, see also [7]) for a detailed study on the congruence of matrices over a field of characteristic 2). We showed that the Toeplitz matrices associated with a pair of binary shifts of commutant index 2 are congruent. This result allows one to show that the corresponding binary shifts are cocycle conjugate. Similar techniques were used in [11] to study certain higher commutant cases. Here we employ an extension of the techniques used in [10, 11] to study the commutant index 3 case. It appears that additional techniques will be required to settle the question of whether there are only finitely many distinct cocycle conjugacy classes of higher commutant index.

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A pair α and β of unital $*$-endomorphisms on R are said to be conjugate in R if there is a $*$-automorphism γ of R such that $\alpha = \gamma^{-1} \circ \beta \circ \gamma$. The notion of *cocycle conjugacy* is derived from A. Connes' notion of outer conjugacy of automorphisms in [3], and is defined as follows.

DEFINITION 1.1. A pair α and β of unital $*$-endomorphisms on R are cocycle conjugate if there exists a unitary operator y in R such that $Ad(y) \circ \alpha$ is conjugate to β.

Next we define what is meant by a binary shift, cf. [8][Definition 3.2]. Let $a_0, a_1, \ldots$ be a fixed sequence of 0's and 1's in $GF(2)$, with $a_0 = 0$. Let $u_0, u_1, \ldots$ be a sequence of self-adjoint unitary operators such that, for all $j, k \in \mathbb{Z}^+$,

$$u_j u_{j+k} = (-1)^{a_k} u_{j+k} u_j. \tag{1.1}$$

We shall call generators with the relations above a spin system (see [1] for results on more general spin systems). In [12] it was shown that the AF-algebra generated by a spin system is simple if and only if the sequence $\ldots, a_2, a_1, a_0, a_1, a_2, \ldots$ is not periodic. In every such case the C^*-algebra generated by the spin system is isomorphic to the CAR algebra, [11][Theorem 3.5] (see also [1]). We shall assume in all that follows that the sequences $a_0, a_1, a_2, \ldots$ we study have this property, and we shall refer to such sequences as the *bitstream* for the spin system. Let τ be the unique tracial state on the CAR algebra. It follows that $\tau(w) = 0$ for any non-trivial word $u = u_0^{k_0} u_1^{k_1} \ldots u_n^{k_n}$ in the u_j's. Using the GNS representation of the CAR algebra A with respect to the trace τ one may consider A as a strongly dense subalgebra of R. In what follows we abuse notation by viewing A as a C^*-subalgebra of R. Thus the set of linear combinations of words in the generators forms a weakly dense submanifold of the algebra R.

The assumption that the commutation relations are translation-invariant makes it possible to define a unital $*$-endomorphism α on $\mathscr{A}$ by setting $\alpha(u_j) = u_{j+1}$ and extending the definition of α to linear combinations of words in the obvious way. The mapping α extends to a unital $*$-endomorphism on R, which we also denote by α. As noted above the subfactor index of $\alpha(R)$ in R is 2, see [8][Section 3].

As shown in [8] the bitstream $a_0, a_1, \ldots$ of a binary shift α is a complete conjugacy invariant, i.e., binary shifts α and β are conjugate if and only if their bitstreams are identical. We conclude this section by presenting two cocycle conjugacy invariants for binary shifts on R.

DEFINITION 1.2. The commutant index of a binary shift α is the first positive integer k (or ∞) such that the relative commutant algebra $\alpha^k(R)' \cap R$ is nontrivial.

It follows from a remark in [5] that $k \geqslant 2$. Examples of binary shifts of commutant k exist for every $k \in \{\infty, 2, 3, \ldots\}$, [11]. For example, fix $k \geqslant 2$ and consider the bitstream $0\ldots010\ldots$ where $a_i = 0$ for $i \neq k-1$ and $a_{k-1} = 1$. It is straightforward to show that α has commutant index k and that u_0 generates the algebra $\alpha^k(R)' \cap R$. At the other extreme, α has infinite commutant index if and only if its bitstream is not eventually periodic (by eventually periodic we mean that there exists a non-negative integer q such that $a_q, a_{q+1}, \ldots$ is a periodic sequence).

THEOREM 1.3. [11][Cor. 5.7] *Let α be a binary shift of finite commutant index k. Then there is a word $u = u_0^{r_0} u_1^{r_1} \dots u_m^{r_m}$, with r_0 necessarily equal to 1, which generates $\alpha^k(R)' \cap R$. In fact for $j \geqslant 0$ the algebra $\alpha^{k+j}(R)' \cap R$ is the 2^{j+1}-dimensional algebra generated by $u, \alpha(u), \dots, \alpha^j(u)$.*

COROLLARY 1.4. [2][Theorem 2.1] *Let α be a binary shift of finite index k. Then its bitstream $a_0, a_1, \dots$ is eventually periodic, i.e. there is a non-negative integer $r \leqslant k$ such that $a_r, a_{r+1}, \dots$ is periodic.*

Proof. Let $u = u_0^{r_0} u_1^{r_1} \dots u_m^{r_m}$ be the word generating $\alpha^k(R)' \cap R$. Since u commutes with the generators $u_k, u_{k+1}, u_{k+2}, \dots$ we obtain the following homogeneous system of equations over $GF(2)$ (where, if $j < 0$ in the system below we define a_j to be $a_{|j|)}$):

$$\begin{aligned} a_k r_0 + a_{k-1} r_1 + a_{k-2} r_2 + \dots + a_{k-m} r_m &= 0 \\ a_{k+1} r_0 + a_k r_1 + a_{k-1} r_2 + \dots + a_{k-m+1} r_m &= 0 \\ a_{k+2} r_0 + a_{k+1} r_1 + a_k r_2 + \dots + a_{k-m+2} r_m &= 0 \\ &\vdots \end{aligned}$$

Since $r_0 = 1$ we may rewrite the system as

$$\begin{aligned} a_k &= a_{k-1} r_1 + a_{k-2} r_2 + \dots + a_{k-m} r_m \\ a_{k+1} &= a_k r_1 + a_{k-1} r_2 + \dots + a_{k-m+1} r_m \\ a_{k+2} &= a_{k+1} r_1 + a_k r_2 + \dots + a_{k-m+2} r_m \\ &\vdots \end{aligned}$$

It follows (see [6][Theorem 6.11]) that the sequence $a_k, a_{k+1}, \dots$ is periodic. $\square$

Let u be the word generating $\alpha^k(R)' \cap R$ in the statement of the theorem above. Let d_j, for $j \geqslant 0$, be the sequence of 0's and 1's satisfying $u\alpha^j(u) = (-1)^{d_j} \alpha^j(u) u$. Since $\alpha^j(u) \in \alpha^k(R)$ for $j \geqslant k$ we have $d_j = 0$ for these j. On the other hand, $\alpha^{k-1}(u)$ has the form $u_{k-1}^{r_0} u_k^{r_1} \dots u_{m+k-1}^{r_m} = u_{k-1} w$, where $w \in \alpha^k(R)$. Since α has commutant index k, u anticommutes with u_{k-1} and commutes with w. Therefore u anticommutes with $\alpha^{k-1}(u)$ and so $d_{k-1} = 1$. Note that the sequence $d_0, d_1, \dots$ has the property that $\dots, d_2, d_1, d_0, d_1, d_2, \dots$ is not periodic, so by [12] the von Neumann algebra R_∞ generated by $u, \alpha(u), \alpha^2(u) \dots$ is also isomorphic to R. It follows that α restricts to a binary shift on R_∞ with bitstream $d_0, d_1, \dots$. We denote the restriction of α to R_∞ by α_∞. Following [2], (see also [4]) α_∞ is called the *derived* shift of α and $d_0, d_1, \dots$ is the derived bitstream. In [2] it is shown that the derived bitstream is a cocycle conjugacy invariant for α, i.e. a necessary condition for α and β to be cocycle conjugate is that their derived shifts α_∞ and β_∞ are conjugate.

It is easy to show that a binary shift α with a finitely non-zero bitstream $a_0, a_1, \dots, a_{k-2}, 1, 0, 0, \dots$, has commutant index k, and in this case α coincides with its derived shift α_∞, as $u = u_0$ generates $\alpha^k(R)' \cap R$. On the other hand any commutant index k binary shift must have a derived shift with a bitstream of the form above. Therefore we have the following.

THEOREM 1.5. [2]. *There are at least* 2^{k-2} *distinct cocycle conjugacy classes of binary shifts of commutant index* k, $k \geqslant 2$.

As mentioned above there is only one class of binary shifts of index 2 up to cocycle conjugacy. The preceding theorem shows that there are at least two cocycle conjugacy classes of binary shifts of commutant index 3. The object of this paper is to show that there are at most five. We believe that there are exactly five but we do not know how to prove this.

We note that nothing is known about the number of cocycle conjugacy classes of binary shifts of commutant index ∞. These are the binary shifts whose bitstreams are never eventually periodic. It is not known, for example, whether all binary shifts of commutant index ∞ are cocycle conjugate to each other or whether there are uncountably many distinct cocycle conjugacy classes.

2. The center sequence

Let $a_0, a_1, \ldots$ be the bitstream of a binary shift α. We define $\mathscr{A}_n$ for each $n \in \mathbb{N}$ to be the $n \times n$ matrix

$$\mathscr{A}_n = \begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_{n-1} \\ a_1 & a_0 & a_1 & \ldots & a_{n-1} \\ a_2 & a_1 & a_0 & \ldots & a_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \ldots & a_0 \end{pmatrix} \tag{2.1}$$

with entries in $GF(2)$, and call $\mathscr{A}_n$ the $n \times n$ Toeplitz matrix associated with α, or the $n \times n$ Toeplitz matrix associated with the bitstream $a_0, a_1, a_2, \ldots$.

For each $n \in \mathbb{N}$ let $c_n = \nu(n)$ be the nullity of $\mathscr{A}_n$. The *center sequence* $c_1, c_2, \ldots$ has the following remarkable property.

THEOREM 2.1. [11][Corollary 2.10] *The center sequence is the concatenation of strings of even length. Its strings are of the form* $1,0$ *or* $1,2,\ldots,j-1,j,j-1,\ldots,0$ *for some* $j \geqslant 2$, *where* j *may vary from one string to the next. In particular,* c_n *is odd if and only if* n *is.*

For example, given the bitstream $011000\ldots$ it is possible to show that the corresponding center sequence is 101210 repeated forever (see Theorem 2.5 (vi)).

DEFINITION 2.2. For $n \in \mathbb{N}$ let A_n be the finite-dimensional von Neumann subalgebra of R generated by $u_0, u_1, \ldots, u_{n-1}$.

Note that A_n has dimension 2^n, consisting of all linear combinations of words of the form $u_0^{k_0} u_1^{k_1} \ldots u_{n-1}^{k_{n-1}}$, with exponents $k_j \in \{0,1\}$. The following result links c_n to the dimension of the center of A_n of R and justifies the name center sequence.

THEOREM 2.3. [11][Lemma 3.3, Theorem 3.4] *The center* $\mathscr{Z}(A_n)$ *of* A_n *is an algebra of dimension* 2^{c_n}. *More precisely, suppose* $c_{r+1}, c_{r+2}, \ldots, c_{r+2j}$ *is a string in the center sequence of the form* $1,2,\ldots,j-1,j,j-1,\ldots,1,0$, *for some* $r \geqslant 0$. *Then there* *is a word z of the form $z = u_0^{s_0} u_1^{s_1} \dots u_r^{s_r}$ in A_n, with $s_0 = 1 = s_r$, such that $\mathscr{Z}(A_{r+q})$ is generated by $z, \alpha(z), \dots, \alpha^{q-1}(z)$ if $1 \leqslant q \leqslant j$ and by $\alpha^{q-j-1}(z), \alpha^{q-j}(z), \dots, \alpha^{j-1}(z)$ if $j < q \leqslant 2j$. The exponents of z read the same backwards as forwards, i.e., $s_r, s_{r-1}, \dots, s_0$ is the same as $s_0, s_1, \dots, s_r$.*

REMARK 2.4. In what follows we shall refer to z as a palindrome since its exponents $s_0, s_1, \dots, s_r$ read the same in reverse order.

In this paper we consider almost exclusively binary shifts of commutant index 3. The following five binary shifts of commutant index 3 will play an important role in the analysis. The notation $\overline{b_1 \dots b_n}$ means that the pattern $b_1 \dots b_n$ repeats forever.

THEOREM 2.5. *Consider the following binary shifts β_1 through β_5, determined by the given bitstreams.*

(i) *β_1 has bitstream* $011\overline{0}$.

(ii) *β_2 has bitstream* $\overline{010}$.

(iii) *β_3 has bitstream* $\overline{001}$.

(iv) *β_4 has bitstream* $001\overline{0}$.

(v) *β_5 has bitstream* $\overline{0110}$.

Each of these binary shifts has commutant index 3.

(vi) *The word $v = v_0$ generates $\beta_1^3(R)' \cap R$ and v anticommutes with $\beta_1(v) = v_1$. Hence β_1 coincides with its derived shift $\beta_{1\infty}$. The center sequence is* $\overline{101210}$.

(vii) *The word $v = v_0 v_3$ generates $\beta_2^3(R)' \cap R$ and v anticommutes with $\beta_2(v)$. Hence its derived shift has bitstream $011\overline{0}$, i.e., $\beta_{2\infty}$ is conjugate to β_1. The center sequence of β_2 is* $10\overline{101210}$.

(viii) *The word $v = v_0 v_3$ generates $\beta_3^3(R)' \cap R$ and v anticommutes with $\beta_3(v)$. Hence its derived shift has bitstream $001\overline{0}$, i.e., $\beta_{3\infty}$ is conjugate to β_1. The center sequence of β_3 is* $\overline{121010}$.

(ix) *The word $v = v_0$ generates $\beta_4^3(R)' \cap R$ and v commutes with $\beta_4(v) = v_1$. Hence β_4 coincides with its derived shift $\beta_{4\infty}$. Its center sequence is* $\overline{1210}$.

(x) *The word $v = v_0 v_1 v_2 v_3$ generates $\beta_5^3(R)' \cap R$ and v commutes with $\beta_5(v)$. Hence its derived shift has bitstream $001\overline{0}$, i.e., $\beta_{5\infty}$ is conjugate to β_4. The center sequence of β_5 is* $10\overline{1210}$.

Proof. We illustrate the proof using $\beta = \beta_2$. We show β has commutant index 3. It is easy to show, using the bitstream for β, that $v = v_0 v_3 \in \beta^3(R)' \cap R$. Since $[R : \beta(R)] = 2$ ([8]) it follows from [5] that $\beta(R)' \cap R$ is trivial. Suppose there is a nontrivial word w in $\beta^2(R)' \cap R$, then by [11], v must be in the $*$-subalgebra generated by w and $\beta(w)$. Hence w must have the form $v_0^{k_0} v_1^{k_1} v_2^{k_2} v_3^{k_3}$ with $k_0 = 1$. If $k_3 = 1$ then

$w = \pm v$, but v does not commute with v_2. Hence $w = v_0 v_1^{k_1} v_2^{k_2}$ and $v = \pm w\beta(w) = \pm v_0 v_1^{1+k_1} v_2^{k_1+k_2} v_3^{k_2}$. This shows that $w = v_0 v_1 v_2$, but this word does not commute with v_2, so we have shown that $\beta^2(R)' \cap R$ is trivial, and therefore β has commutant index 3.

We now show that the center sequence for β is of eventual period 6 and has the form $10\overline{101210}\ldots$. Using the bitstream $\overline{010}$ for β, easy calculations show that the first five entries c_0, c_1, c_2, c_3, c_4 of the center sequence for β are $1,0,1,0,1$ (see Theorem 2.3). (Alternatively, one can use the nullity sequence corresponding to the Toeplitz matrices for β to show that this is so.) For each $n \in \mathbb{N}$ of the form $n = 6k-2, k \in \mathbb{N}$, we will show that the center $\mathscr{Z}(\mathscr{B}_n)$ of the algebra $\mathscr{B}_n = \{v_0, v_1, \ldots, v_n\}''$ is generated by the word $v = v_0 v_1 \ldots v_n$ and $\mathscr{Z}(\mathscr{B}_{n+1})$ is generated by v and $\beta(v)$. It is trivial to show this by direct calculation for $n = 4$. Suppose the result holds for $n = 6k-2$ for some $k \geqslant 1$. Consider the word $v' = v_0 v_1 \ldots v_{n+6} = (v_0 v_1 \ldots v_n)(v_{n+1} \ldots v_{n+6}) = v(v_{n+1} \ldots v_{n+6}) = -v(v_{n+1} v_{n+4})(v_{n+2} v_{n+5})(v_{n+3} v_{n+6})$. Since $v_0 v_3 \in \beta^3(R)' \cap R$ it follows from the symmetry of the commutation relations that the words $v_{n+1} v_{n+4}, v_{n+2} v_{n+5}$ and $v_{n+3} v_{n+6}$ all commute with v_0 through v_{n+1}. But $v \in \mathscr{Z}(\mathscr{B}_{n+1})$, so v_0 through v_{n+1} also commute with v. Hence v_0 through v_{n+1} all commute with v', by the induction assumption. By the symmetry of v', moreover, it follows, since v_0 through v_6 all commute with v' and $\beta(v')$, that v_{n+7} down through v_{n+1} all commute with v'. Hence v' is in the center of both B_{n+6} and B_{n+7}. Similarly using the assumption that $\beta(v) \in \mathscr{Z}(\mathscr{B}_{n+1})$, it follows that $\beta(v')$ is in the center of B_{n+7}. Therefore c_{n+6} is at least 1 and c_{n+7} is at least 2.

Next note that v_0 anticommutes with $\beta^2(v') = v_2 \ldots v_{n+8}$ because v_0 commutes with $\beta(v')$, commutes with v_{n+8} (because $n+8 = 6(k+1)-2+2 = 6(k+1)$ and $a_{6j} = 0$ for all j) and anticommutes with v_1: therefore $\beta^2(v')$ is not in $\mathscr{Z}(\mathscr{B}_{n+8})$, so by Theorem 2.3, $c_{n+8} < c_{n+7}$.

Next observe that for any $r \geqslant 3$ the center sequence term c_r satisfies $c_r < 3$. For suppose r is the first $r \geqslant 3$ such that $c_r = 3$. By the observations made about c_0, c_1, c_2, c_3, c_4 in the first paragraph of the proof, $r \geqslant 6$. Then $c_{r-3}, c_{r-2}, c_{r-1}$ must be $0,1,2$ respectively, by Theorem 2.1. Then by Theorem 2.2 there is an element $z \in \mathscr{B}_{r-2}$ such that $z, \beta(z)$ and $\beta^2(z)$ are in $\mathscr{Z}(\mathscr{B}_r)$ and z is a word beginning with v_0. Hence $\beta^2(z)$ begins with v_2. Since $v_0 v_3 \in \mathscr{B}_r$ generates $\beta^3(R)' \cap R$ and also anticommutes with v_2, however, it follows that $v_0 v_3$ anticommutes with $\beta^2(z)$. Therefore $\beta^2(z)$ is not in the center of $\mathscr{B}_r$, a contradiction, and we have established our claim.

Combining the observations of the last two paragraphs together with Theorem 2.1, we see that $c_n, c_{n+1}, c_{n+2}, c_{n+3}, c_{n+4}$ is $1,2,1,0,1$. Then either $c_{n+5} = 0$ or $c_{n+5} = 2$. If $c_{n+5} = 2$ it follows from the bound $c_r < 3$ for $r \geqslant 3$ that $c_{n+6} = 1$ and $c_{n+7} = 0$. But we have shown that $c_{n+7} \geqslant 2$. Hence $c_{n+5} = 0, c_{n+6} = 1$. This proves the assertions about the form of the center sequence for β.

Finally, the claims about the bitstreams of the derived shifts $\beta_{j\infty}$ for each of the β_j's are easily verified, using the Powers' result that two shifts are conjugate if and only if they have the same bitstream, [8][Theorem 3.6]. $\square$

THEOREM 2.6. *Let σ be a binary shift of commutant index* 3*. Then its center sequence eventually coincides with the center sequence of one of the shifts β_j, $1 \leqslant j \leqslant 5$.*

Proof. The derived shift σ_∞ must have bitstream either $001\overline{0}$ or $011\overline{0}$. By the theorem above the center sequence of σ_∞ has period 4 or 6. From [13][Theorem 3.7] the center sequence of σ is eventually periodic with period an even integer dividing the period of σ_∞. Therefore the possible periods are 2, 4 or 6. If the center sequence of σ has eventual period 4 or 6 it follows from Theorem 2.1 that its center sequence must eventually agree with that of one of the five shifts β_j, $1 \leqslant j \leqslant 5$. We next rule out the possibility that the center sequence of σ has eventual period 2.

Let $w = v_0^{r_0} v_1^{r_1} \dots v_m^{r_m}$ be the nontrivial word which generates $\sigma^3(R)' \cap R$. Then by Theorem 1.3, w must anticommute with v_2 and r_0 must equal 1.

For each positive integer p let A_p be the 2^p-dimensional algebra generated by the spin generators v_0 through v_{p-1} of σ. Suppose the center sequence eventually has period 2. Fix an even positive integer n such that $n > m$ and $c_n, c_{n+1}, c_{n+2}, \dots$ is periodic with period 2. Then $c_n = c_{n+2} = \dots = 0$ and $c_{n+1} = c_{n+3} = \dots = 1$. Let z_n (resp., z_{n+2}) be a nontrivial word generating $\mathscr{Z}(A_{n+1})$ (resp. $\mathscr{Z}(A_{n+3})$). We know by Theorem 2.3 that both z_n and z_{n+2} "start" with v_0 and that z_n (resp., z_{n+2}) ends in v_n, (resp., in v_{n+2}).

For the remainder of the proof we will use the notation $x \sim y$ for words x and y in the generators $v_0, v_1, \dots$ to indicate that $x = \pm y$. Note, for example, that if $y = v_0^{k_0} v_1^{k_1} \dots v_r^{k_r}$ is any word in the v_j's then $y^* = v_r^{k_r} \dots v_1^{k_1} v_0^{k_0}$ and $y \sim y^*$.

Consider $x = \sigma^2(z_n) z_{n+2}$, a word which begins with v_0. Note that $x \in A_{n+2}$ because because both $\sigma^2(z_n)$ and z_{n+2} end in v_{n+2}, and therefore x ends in v_{n+1} or earlier. $\sigma^2(z_n)$ commutes with v_2 through v_{n+2}, since z_n commutes with v_0 through v_n. Also z_n anticommutes with v_{n+1}, otherwise we would conclude that $z_n \in \mathscr{Z}(A_{n+2})$, a contradiction since $c_{n+2} = 0$. Since z_n anticommutes with v_{n+1} it follows from the fact z_n is a palindrome (Theorem 2.3), that $\sigma^2(z_n)$ anticommutes with v_1. Since $z_{n+2} \in \mathscr{Z}(A_{n+3})$ the facts about $\sigma^2(z_n)$ imply that $\sigma^2(z_n) z_{n+2}$ anticommutes with v_1 and commutes with v_2 through v_{n+2}.

Next consider the word wx, which commutes with $v_3, \dots, v_{n+2}$, anticommutes with v_2, and starts with a generator after v_0. Hence we can define y by $y = \sigma^{-1}(wx)$. The word y commutes with $v_2, \dots, v_{n+1}$ and anticommutes with v_1. Also $y \in A_{n+1}$ since $x \in A_{n+2}$ and $m < n$, so $w \in A_{n+1}$.

We claim that y starts with the generator v_0. For if y starts with v_2 or higher, $\sigma^{-2}(y)$ commutes with v_0 through v_{n-1} and lies in A_{n-1}. Hence $\sigma^{-2}(y) \in A_n^c \cap A_{n-1} \subset \mathscr{Z}(A_n)$ which is trivial, since $c_n = 0$. If y starts with v_1 then since it anticommutes with v_1 and commutes with v_2 through v_{n+1} we conclude that y anticommutes with itself, a contradiction. Hence we have determined that y starts with v_0.

Since both x and y start with v_0 we can form $\sigma^{-1}(xy)$, which commutes with v_0 since both x and y anticommute with v_1. Hence $\sigma^{-1}(xy)$ commutes with v_0 through v_n, i.e., $\sigma^{-1}(xy) \in \mathscr{Z}(A_{n+1})$. Therefore either $\sigma^{-1}(xy) \sim z_n$ or $\sigma^{-1}(xy) \sim I$.

First suppose $\sigma^{-1}(xy) \sim z_n$. Then $xy \sim \sigma(z_n)$, or $x\sigma^{-1}(wx) \sim \sigma(z_n)$, or $\sigma(x)wx \sim \sigma^2(z_n)$, or $\sigma(x)w\sigma^2(z_n)z_{n+2}$, or $\sigma(x)wz_{n+2} \sim I$, so $wz_{n+2} \sim \sigma(x)$. Since w commutes with v_{n+3} and z_{n+2} does not (otherwise $z_{n+2} \in \mathscr{Z}(A_{n+4})$, which is trivial) the word wz_{n+2} anticommutes with v_{n+3}. But $\sigma(x)$ commutes with v_{n+3}, a contradiction. So we have ruled out the possibility that $\sigma^{-1}(xy) \sim z_n$.

Next suppose $\sigma^{-1}(xy) \sim I$. Then $xy \sim I$, so $x \sim y = \sigma^{-1}(wx)$, so $\sigma(x) \sim wx$, or $w \sim x\sigma(x)$. Therefore $x\sigma(x)$ commutes with v_j, for all $j \geqslant 3$. Since x commutes with v_2 through v_{n+2}, $\sigma(x)$ commutes with v_3 through v_{n+3}. Since both w and $\sigma(x)$ commute with v_{n+3}, so must x. Continuing in this way we conclude that x commutes with v_j for all $j \geqslant 2$. Then $x \in \sigma^2(R)' \cap R$, which is trivial. But x starts with v_0 and so is not trivial. This contradiction shows that $xy \nsim I$. Hence we have ruled out the possibility that the center sequence has eventual period 2.

As we have ruled out the possibility that a shift of commutant index 3 could have a center sequence of eventual period 2, we see from the first paragraph of the proof that if the center sequence of the derived shift σ_∞ has eventual period 4 then so does σ. Similarly if the eventual period of the center sequence of σ_∞ is 6 then the same is true for σ. An application of Theorem 2.3 on the form of strings of a center sequence now establishes the result. □

When the bitstream of σ_∞ has the form $001\overline{0}$ (see Theorem 2.5, see also [13][Theorem 2.10]), i.e., when the word v that generates $\sigma^3(R)' \cap R$ commutes with $\sigma(v)$ then the eventual period of both σ and σ_∞ is 4. In the case when σ is a binary shift for which v, the generator of $\sigma^3(R)' \cap R$, anticommutes with $\sigma(v)$, the eventual period of the center sequence of σ_∞ and of σ is 6. Hence we have established the following.

COROLLARY 2.7. *Let σ be a binary shift of commutant index 3. If σ_∞ has bitstream $001\overline{0}$ then the center sequences of both σ_∞ and σ have eventual period 4. If σ_∞ has bitstream $011\overline{0}$ then the center sequences of both σ and σ_∞ have eventual period 6.*

3. Toeplitz matrices and congruence

As we have seen, the Toeplitz matrix associated with a bitstream contains important information about the corresponding binary shift. In this section we show that if a pair of binary shifts of commutant index 3 have center sequences which eventually coincide, then their associated Toeplitz matrices are congruent. We first recall the notion of congruence of a pair of $n \times n$ matrices. See [7][Chapter IV] for details.

DEFINITION 3.1. A pair of $n \times n$ matrices $\mathscr{A}$ and $\mathscr{B}$ are congruent if there is a unitary matrix U such that $U^t \mathscr{A} U = \mathscr{B}$, where U^t is the transpose of U.

It is clear that congruence of matrices is an equivalence relation and that congruent matrices have the same rank.

As it will be useful to consider infinite Toeplitz matrices (see below) we will develop a notion of congruence in this context. Before we do so we introduce some notation. Given a binary shift α of commutant index 3, with corresponding bitstream $a_0, a_1, a_2, \ldots$, let $\mathscr{A}$ be the semi-infinite Toeplitz matrix over $GF(2)$ determined by the bitstream for α, i.e.,

$$\mathscr{A} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\ a_1 & a_0 & a_1 & a_2 & a_3 & \dots \\ a_2 & a_1 & a_0 & a_1 & a_2 & \dots \\ a_3 & a_2 & a_1 & a_0 & a_1 & \dots \\ a_4 & a_3 & a_2 & a_1 & a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{3.1}$$

Note that for $n \geqslant 1$, $\mathscr{A}_n$ from (2.1) is the $n \times n$ upper left block of $\mathscr{A}$. For convenience in subsequent calculations the rows and columns of $\mathscr{A}_n$ are numbered from 0 to $n-1$. Let $u_0, u_1, u_2, \dots$ be the generators of α satisfying the commutation relations

$$u_i u_{i+k} = (-1)^{a_k} u_{i+k} u_i,\ i,k \in \mathbb{Z}^+.$$

Let A_n be the 2^n-dimensional C^*-subalgebra generated by the words in the spin generators $u_0, u_1, \dots, u_{n-1}$. Let $w = u_0^{r_0} u_1^{r_1} \dots u_m^{r_m}$ be the word generating $\alpha^3(R)' \cap R$.

DEFINITION 3.2. For fixed $n \geqslant 2$ and $i, j \in \{0, \dots, n-1\}$ with $i \neq j$, let E_{ij} be the $n \times n$ elementary matrix with 1's along the main diagonal, a 1 in the (i, j) position of the matrix, and 0's elsewhere.

We will always be able to determine the size of the matrix E_{ij} from the context in which it appears. The following properties of E_{ij} are easily verified.

PROPOSITION 3.3. *Let $\mathscr{B}$ be an $n \times n$ matrix over $GF(2)$. Then*

1. *$\mathscr{B}E_{ij}$ is the matrix obtained from $\mathscr{B}$ by adding column i to column j.*
2. *$E_{ji}\mathscr{B}$ is the matrix obtained from $\mathscr{B}$ by adding row i to row j.*
3. *$E_{ji} = E_{ij}^t$, i.e., E_{ji} is the transpose of E_{ij}.*
4. *$E_{ij}^{-1} = E_{ij}$.*

The following result is immediate from combining the first two properties of the preceding Proposition and the fact that the matrices $\mathscr{A}_n, n \in \mathbb{N}$ over $GF(2)$ have 0 diagonal.

COROLLARY 3.4. *If $\mathscr{A}_n$ is the $n \times n$ corner matrix of $\mathscr{A}$ and E_{ij} is an $n \times n$ elementary matrix then $E_{ij}^t \mathscr{A}_n E_{ij}$ has 0 diagonal.*

Let β be another binary shift of commutant index 3, with bitstream $b_0, b_1, \dots$, Toeplitz matrix $\mathscr{B}$, whose center sequence eventually agrees with that of α. We may then conclude from the paragraph preceding Corollary 2.7 that the bitstreams of their derived shifts α_∞ and β_∞ coincide. We use the notation $d_0 d_1 d_2 \overline{0}$ for the bitstream of α_∞ and β_∞, where $d_2 = 1$ and d_1 is 0 or 1, depending upon whether the center sequence of α has period 4 or 6.

We will show for n sufficiently large that $\mathscr{A}_n$ and $\mathscr{B}_n$ are congruent. We will establish this congruence with the use of products of $n \times n$ elementary matrices. We first show that for n sufficiently large $\mathscr{A}_n$ is congruent to a matrix of a special form.

A similar result will follow for $\mathscr{B}_n$. We will make use of products of the form $\mathscr{E}_j = E^{r_m}_{j-m,j} E^{r_{m-1}}_{j-m+1,j} \dots E^{r_1}_{j-1,j}$. By Theorem 2.5 there are infinitely many $p \in \mathbb{N}$ such that the string $c_p c_{p+1} c_{p+2} c_{p+3} c_{p+4}$ is 01210. Fix $n > p+4 > p > m$. Using the fact that $c_{p+2} = 2$, [11][Corollary 6.5] shows that $(\mathscr{E}_{n-1}\mathscr{E}_{n-2}\dots\mathscr{E}_{p+2})^t \mathscr{A}_n \mathscr{E}_{n-1}\mathscr{E}_{n-2}\dots\mathscr{E}_{p+2} = \mathscr{F}_n$, where $\mathscr{F}_n$ is the matrix

$$\left(\begin{array}{ccccccccccccccc}
 & & & & & & 0 & & & & & & \dots & & 0 \\
 & & & & & & \vdots & & & & & & & & \vdots \\
 & & \mathscr{A}_{p+2} & & & & 0 & & & & & & \dots & & 0 \\
 & & & & & & e_2 & 0 & & & & & \dots & & 0 \\
 & & & & & & e_1 & e_2 & 0 & & & & \dots & & 0 \\
\hdotsfor{15} \\
0 & 0 & \dots & 0 & e_2 & e_1 & 0 & d_1 & d_2 & 0 & & & \dots & & 0 \\
0 & 0 & \dots & 0 & 0 & e_2 & d_1 & 0 & d_1 & d_2 & 0 & & \dots & & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & d_2 & d_1 & 0 & d_1 & d_2 & 0 & \dots & & 0 \\
\vdots & \vdots & & & & \vdots & & & & \ddots & & & & & \vdots \\
 & & & & & & & & & & \ddots & & & & \\
 & & & & & & & & & & & \ddots & & & \\
 & & & & & & & & & & & & \ddots & & \\
0 & 0 & \dots & & & 0 & 0 & \dots & & & 0 & d_2 & d_1 & 0 & d_1 \\
0 & 0 & \dots & & & 0 & 0 & \dots & & & 0 & 0 & d_2 & d_1 & 0
\end{array}\right), \qquad (3.2)$$

and where $d_2 = e_2 = 1$.

Since the matrices $\mathscr{A}_p, \mathscr{A}_{p+1}, \mathscr{A}_{p+2}$ have nullities $0,1,2$ respectively, there is, by Theorem 2.3, an element $z = u_0^{s_0} u_1^{s_1} \dots u_p^{s_p}$ with the following properties: z generates $\mathscr{Z}(A_{p+1})$, $z, \alpha(z)$ generate $\mathscr{Z}(A_{p+2})$, and $\alpha(z)$ generates $\mathscr{Z}(A_{p+3})$. Also $s_0 = 1$ and the vector of exponents, $\mathbf{s} = [s_0, s_1, \dots, s_p]$, reads the same backwards as forwards. Since $\alpha(z)$ is in the center $\mathscr{Z}(A_{p+2})$ of A_{p+2} it follows that the dot product (over $GF(2)$) of the vector $[0, s_0, s_1, \dots, s_p]$ with all of the rows of $\mathscr{A}_{p+2}$ gives 0. The same holds for the dot product of this vector with $[0, 0, \dots, 0, e_2, e_1]$ in the row below the corner matrix $\mathscr{A}_{p+2}$. To see this, observe that the latter vector is a linear combination of the rows of $\mathscr{A}_{p+2}$ and the row vector $[a_{p+2}, a_{p+1}, \dots, a_0]$. But $[0, s_0, s_1, \dots, s_p]$ annihilates the rows of $\mathscr{A}_{p+2}$ and $[s_0, s_1, \dots, s_p]$ annihilates the last row $[a_p, a_{p-1}, \dots, a_0]$ of $\mathscr{A}_{p+1}$. These observations establish the claim. Hence if $\mathscr{D}_{p+1} = E^{c_{p-1}}_{p,p+1} E^{c_{p-2}}_{p-1,p+1} \dots E^{c_0}_{1,p+1}$, then via $\mathscr{D}_{p+1}$ the matrix $\mathscr{F}_n$ is congruent to the matrix

$$\left(\begin{array}{ccccccccccccccc}
 & & & & & 0 & 0 & & & & & & \dots & & 0 \\
 & & & & & \vdots & \vdots & & & & & & & & \\
 & & & & & & & & & & & & & & \vdots \\
 & & \mathscr{A}_{p+1} & & & & & & & & & & & & \\
 & & & & & & & & & & & & & & \\
 & & & & & 0 & 0 & & & & & & \dots & & 0 \\
 & & & & & 0 & e_2 & 0 & & & & & \dots & & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & e_2 & 0 & & & & \dots & & 0 \\
\hdotsfor{15} \\
0 & 0 & \dots & 0 & e_2 & e_1 & 0 & d_1 & d_2 & 0 & & & \dots & & 0 \\
0 & 0 & \dots & 0 & 0 & e_2 & d_1 & 0 & d_1 & d_2 & 0 & & \dots & & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & d_2 & d_1 & 0 & d_1 & d_2 & 0 & \dots & & 0 \\
\vdots & \vdots & & & & \vdots & & & & \ddots & & & & & \vdots \\
 & & & & & & & & & & \ddots & & & & \\
 & & & & & & & & & & & \ddots & & & \\
 & & & & & & & & & & & & \ddots & & \\
0 & 0 & \dots & & & 0 & 0 & \dots & & & 0 & d_2 & d_1 & 0 & d_1 \\
0 & 0 & \dots & & & 0 & 0 & \dots & & & 0 & 0 & d_2 & d_1 & 0
\end{array}\right),$$

obtained from $\mathscr{F}_n$ by changing the last row and column of the corner matrix $\mathscr{A}_{p+2}$ to 0's, and replacing e_1 with 0 in row $p+1$ and column $p+1$ of $\mathscr{F}_n$.

Next note that since $\mathbf{s}$ annihilates the rows of $\mathscr{A}_{p+1}$ it follows that if we set $\mathscr{D}_p$ to be $E_{p-1,p}^{c_{p-1}} \dots E_{1,p}^{c_1}$ (note that $E_{0,p}^{c_0} = E_{0,p}$ is “missing” from this expression) then via $\mathscr{D}_p$ the matrix above is congruent to the matrix

$$
\left(\begin{array}{ccccccccccccccc}
 & & & & a_0 & 0 & 0 & & & & & & \dots & & 0 \\
 & & & & \vdots & \vdots & \vdots & & & & & & & & \vdots \\
 & & \mathscr{A}_p & & & & & & & & & & & & \\
 & & & & a_{p-2} & 0 & 0 & & & & & & \dots & & 0 \\
a_0 & a_1 & \dots & a_{p-2} & 0 & 0 & e_2 & 0 & & & & & \dots & & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & e_2 & 0 & & & & \dots & & 0 \\
\multicolumn{15}{c}{\dotfill} \\
0 & 0 & \dots & 0 & e_2 & e_1 & 0 & d_1 & d_2 & 0 & & & \dots & & 0 \\
0 & 0 & \dots & 0 & 0 & e_2 & d_1 & 0 & d_1 & d_2 & 0 & & \dots & & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & d_2 & d_1 & 0 & d_1 & d_2 & 0 & \dots & & 0 \\
\vdots & \vdots & & & & \vdots & & & & \ddots & & & & & \vdots \\
 & & & & & & & & & & \ddots & & & & \\
 & & & & & & & & & & & \ddots & & & \\
 & & & & & & & & & & & & \ddots & & \\
0 & 0 & \dots & & & 0 & 0 & \dots & & & 0 & d_2 & d_1 & 0 & d_1 \\
0 & 0 & \dots & & & 0 & 0 & \dots & & & 0 & 0 & d_2 & d_1 & 0
\end{array}\right),
$$

Applying techniques from [11], and using the assumption that $\mathscr{A}_p$ is invertible, there is a product $\mathscr{D}_0$ of elementary matrices E_{jk} with $0 < j < p-1$ and $0 < k \leqslant p-1$ which implements the congruence between $\mathscr{A}_p$ and $\mathscr{J}_p$, where $\mathscr{J}_p$ is the Toeplitz matrix with 1's along its secondary diagonals and 0's elsewhere. It follows, by comparing the first and last columns of the matrix

$$
\mathscr{R}_{p+1} = \left(\begin{array}{cccc}
 & & & a_0 \\
 & & & a_1 \\
 & & & \vdots \\
 & \mathscr{A}_p & & \\
 & & & \vdots \\
 & & & a_{p-2} \\
a_0 \ \dots & & \dots \ a_{p-2} & 0
\end{array}\right),
$$

that $\mathscr{D}_0^t \mathscr{R}_{p+1} \mathscr{D}_0$ is the matrix

$$\begin{pmatrix} & & & & 0 \\ & & & & 1 \\ & & & & 0 \\ & & & & \vdots \\ & & \mathscr{J}_p & & \\ & & & & \vdots \\ & & & & 0 \\ 0\ 1\ 0 & \dots & & \dots & 0\ 0 \end{pmatrix}.$$

The latter matrix is congruent, via the product $\mathscr{K} = E_{2p}E_{4p}\dots E_{p-2,p}$, to the matrix $\mathscr{J}_{p+1}$. Hence A_n is congruent via $\mathscr{E}_{n-2}\dots\mathscr{E}_{p+2}\mathscr{D}_{p+1}\mathscr{D}_p\mathscr{D}_0\mathscr{K}$ to the matrix $\mathscr{C}_n$:

Let β be any other binary shift of commutant index 3 whose center sequence eventually coincides with the center sequence of α. If β has bitstream $b_0b_1b_2\dots$ let $\mathscr{B}_n$ denote the $n\times n$ Toeplitz matrix whose first row is $b_0, b_1, \dots, b_{n-1}$. Then following the procedure above shows that for n sufficiently both $\mathscr{B}_n$ and $\mathscr{A}_n$ are congruent to $\mathscr{C}_n$. We will apply this observation to show that the two binary shifts α and β are cocycle conjugate.

We will use the notation $\mathscr{M}_\alpha(n)$ to denote the matrix $\mathscr{E}_{n-1}\dots\mathscr{E}_{p+2}\mathscr{D}_{p+1}\mathscr{D}_p\mathscr{D}_0\mathscr{K}$ for α which implements the congruence between $\mathscr{A}_n$ and $\mathscr{C}_n$. (We will also simplify the notation by writing $\mathscr{M}_\alpha(n)$ as $\mathscr{E}_{n-1}\dots\mathscr{E}_{p+2}\mathscr{F}$ in what follows.) Similarly $\mathscr{B}_n$ and $\mathscr{C}_n$ are congruent via the same procedure and we shall use $\mathscr{M}_\beta(n)$ to denote the matrix implementing this congruence.

Denote by F^n, (resp., F^∞) the vector space of n-tuples $[t_0, t_1, \dots, t_{n-1}]$ (resp. of finitely non-zero ∞-tuples) over the field $GF(2)$. It will be convenient to view F^k as a subspace of F^n for non-negative integers $k < n$, and F^n as a subspace of F^∞, for all n. Let $\{e_j : j \geqslant 0\}$ be the standard basis for F^∞, i.e. e_j has a 1 in the j^{th} spot and 0's elsewhere. We shall also use the notation F_0^n, (resp., F_0^∞) for the subspace of F^n (resp., of F^∞) consisting of all vectors whose first entry is 0.

For $0 \leqslant j \leqslant j+k$ the identities

$$(e_j)^t\mathscr{A}_n e_{j+k} = a_k \tag{3.3}$$

are easily verified.

The next result follows from an analysis of the transformations $\mathscr{M}_\alpha(n)$. Details of the proof may be found in [11]. In the statement of the following lemma we shall assume that p is a fixed integer chosen so that $p > m$, where m is the length of the word $w = u_0^{r_0}\dots u_m^{r_m}$ generating $\alpha^3(R)'\cap R$ and $\mathscr{A}_p$ and $\mathscr{A}_{p+4}$ are both invertible. Also we shall assume $n \in \mathbb{N}$ is chosen so that $n > p+4$.

LEMMA 3.5. (cf. [11][Theorem 6.12]) *Let n and p be as above. Then*

(o) $M_\alpha(n)^t\mathscr{A}_n\mathscr{M}_\alpha(n) = \mathscr{C}_n$,

(i) *for any j such that $0 \leqslant j \leqslant n-1$, and any $k \geqslant 1$, $\mathscr{M}_\alpha(n+k)e_j = \mathscr{M}_\alpha(n)e_j$ and $\mathscr{M}_\alpha(n+k)^{-1}e_j = \mathscr{M}_\alpha(n)^{-1}e_j$,*

(ii) $\mathscr{M}_\alpha(n)e_0 = e_0 = \mathscr{M}_\alpha(n)^{-1}e_0$,

(iii) *if* $\mathbf{v} \in F_0^n$, *then* $\mathscr{M}_\alpha(n)\mathbf{v}$ *and* $\mathscr{M}_\alpha(n)^{-1}\mathbf{v}$ *lie in* F_0^n,

(iv) *if* $\mathbf{v} \in F_0^{p+2}$ *then* $\mathscr{M}_\alpha(n)\mathbf{v}$ *and* $\mathscr{M}_\alpha(n)\mathbf{v}$ *lie in* F_0^{p+2}.

(v) *if* $j \geqslant p+2$ *then both* $\mathscr{M}_\alpha(n)e_j$ *and* $\mathscr{M}_\alpha(n)^{-1}e_j$ *lie in* F_0^{j+1} *(the linear span of* $\{e_1,\ldots,e_j\}$.

We remark that it follows from the form of $\mathscr{M}_\alpha(n)$ for $n > p+4$ that if $p+4 < k \leqslant n$ then $\mathscr{M}_\alpha(n)e_k$ and $\mathscr{M}_\alpha(n)^{-1}e_k$ are both in F^k but not in F^{k-1}, i.e. that both of these vectors end in e_k.

Given the results of the lemma it makes sense to define an invertible transformation $\mathscr{M}_\alpha$ on F_0^∞ by setting

$$\mathscr{M}_\alpha e_j = \underset{n\to\infty}{\underrightarrow{\ }} \lim \mathscr{M}_\alpha(n)e_j,$$

for $j \geqslant 1$, and extending $\mathscr{M}_\alpha$ to all of F_0^∞ by linearity. Similarly for M_α^{-1}.

From now on let β be another binary shift of commutant index 3 whose center sequence eventually agrees with that of α. Let $v_0, v_1, \ldots$ be the spin generators for β, and let $w' = v_0^{s_0} \ldots v_{m'}^{s_{m'}}$ be the word generating $\beta^3(R)' \cap R$. Let $b_0 b_1 \ldots$ be the bitstream defining the commutation relations among the generators. Let $\mathscr{B}$ be the Toeplitz matrix corresponding to this bitstream with upper $n \times n$ corners denoted by $\mathscr{B}_n$. Finally let $\mathscr{W}(n) = \mathscr{W}_{\alpha,\beta}(n)$ be the invertible linear transformation $\mathscr{M}_\alpha(n)\mathscr{M}_\beta^{-1}(n)$ on F_n. Note from the lemma that $\mathscr{W}(n)$ restricts to an invertible transformation on F_0^n.

For the remainder of this section we assume that p has been chosen so that $p > m_0 = \max\{m, m'\}$, and such that the center sequences of both α and β agree and coincide with one of the center sequences in Theorem 2.5 from position p and above. We assume p has also been chosen so that the center sequences for both α and β take the values 01210 for $k = p$ through $k = p+4$. In particular $\mathscr{A}_p, \mathscr{B}_p, \mathscr{A}_{p+4}, \mathscr{B}_{p+4}$ are all invertible. Finally, we shall assume $n \in \mathbb{N}$ has been chosen so that $n > p+4$.

The following result is obtained as an application of the lemma. The proof uses the fact that $E_{ij}^{-1} = E_{ij}$.

THEOREM 3.6. *Let* $\mathscr{S}$ *denote the unilateral shift on* F^∞. *Under the standing assumptions of the preceding,*

(0) $\mathscr{W}(n)^t \mathscr{A}_n \mathscr{W}(n) = \mathscr{B}_n$,

(i) *for any* j *such that* $0 \leqslant j \leqslant n-1$,*and any* $k \geqslant 1$, $\mathscr{W}(n+k)e_j = \mathscr{W}(n)e_j$ *and* $\mathscr{W}(n+k)^{-1}e_j = \mathscr{W}(n)^{-1}$,

(ii) $\mathscr{W}(n)e_0 = e_0 = \mathscr{W}(n)^{-1}e_0$,

(iii) *if* $\mathbf{v} \in F_0^n$ *then* $\mathscr{W}(n)\mathbf{v}$ *and* $\mathscr{W}(n)^{-1}\mathbf{v}$ *lie in* F_0^n,

(iv) *if* $\mathbf{v} \in F_0^{p+2}$ *then* $\mathscr{W}(n)\mathbf{v}$ *and* $\mathscr{W}(n)^{-1}\mathbf{v}$ *lie in* F_0^{p+2},

(v) *if* $j \geqslant p+2$ *then both* $\mathscr{W}_\alpha(n)e_j$ *and* $\mathscr{W}_\alpha(n)^{-1}e_j$ *lie in* F_0^{j+1} *(the linear span of* $\{e_1,\ldots,e_j\}$,

(vi) *for any* k *such that* $n > k > p$, *and for any* $\mathbf{v}$ *of the form* $s_{m'}e_{k-m'} + s_{m'-1}e_{k-(m'-1)} + \ldots + s_1e_{k-1} + e_k$, $\mathscr{W}(n)\mathscr{S}\mathbf{v} = \mathscr{S}\mathscr{W}(n)\mathbf{v} = r_me_{k+1-m} + r_{m-1}e_{k+1-(m-1)} + \ldots + r_1e_k + e_{k+1}$.

Proof. All but the last statement follow from their counterparts in the lemma. For (vi) note that if $\mathscr{E}'_j = E^{s_{m'}}_{j-m',j}E_{j-(m'-1,j)})^{s_{m'-1}} \ldots E^{s_1}_{j-1,j}$ for $j > m_0$, then $\mathscr{M}_\beta(n) = \mathscr{E}'_{n-1}\mathscr{E}'_{n-2}\ldots\mathscr{E}'_{p+2}\mathscr{F}'$, where $\mathscr{F}'$ is the counterpart for β of $\mathscr{F}$ in the expression for $\mathscr{M}_\alpha(n)$. Noting that $(\mathscr{E}'_k)^{-1}\mathbf{v} = e_k$ and that $(\mathscr{E}'_j)^{-1}\mathbf{v} = \mathbf{v}$ for $j > k$, it follows that

$$\begin{aligned}
\mathscr{W}(n)\mathbf{v} &= \mathscr{M}_\alpha(n)\mathscr{M}_\beta(n)^{-1}\mathbf{v}\\
&= \mathscr{M}_\alpha(n)(\mathscr{E}'_{n-1}\mathscr{E}'_{n-2}\ldots\mathscr{E}'_{p+2}\mathscr{F}')^{-1}\mathbf{v}\\
&= \mathscr{M}_\alpha(n)(\mathscr{F}')^{-1}(\mathscr{E}'_{p+2})^{-1}\ldots(\mathscr{E}'_{n-1})^{-1}\mathbf{v}\\
&= \mathscr{M}_\alpha(n)(\mathscr{F}')^{-1}(\mathscr{E}'_{p+2})^{-1}\ldots(\mathscr{E}'_{k})^{-1}\mathbf{v}\\
&= \mathscr{M}_\alpha(n)(\mathscr{F}')^{-1}(\mathscr{E}'_{p+2})^{-1}\ldots(\mathscr{E}'_{k-1})^{-1}e_k\\
&= \mathscr{M}_\alpha(n)(\mathscr{F}')^{-1}e_k\\
&= \mathscr{M}_\alpha(n)e_k\\
&= \mathscr{E}_{n-1}\mathscr{E}_{n-2}\ldots\mathscr{E}_{p+2}\mathscr{F}e_k\\
&= \mathscr{E}_{n-1}\mathscr{E}_{n-2}\ldots\mathscr{E}_{p+2}e_k\\
&= \mathscr{E}_{n-1}\mathscr{E}_{n-2}\ldots\mathscr{E}_{k}e_k\\
&= \mathscr{E}_{n-1}\mathscr{E}_{n-2}\ldots\mathscr{E}_{k+1}(r_me_{k-m} + r_{m-1}e_{k-(m-1)} + \ldots + r_1e_{k-1} + e_k)\\
&= r_me_{k-m} + r_{m-1}e_{k-(m-1)} + \ldots + r_1e_{k-1} + e_k.
\end{aligned}$$

The last statement of the theorem follows from this calculation. □

From the theorem it makes sense to define $\mathscr{W}$ on F^∞ as $\mathscr{W} = \lim_{\overrightarrow{n\to\infty}} \mathscr{W}(n)$.

LEMMA 3.7. (cf. [11][Theorem 6.12]) *For all* $j > 0$ *and for all* $k \geqslant 0$ *it follows that* $(\mathscr{W}^{-1}e_j)^t\mathscr{B}(\mathscr{W}^{-1}e_{j+k}) = a_k$.

Proof. We have

$$\begin{aligned}
(\mathscr{W}^{-1}e_j)^t\mathscr{B}\mathscr{W}^{-1}e_{j+k} &= (\mathscr{M}_\beta\mathscr{M}_\alpha^{-1}e_j)^t\mathscr{B}\mathscr{M}_\beta\mathscr{M}_\alpha^{-1}e_{j+k}\\
&= e_j^t(\mathscr{M}_\alpha^{-1})^t\mathscr{M}_\beta^t\mathscr{B}\mathscr{M}_\beta\mathscr{M}_\alpha^{-1}e_{j+k}\\
&= e_j^t(\mathscr{M}_\alpha^{-1})^t\mathscr{C}\mathscr{M}_\alpha^{-1}e_{j+k}, \text{ where } \mathscr{C} = \text{“lim”}\mathscr{C}_n\\
&= e_j^t\mathscr{A}e_{j+k}\\
&= a_k. \qquad \square
\end{aligned}$$

As we shall see in the next section, the linear transformation $\mathscr{W}^{-1}\mathscr{S}\mathscr{W}\mathscr{S}^{-1}$ on F_0^∞ in the following lemma is closely related to a unitary operator in R which implements the cocycle conjugacy between α and β.

LEMMA 3.8. *For $j>0$ and $k \geqslant 0$, $(\mathscr{W}^{-1}\mathscr{S}\mathscr{W}\mathscr{S}^{-1}e_j)^t\mathscr{B}\mathscr{W}^{-1}\mathscr{S}\mathscr{W}\mathscr{S}^{-1}e_{j+k} = b_k$.*

Proof. We calculate

$$\begin{aligned}(\mathscr{W}^{-1}\mathscr{S}\mathscr{W}\mathscr{S}^{-1}e_j)^t\mathscr{B}\mathscr{W}^{-1}\mathscr{S}\mathscr{W}\mathscr{S}^{-1}e_{j+k} &= (\mathscr{W}^{-1}\mathscr{S}\mathscr{W}e_{j-1})^t\mathscr{B}\mathscr{W}^{-1}\mathscr{S}\mathscr{W}e_{j+k-1}\\ &= e^t_{j-1}\mathscr{W}^t\mathscr{S}^t(\mathscr{W}^{-1})^t\mathscr{B}\mathscr{W}^{-1}\mathscr{S}\mathscr{W}e_{j+k-1}\\ &= e^t_{j-1}\mathscr{W}^t\mathscr{S}^t\mathscr{A}\mathscr{S}\mathscr{W}e_{j+k-1}\\ &= e^t_{j-1}\mathscr{W}^t\mathscr{A}\mathscr{W}e_{j+k-1}\\ &= e^t_{j-1}\mathscr{B}e_{j+k-1}\\ &= b_k,\end{aligned}$$

where we have used the fact that $\mathbf{v}\mathscr{S}^t\mathscr{A}\mathscr{S}\mathbf{w} = \mathbf{v}\mathscr{A}\mathbf{w}$ for any vectors $\mathbf{v}$ and $\mathbf{w}$ in F^∞. $\square$

REMARK 3.9. From the proof of (vi) in the theorem note that $M_\beta^{-1}\mathbf{v} = e_k$ for $k>p$, where $\mathbf{v} = s_{m'}e_{k-m'} + s_{m'-1}e_{k-(m'-1)} + \ldots + s_1e_{k-1} + e_k$.

COROLLARY 3.10. *The mapping $\varphi = \mathscr{W}^{-1}\mathscr{S}\mathscr{W}\mathscr{S}^{-1}$ is well-defined as a linear transformation on F_0^∞ and is in fact an isomorphism on F_0^∞. Moreover, when restricting $\mathscr{B}$ to F_0^∞,*

$$(\mathscr{W}^{-1}\mathscr{S}\mathscr{W}\mathscr{S}^{-1})^t\mathscr{B}\restriction_{F_0^\infty}(\mathscr{W}^{-1}\mathscr{S}\mathscr{W}\mathscr{S}^{-1}) = \mathscr{B}\restriction_{F_0^\infty}.$$

Therefore $\varphi(e_j)^t\mathscr{B}\varphi(e_k) = e_j^t\mathscr{B}e_k$ for all $j,k\in\mathbb{N}$.

Proof. Note by $(ii),(iii)$ and (iv) of the theorem, $\varphi(e_1) = \mathscr{W}^{-1}\mathscr{S}\mathscr{W}e_0 = \mathscr{W}^{-1}\mathscr{S}e_0 = \mathscr{W}^{-1}e_1 \in F_0^\infty$, and for $k>1$ parts (iii) and (iv) of the theorem show that $\varphi(e_k)\in F_0^\infty$ as well. In particular it follows from (iii) that $\varphi\restriction_{F_0^n}$ is an isomorphism on F_0^n for all $n\geqslant p$, hence φ itself is actually an isomorphism on F_0^∞.

It is straightforward to see, from the symmetry of $\mathscr{A}$, that for $j,k\in\mathbb{N}$, $e_j^t\mathscr{A}e_k = e^t_{j-1}\mathscr{A}e_{k-1}$. Therefore, on F_0^∞,

$$\begin{aligned}(\mathscr{W}^{-1}\mathscr{S}\mathscr{W}\mathscr{S}^{-1})^t\mathscr{B}(\mathscr{W}^{-1}\mathscr{S}\mathscr{W}\mathscr{S}^{-1}) &= \mathscr{S}^{-1^t}\mathscr{W}^t\mathscr{S}^t\mathscr{W}^{-1^t}\mathscr{B}\mathscr{W}^{-1}\mathscr{S}\mathscr{W}\mathscr{S}^{-1}\\ &= \mathscr{S}^{-1^t}\mathscr{W}^t\mathscr{S}^t\mathscr{A}\mathscr{S}\mathscr{W}\mathscr{S}^{-1}\\ &= \mathscr{S}^{-1^t}\mathscr{W}^t\mathscr{A}\mathscr{W}\mathscr{S}^{-1}\\ &= \mathscr{S}^{-1^t}\mathscr{B}\mathscr{S}^{-1}\\ &= \mathscr{B}. \qquad \square\end{aligned}$$

From now on we will specialize to the case where β is the binary shift β_2 from Theorem 2.5 above with generators $v_0, v_1, \ldots$ and bitstream $01001001001\ldots$. We have shown that β has commutant index 3 and that v_0v_3 is the word generating $\beta^3(R)'\cap R$.

Our goal is to show that if α is any other binary shift of commutant index 3, whose center sequence eventually coincides with that of β, then β and α are cocycle conjugate.

We record the following remark, which follows immediately as a special case of the remark above and part (vi) of Theorem 3.6.

REMARK 3.11. For $k > p$, $M_\beta^{-1}(e_k + e_{k-3}) = e_k$, and $\mathscr{W}\mathscr{S} = \mathscr{S}\mathscr{W}$ on the vector space of finitely non-zero linear combinations of $\{e_{p+3} + e_p, e_{p+4} + e_{p+1}, \dots\}$.

4. Cocycle conjugacy results

As above we will assume in this section that $\beta = \beta_2$ is the binary shift of commutant index 3 from Theorem 2.5. Below we follow the approach of [10, 11] to define an automorphism π on $\beta(R)$ related to the map φ on F_0^∞ of the previous section. As we will see, π is "nearly" an inner automorphism in the sense that for sufficiently large n there is a unitary operator y in B_n (more specifically, in $B_n \cap NN(\beta)$, (see the paragraph following Theorem 4.5 below for the definition of $NN(\beta)$) such that, for any word v in the generators $v_0, v_1, \dots$, $\pi(v) = \pm y^* v y$. Using y we will be able to show that β and α above are cocycle conjugate, i.e., if α is any binary shift of commutant index 3 whose center sequence eventually coincides with the center sequence $10\overline{101210}$ of β, then α and β are cocycle conjugate (see Theorem 4.12).

A similar analysis can be carried out to show that if α is a binary shift of commutant index 3 whose center sequence eventually coincides with the center sequence of β_3 of Example 3 (respectively, of β_5 of Example 5) then α is cocycle conjugate to β_3 (respectively, to β_5).

Recall that the derived shift of β_2 is conjugate to β_1, as is the derived shift of β_3, whereas the derived shift of β_5 is conjugate to β_4. Therefore the center sequences of each of the binary shifts β_2, β_3 and β_5 do not eventually coincide with the center sequences of the corresponding derived shift. On the other hand, in [11][Theorem 7.12] it was shown that if α is a binary shift of finite commutant index whose center sequence eventually agrees with the center sequence of its derived shift α_∞ then α and α_∞ are cocycle conjugate.

As the center sequence of any shift of commutant index 3 must eventually coincide with the center sequence of one of the binary shifts $\beta_i, i = 1,2,3,4,5$, we can combine our results from this section and from [11] to conclude that there are at most 5 cocycle conjugacy classes of shifts of commutant index 3.

DEFINITION 4.1. Given a vector $\mathbf{s} = s_0e_0 + s_1e_1 + \dots + s_qe_q \in F^\infty$, let $\chi(\mathbf{s})$ be the word $v_0^{s_0} v_1^{s_1} \dots v_q^{s_q}$ in R.

Next we use the mapping χ to define a mapping π on $\beta(R)$. First note by an application of (3.3) it follows the words $\chi(\mathbf{s})$ and $\chi(\mathbf{t})$ commute if and only if $\mathbf{s}^t \mathscr{B} \mathbf{t} = 0$. From Corollary 3.10 we have

$$\varphi(e_j)^t \mathscr{B} \varphi(e_k) = e_j^t \mathscr{B} e_k$$

for all $j,k \in \mathbb{N}$. It therefore follows that if we define words x_j in $\beta(R)$ by $x_j = \chi(\varphi(e_j))$, for $j \geqslant 1$, then for $j,k \geqslant 1$, x_j and x_k commute if and only if v_j and v_k do.

We define π on the v_j's, for $j \geqslant 1$, by

$$\pi(v_j) = \begin{cases} x_j, & \text{if} \quad x_j = x_j^*, \quad \text{and} \\ \sqrt{-1}x_j, & \text{if} \quad x_j = -x_j^* \end{cases}$$

For convenience we will write $\pi(v_j) = w_j$ for all $j \in \mathbb{N}$.

By Corollary 3.10 the mapping φ is an isomorphism of F_0^∞. It follows that the set of linear combinations of words in the w_j's, for $j \geqslant 1$ is weakly dense in $\beta(R)$. Hence by defining π on words $v = v_1^{t_1} v_2^{t_2} \dots v_r^{t_r}$ according to

$$\pi(v) = w_1^{t_1} w_2^{t_2} \dots w_r^{t_r},$$

π extends to a $*$-isomorphism on $\beta(R)$.

The following notation will be useful.

DEFINITION 4.2. For any $n \in \mathbb{N}$ let B_n^0 be the C^*-subalgebra of B_n generated by $v_1, \dots, v_n$.

We now wish to show that we can assume that π fixes the words $v_n v_{n+3}$ for all $n > p+2$. To see this note first from Theorem 3.6(vi) that φ fixes $e_n + e_{n+3}$, so π fixes $v_n v_{n+3}$ up to multiplication by a scalar, i.e. $\pi(v_n v_{n+3}) = b_n v_n v_{n+3}$ for some $b_n \in \mathbb{C}$ of modulus one. On the other hand, since φ is an isomorphism of F_0^n for $n \geqslant p+2$ it follows that π restricts to a $*$-automorphism of B_n^0. Fix $n \geqslant p+2$. From the paragraph following Lemma 3.5 we see that we can assume $\varphi(e_n)$ "ends" with e_n, hence $\pi(v_n)$ "ends" with v_n, i.e. there is a unitary operator w, in the algebra generated by v_1 through v_{n-1}, such that $\pi(v_n) = w v_n$. Since $\pi(v_n v_{n+3}) = b_n v_n v_{n+3}$, $\pi(v_{n+3}) = c w v_{n+3}$ for some scalar c. Since the word $w v_n = \pi(v_n)$ is hermitian, $w v_n = (w v_n)^* = v_n w^*$, so $v_n w v_n = w^*$. Then

$$\begin{aligned} v_{n+3} w v_{n+3} &= v_{n+3} v_n (v_n w v_n) v_n v_{n+3} \\ &= v_{n+3} v_n w^* v_n v_{n+3} \\ &= v_{n+3} v_n v_n v_{n+3} w^* \\ &= w^*, \end{aligned}$$

where the next to last equality holds because $v_0 v_3$ commutes with $v_3, v_4, \dots$ and therefore, by symmetry $v_n v_{n+3}$ commutes with $v_n, v_{n-1}, \dots, v_1$. Hence $w v_{n+3}$ is hermitian if $w v_n$ is. Therefore, having defined $\pi(v_n)$ as $w v_n$ we can define $\pi(v_{n+3})$ as $w v_{n+3}$, if $w v_n w = v_n$, and as $-w v_{n+3}$ if $w v_n w = -v_n$. In either case we have $\pi(v_n v_{n+3}) = v_n v_{n+3}$. Therefore, having defined $\pi(v_{p+2}), \pi(v_{p+3})$ and $\pi(v_{p+4})$ we can define v_j for $j \geqslant p+5$ such that π fixes $v_n v_{n+3}$ for all $n \geqslant p+2$. Hence we have established the following result.

LEMMA 4.3. *There is a $*$-automorphism π of $\beta(R)$ such that $\pi(v_j)$ is a scalar multiple of $\chi(\varphi(e_j))$, for all $j \in \mathbb{N}$, and for all $n \geqslant p+2$, π fixes $v_n v_{n+3}$.*

REMARK 4.4. Note from Theorem 3.6(vi) that for $n \geqslant p+2$, $\varphi(e_n)$ is a vector which ends in e_n. Hence $\pi(v_n)$ is a scalar multiple of a word which ends in v_n.

In [8][Lemma 3.3] Powers obtained the following characterization of the normalizer $N(\beta)$ of β, i.e. the subgroup of unitary operators w in R such that $w^* x w \in \beta(R)$ for all $x \in \beta(R)$.

THEOREM 4.5. *A unitary operator w is in $N(\beta)$ if and only w has the form λI or $\lambda v_{k_1}v_{k_2}\dots v_{k_s}$ where $0 \leqslant k_1 < k_2 < \dots < k_s$.*

In [9][Theorem 3.7] it was shown that if $y \in R$ is a unitary operator with the property that $Ad(y)$ maps words in the v_j's into other words, then y is a finite product of words and operators of the form $\Gamma_\pm(w)$, for $w \in N(\beta)$, where $\Gamma_\pm = (1/\sqrt{2})(I \pm iw)$, if $w = w^*$ and $(I \pm w)/\sqrt{2}$, if $w = -w^*$. We use the notation $NN(\beta)$ (the *normalizer* of the normalizer group $N(\beta)$) to denote the group of such operators. The proof can easily be adapted to show that the result holds for factors B_n as well as for R, i.e., if $y \in B_n$, where B_n is a factor and $Ad(y)$ leaves $N(\beta) \cap B_n$ invariant, then y is of the form above. We shall use this result in the proof of the following theorem.

THEOREM 4.6. *For any $n > p+2$, π restricts to an automorphism of B_n^0. If, in addition, B_n^0 is a factor, then there is a unitary operator $y_n \in B_n^0 \cap NN(\beta)$ such that, for all words $v = v_{k_1}v_{k_2}\dots v_{k_s}$ in B_n^0, $\pi(v) = y_n^* v y_n$.*

Proof. For $n > p+2$ it follows that φ restricts to an isomorphism of F_0^n and therefore, $\pi(v_j) = \chi(\varphi(e_j)), 1 \leqslant j \leqslant n$ is in B_n^0, hence π is an automorphism of B_n^0. If B_n^0 is a factor then the automorphism $\pi\restriction_{B_n^0}$ is inner. Let y_n be a unitary operator implementing this automorphism.

To show that $y_n \in NN(\beta)$ note by Lemma 4.3 that π maps words $v \in B_n^0$ in the v_j's into scalar multiples of words in the v_j's. Therefore $Ad(y_n)(v)$ is a word in the v_j's. It follows that $y_n \in NN(\beta)$, from the remark in the paragraph preceding the theorem. $\square$

REMARK 4.7. Note that for $n \in \mathbb{N}$, B_n^0 is a factor if and only if B_{n-1} is a factor, since $B_{n-1} = \{v_0, \dots, v_{n-1}\}''$ and $B_n^0 = \beta(B_{n-1}) = \{v_1, \dots, v_n\}''$ are isomorphic.

COROLLARY 4.8. *For every $n > p+2$ such that B_n^0 is a factor, let $y_n \in B_n^0$ satisfy $\pi_{\restriction B_n^0} = Ad(y_n)$ as above. Then B_{n+6}^0 is also a factor and $y_n y_{n+6}^* \in NN(\beta) \cap (B_n^0)' \cap \{v_n v_{n+3}, v_{n+1}v_{n+4}, v_{n+2}v_{n+5}, v_{n+3}v_{n+6}\}' \cap B_{n+6}^0$.*

Proof. Since the center sequence is eventually periodic with period 6 it follows that B_{n+6}^0 is also a factor. Since $\pi_{\restriction B_n^0} = Ad(y_n)$ and $\pi_{\restriction B_{n+6}^0} = Ad(y_{n+6})$, $y_n^* x y_n = y_{n+6}^* x y_{n+6}$ for all $x \in B_n^0$, so that $y_n y_{n+6}^*$ commutes with B_n^0. Since $v_0 v_3 \in \beta^3(R)' \cap R$, it follows that v_0v_3, v_1v_4, v_2v_5 and v_3v_6 all commute with $\{v_6, v_7, v_8, \dots\}''$. By the symmetry of the commutation relations for the spin system corresponding to β it follows that the operators $v_n v_{n+3}, v_{n+1}v_{n+4}, v_{n+2}v_{n+5}$ and $v_{n+3}v_{n+6}$ all commute with B_n^0 and hence with $y_n \in B_n^0$. On the other hand, y_{n+6} commutes with each of these four operators, since π fixes them.

Since π maps words in $\beta(R)$ into words in $\beta(R)$, and since $Ad(y_n)_{\restriction B_n^0} = \pi_{\restriction B_n^0}$, it follows from the argument in the paragraph preceding Theorem 4.6 that $y_n \in NN(\beta)$. Similarly for y_{n+6}. Hence $y_n y_{n+6}^* \in NN(\beta)$ also. $\square$

The following is an immediate consequence of the proof of Theorem 2.5.

PROPOSITION 4.9. *Let $n \geqslant p+2$ be such that B_n^0 is a factor and the center $Z(B_{n+1}^0)$ of B_{n+1}^0 (respectively, $Z(B_{n+2}^0)$ of B_{n+2}^0) is generated by one (respectively, two) words. Then $Z(B_{n+1}^0) = \{z\}''$, where $z = v_1 v_2 \dots v_{n+1}$ and $Z(B_{n+2}^0) = \{z, \beta(z)\}''$.*

THEOREM 4.10. *Fix $n > p+2$ such that $v_n = 0, v_{n+1} = 1$ and $v_{n+2} = 2$. Then $C = (B_n^0)' \cap \{v_n v_{n+3}, v_{n+1} v_{n+4}, v_{n+2} v_{n+5}, v_{n+3} v_{n+6}\}' \cap B_{n+6} = \{z_0 z_1 z_2, z_1 z_2 z_3\}''$ where $z_0 = v_n v_{n+3}$ and $z_j = \beta^j(z_0)$,for $j = 0,1,2,3$.*

Proof. It is straightforward to see that C is generated by the words that it contains. Suppose $w \in C$ is a word. Noting that $B_{n+6}^0 = B_{n+2}^0 \vee \{z_0, z_1, z_2, z_3\}''$ we can write $w = (v_1^{t_1} v_2^{t_2} \dots v_{n+2}^{t_{n+2}})(z_0^{p_0} z_1^{p_1} z_2^{p_2} z_3^{p_3})$ where the exponents are 0's or 1's. Since w and z_0 through z_3 commute with B_n^0 it follows that $\tilde{w}$ does too, where $\tilde{w} = v_1^{t_1} v_2^{t_2} \dots v_{n+2}^{t_{n+2}}$. We will show that $\tilde{w}$ is a scalar multiple of a word of the form $u^{s_0}\beta(u)^{s_1}$, where u is the word generating the center of B_{n+1}^0 (and therefore, by Proposition 4.9, u and $\beta(u)$ are the words generating the center of B_{n+2}^0). Assume that $\tilde{w}$ is a nontrivial word, then $\tilde{w} \notin B_n^0$ since B_n^0 has trivial center. Therefore $\tilde{w}$ is a word that ends with either v_{n+1} or v_{n+2} and so, since u ends with v_{n+1}, by Theorem 2.3, there is a word of the form $u^{s_0}\beta(u)^{s_1}$ such that $\tilde{w}u^{s_0}\beta(u)^{s_1} \in B_n$ and commutes with B_n^0. Since B_n^0 has trivial center, $\tilde{w}$ must be a scalar multiple of $u^{s_0}\beta(u)^{s_1}$. Therefore we may assume that w has the form $u^{s_0}\beta(u)^{s_1} z_0^{p_0} z_1^{p_1} z_2^{p_2} z_3^{p_3}$.

From the commutation relations associated with the bitstream for β it follows that z_0 anticommutes with both z_1 and z_2 and commutes with z_3. Also note from the commutation relations for β that $v_0 v_3$ commutes with v_0, anticommutes with both v_1 and v_2, and commutes with $v_3, v_4, \dots$. Therefore we can use the symmetry of the commutation relations to conclude that z_0 anticommutes with v_{n+1} and v_{n+2} and commutes with v_j for $1 \leqslant j \leqslant n$. We also have the result from the preceding proposition that $u = v_1 v_2 \dots v_{n+1}$. Using the observations above we arrive at the following equations over $GF(2)$, from w commuting with z_0 through z_3.

$$\begin{aligned} s_0 + p_1 + p_2 &= 0 \\ s_1 + p_0 + p_2 + p_3 &= 0 \\ p_0 + p_1 + p_3 &= 0 \\ p_1 + p_2 &= 0 \end{aligned}$$

Then $s_0 = 0$, $s_1 = 0$, $p_1 = p_2$ and $p_3 = p_0 + q$, where $q = p_1 = p_2$. This establishes the claim. □

Using the preceding results we can show that the $*$-automorphism π of $\beta(R)$ is ”nearly” inner.

COROLLARY 4.11. *Let n and $y = y_n$ as above. Then for any word z in the generators $v_1, v_2, \dots$, $\pi(z) = \pm y^* z y$.*

Proof. By assumption $y^* z y = \pi(z)$ for all words $z \in B_n^0$. Since B_{n+6}^0 is generated by B_n^0 and the words $z_0, z_1, z_2, z_3, z_{-1} = v_{n-1} v_{n+2}$ and $z_{-2} = v_{n-2} v_{n+1}$, we may assume that z is one of these words. Since y commutes with z_0 through z_3

we have $y^*z_jy = z_j = \pi(z_j)$ for $0 \leqslant j \leqslant 3$. Let $w = y^*_{n+6}y$. Since y and y_{n+6} are in $NN(\beta)$, so is w. Then if $j = -1$ or -2 we have, since both π and $Ad(y_{n+6})$ fix z_j, $y^*z_jy = w^*y^*_{n+6}z_jy_{n+6}w = w^*z_jw$. Since $w \in C$, where C is as in the previous theorem, it follows from the theorem and paragraph describing $NN(\beta)$ following Theorem 4.5 that w^*z_jw must be a scalar multiple of one of the following words: $z_j, z_jz_0z_1z_2, z_jz_1z_2z_3$ or $z_jz_0z_3$. But $y^*z_jy \in B^0_{n+2}$ whereas z_j is the only word of the four above that is in B^0_{n+2}. Therefore we have shown that $y^*z_jy = \pm y^*_{n+6}z_jy_{n+6}$ and that therefore $Ad(y)$ agrees with π on words in B^0_{n+6}, up to multiplication by ± 1.

Similarly $Ad(y_{n+6})$ agrees with π on words in B^0_{n+12}, up to scalar multiplication by ± 1. But since B^0_{n+12} is generated by B^0_n and z_j, for $-2 \leqslant j \leqslant 9$, and since $Ad(y), Ad(y_{n+6})$ and $Ad(y_{n+12})$ all fix the z_j's up to multiplication by ± 1; and since $Ad(y_{n+12})$ agrees with π on B^0_{n+12}, it follows that $Ad(y)$ agrees with π on words in B^0_{n+12} up to a multiple of -1. Continuing inductively establishes the result.

THEOREM 4.12. *Let α be a binary shift on R of commutant index 3 and center sequence that eventually coincides with the center sequence of β. Then β and α are cocycle conjugate.*

Proof. Let $\mathscr{W} = \mathscr{M}_\alpha \mathscr{M}_\beta^{-1}$ be the invertible linear transformation defined in the paragraph preceding Lemma 3.7. From Lemma 3.7 it follows that for any $j \in \mathbb{N}$ and $k \in \mathbb{Z}^+$, $(\mathscr{W}^{-1}e_j)^t \mathscr{B} \mathscr{W}^{-1} e_{j+k} = a_k = e^t_j \mathscr{A} e_{j+k}$. Hence if we define $x_j, j \in \mathbb{Z}^+$ by $x_j = \chi(\mathscr{W}^{-1}e_j)$, the x_j's satisfy the same commutation relations as do the spin generators for α.

Since $\mathscr{W}^{-1}$ is an invertible linear transformation on F^∞ it follows that F^∞ is spanned by $\{\mathscr{W}^{-1}e_j : j \geqslant 0\}$. From the definition of the x_j's in the preceding paragraph we may therefore conclude that every generator w_k is a word in the x_j's. Hence the von Neumann algebra generated by the x_j's coincides with R.

Let y be the unitary operator defined in the previous result. Then y satisfies $Ad(y)(v) = \pm\pi(v)$ for every word v in the v_j's. We will show that $Ad(y) \circ \beta$ is conjugate to α, cf. [10]. We shall do this by demonstrating that $Ad(y) \circ \beta(x_j) = \pm x_{j+1}$ for all $j \geqslant 0$. To begin note that $\mathscr{W}^{-1}e_0 = e_0$ from Theorem 3.6(ii), $x_0 = \chi(\mathscr{W}^{-1}e_0) = \chi(e_0) = v_0$. But then

$$
\begin{aligned}
y^*\beta(x_0)y &= y^*v_1y \\
&= \pm\pi(v_1) \\
&= \pm\chi(\varphi(e_1)) \\
&= \pm\chi(\mathscr{W}^{-1}\mathscr{S}\mathscr{W}\mathscr{S}^{-1}e_1) \\
&= \pm\chi(\mathscr{W}^{-1}\mathscr{S}\mathscr{W}e_0) \\
&= \pm\chi(\mathscr{W}^{-1}e_1) = x_1
\end{aligned}
$$

Suppose $y^*\beta(x_j)y = \pm x_{j+1}$ for $0 \leqslant j \leqslant k-1$. Since $\beta\circ\chi = \chi\circ\mathscr{S}$ on F_0^∞,

$$\begin{aligned} y^*\beta(x_k)y &= y^*\beta(\chi(\mathscr{W}^{-1}e_k))y \\ &= \pm y^*\chi(\mathscr{S}\mathscr{W}^{-1}e_k)y \\ &= \pm\pi(\chi(\mathscr{S}\mathscr{W}^{-1}e_k)) \end{aligned}$$

and since $\pi\circ\chi = \chi\circ\varphi$,

$$\begin{aligned} y^*\beta(x_k)y &= \pm\chi(\varphi(\mathscr{S}\mathscr{W}^{-1}e_k) \\ &= \pm\chi(\mathscr{W}^{-1}\mathscr{S}\mathscr{W}\mathscr{S}^{-1}\mathscr{S}\mathscr{W}^{-1}e_k) \\ &= \pm\chi(\mathscr{W}^{-1}e_{k+1}) = \pm x_{k+1}. \end{aligned}$$

Define $x'_j, j\in\mathbb{N}\cup\{0\}$ inductively by $x'_0 = x_0$ and for $j\geqslant 0$, $x'_{j+1} = Ad(y)\circ\beta(x'_j)$. Then $x'_j = \pm x_j$ for all j and therefore the x'_j's satisfy the same commutation relations as the w_j's. Therefore we have shown that $Ad(y)\circ\beta$ is conjugate to α from which we can conclude that α and β are cocycle conjugate. □

We suspect that for any $k\geqslant 2$ there are only finitely many cocycle conjugacy classes of binary shifts of commutant index k. The proof that we have used to establish the result for commutant index 3 does not immediately generalize, however. In the proof above we relied on the fact that if α has commutant index 3 its center sequence contains infinitely many strings of the form 1210. The analogous result is not necessarily true for the higher commutant cases, i.e., it is not always true that the center sequence of a binary shift of commutant index k with $k\geqslant 4$ contains has the property that $c_n = k-1$ for infinitely many n. In [13][Example 4.2], for example, a binary shift of commutant index 4 is identified whose center sequence eventually has period 2. The proof of Theorem 4.12 does not seem to generalize to cases such as this. We also suspect that the eventual pattern of the center sequences of binary shifts of a fixed commutant index is a complete cocycle conjugacy invariant. As evidence to support this conjecture R. T. Powers and the author showed in [9] that if $Ad(y)\circ\beta$ and α are conjugate with $y\in NN(\beta)$ the center sequences of β and α must eventually coincide.

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