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ON THE CLOSED SUBIDEALS OF $L(\ell_p \oplus \ell_q)$

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Abstract. In this paper we first review the known results about the closed subideals of the space of bounded operator on $\ell_p \oplus \ell_q$, $1 < p < q < \infty$, and then construct several new ones.

1. Introduction

For very few Banach spaces X all the closed subideals of $L(X)$, the algebra of all bounded and linear operators on X, are determined. In 1941 Calkin [6] showed that the only proper, non-trivial and closed ideal of $L(\ell_2)$ is the ideal of compact operators. The same was shown to be true for ℓ_p $(1 \leqslant p < \infty)$ and c_0 in [13]. Until very recently it was open if there are any other infinite dimensional Banach spaces X, for which the compact operators are the only proper, non-trivial and closed subideal of $L(X)$. We call such spaces *simple*. Then Argyros and Haydon [3] established the existence of Banach spaces with a basis on which all operators are a compact perturbation of a scalar multiple of the identity. It follows immediately that such spaces are simple. But it is not known whether or not there are any other simple spaces admitting an unconditional basis (and thus having a rich structure of operators on them).

The structure of the closed ideals of operators on non separable Hilbert spaces was independently obtained by Gramsch [14] and Luft [21]. Recently Daws [7] extended their results to non separable ℓ_p-spaces, $1 \leqslant p < \infty$, and non separable c_0-spaces.

Beyond these spaces the complete structure of closed ideals in $L(X)$ was described in [16] for $X = \big(\bigoplus_{n=1}^\infty \ell_2(n)\big)_{c_0}$ and in [18] for $X = \big(\bigoplus_{n=1}^\infty \ell_2(n)\big)_{\ell_1}$. In both cases, there are exactly two nested proper non-zero closed ideals, namely the compacts and the closure of all operators factoring through c_0, or ℓ_1, respectively. Apart from those mentioned above, there are no other separable Banach spaces X for which the structure of the closed ideals in $L(X)$ is completely known. It is still open whether or not the closed subideals of the operators on the spaces $(\oplus_{n=1}^\infty \ell_1(n))_{c_0}$ and $(\oplus_{n=1}^\infty \ell_\infty(n))_{\ell_1}$ admit the same sublattice structure (for partial results see [17]). An interesting space for studying the closed subideals of its bounded linear operators is the space X introduced in [26]. This space is *complementably minimal* [1], which means that every infinite dimensional closed subspace of X contains a further subspace which is complemented in X and isomorphic to X. This implies that the strictly singular operators (see the definition at the end of this section) is the only maximal proper closed subideal of $L(X)$. As shown in [2], X admits strictly singular but not compact operators, and it is conjectured

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that $L(X)$ contains infinitely many closed subideals, all of which have to lie between the ideal of compact operators, and the ideal of strictly singular operators.

A space whose closed ideals of operators attracted the attention of several researches is the pth quasi reflexive James J_p, with $1<p<\infty$. Edelstein and Mityagin [10] observed that the ideal of weakly compact operators on J_p is the only maximal proper subideal of $L(J_p)$. In [20], for $p=2$, and in [15], for general $p\in(1,\infty)$, it was shown that the closure of the operators on J_p factoring through ℓ_2 contains strictly the ideal of compact operators and is strictly contained in the ideal of weakly compact operators. Very recently Bird, Jameson and Laustsen [5] found a new closed sub ideal of $L(J_p)$ and proved that the closure of the ideal of operators factoring through the ℓ_p-sum of $\ell_\infty(n)$, $n\in\mathbb{N}$, is strictly larger than the closure of the ideal of operators factoring through ℓ_p and strictly smaller then the ideal of weakly compact operators.

Although studied in several papers (cf. [22], [24] and [25]) the structure of the closed subideals of $L(\ell_p\oplus\ell_q)$, $1<p<q<\infty$ remains a mystery. It is not even known whether or not $L(\ell_p\oplus\ell_q)$, contains infinitely many subideals. There were several results proved in the 1970's concerning various special ideals or special cases of p and q. We refer the reader to the book by Pietsch [24, Chapter 5] for details. In particular, [24, Theorem 5.3.2] asserts that $L(\ell_p\oplus\ell_q)$, with $1\leqslant p<q$, has exactly two proper maximal ideals (namely, the ideal of operators which factor through ℓ_p and the ideals of operators which factor through ℓ_q), and establishes a one-to-one correspondence between the non-maximal proper subideals of $L(\ell_p\oplus\ell_q)$ and the closed ideals in $L(\ell_p,\ell_q)$. By proving that the formal identity $I(p,q):\ell_p\to\ell_q$ is *finitely strictly singular* (see the definition at the end of this section) and establishing the existence of an operator $T:\ell_p\to\ell_q$ which is not finitely strictly singular Milman [22] concluded that $L(\ell_p,\ell_q)$ contains at least two non trivial, proper and closed subideals. In [25] the study of the structure of the closed subideals of $L(\ell_p,\ell_q)$ was continued, and, among other results, it was discovered that the lattice of subideals of $L(\ell_p,\ell_q)$ is not linearly ordered, and contains at least 4 nontrivial, proper and closed subideals if $1<p<2<q<\infty$. In this paper we increase this number to 7.

In Section 2 we will recall the known results on the closed subideals of $L(\ell_p\oplus\ell_q)$ and $L(\ell_p,\ell_q)$, and sketch the proof of several of them. In Section 3 we will formulate and prove our main result (see Theorem 3.1).

Let us first recall some necessary notation.

If X and Y are Banach spaces, $L(X,Y)$ denotes the space of bounded linear operators $T:X\to Y$, and if $X=Y$ we write $L(X)$ instead of $L(X,X)$. A linear subspace $\mathscr{J}\subset L(X,Y)$, is called a *subideal of* $L(X,Y)$, if for all $A\in L(Y)$, $B\in L(X)$, and $T\in\mathscr{J}$ also $A\circ T\circ B\in\mathscr{J}$. A *closed subideal of* $L(X,Y)$ is a subideal which is closed in the operator norm. We say that a subideal $\mathscr{J}\subset L(X,Y)$ is *non trivial* if it is not the *zero ideal* $\{0\}$ and proper if it is not all of $L(X,Y)$.

The following is a list of some important closed subideals of $L(X,Y)$.

$\mathscr{F}\mathscr{D}(X,Y)$ is the closure of the ideal of operators with finite dimensional rank. Note that any nontrivial closed subideal $\mathscr{J}$ in $L(X,Y)$ contains all of $\mathscr{F}\mathscr{D}(X,Y)$. This follows from the fact that $\mathscr{J}$ is closed under taking sums, under multiplication by elements of $L(X)$ from the right, under multiplication from the left by elements of $L(Y)$, and that it must contain a non zero operator (and thus a rank 1 operator). Thus, for all infinite dimensional Banach spaces X and Y the ideal $\mathscr{F}\mathscr{D}(X,Y)$ is the minimal nontrivial closed subideal of $L(X,Y)$.

$\mathscr{K}(X,Y)$ denotes the ideal of compact operators . All the spaces we consider are spaces with a basis. Thus, these spaces have the approximation property, which means that $\mathscr{F}\mathscr{D}(X,Y) = \mathscr{K}(X,Y)$.

$\mathscr{S}t\mathscr{S}i(X,Y)$ is the closed ideal of operators $T : X \to Y$ which are *strictly singular*, i.e. on no infinite dimensional subspace Z of X is the restriction of T onto Z an isomorphism.

$\mathscr{F}\mathscr{S}\mathscr{S}$ is the closed ideal of *finitely strictly singular operators*. A linear bounded operator $T : X \to Y$, is called *finitely strictly singular* if for all $\varepsilon > 0$ there is an $n = n_\varepsilon \in \mathbb{N}$ so that for any n-dimensional subspace E of X, there is an $x \in E$, with $\|x\| = 1$, so that $\|T(x)\| \leqslant \varepsilon$.

If W and Z are Banach spaces and $S : W \to Z$ a bounded linear operator, we denote by $\mathscr{J}^S(X,Y)$ the closure of the ideal generated by all operators $T \in L(X,Y)$, which factor through S, thus $T = A \circ S \circ B$, with $A \in L(Z,Y)$ and $B \in L(X,W)$. In general the set $\{A \circ S \circ B, A \in L(Z,Y) \text{ and } B \in L(X,W)\}$ is not closed under addition and therefore not an ideal. But if the operator

$$S \oplus S : W \oplus W \to Z \oplus Z, \qquad (w_1, w_2) \mapsto (S(w_1), S(w_2)),$$

factors through S, then $\{A \circ S \circ B, A \in L(Z,Y) \text{ and } B \in L(X,W)\}$ is an ideal and we conclude in that case that

$$\mathscr{J}^S(X,Y) = \overline{\{A \circ S \circ B : A \in L(Z,Y) \text{ and } B \in L(X,W)\}}. \tag{1}$$

Let $I(p,q) : \ell_p \to \ell_q$ be the formal inclusion (using that ℓ_p is a subset of ℓ_q), for $1 \leqslant p < q \leqslant \infty$. It is easily seen that $I(p,q) \oplus I(p,q)$ factors through $I(p,q)$ and we conclude that $\mathscr{J}^{I(p,q)}(X,Y) = \overline{\{A \circ I(p,q) \circ B : A \in L(\ell_q, Y) \text{ and } B \in L(X, \ell_p)\}}$.

If I_Z is the identity on some Banach space Z we write $\mathscr{J}^Z$ instead of $\mathscr{J}^{I_Z}$, and we note that if Z is isomorphic to $Z \oplus Z$ it follows that

$$\mathscr{J}^Z(X,Y) = \overline{\{A \circ S \circ B : A \in L(Z,Y) \text{ and } B \in L(X,Z)\}}. \tag{2}$$

If $X = Y$ we will write $\mathscr{K}(X)$, $\mathscr{F}\mathscr{S}\mathscr{S}(X)$ etc. instead of $\mathscr{K}(X,X)$, $\mathscr{F}\mathscr{S}\mathscr{S}(X,X)$ etc.

For $1 \leqslant p < \infty$, we denote the unit vector basis of $\ell_p = \ell_p(\mathbb{N})$ by $(e_{(p,j)} : j \in \mathbb{N})$ (if $p = \infty$ we consider c_0 instead of ℓ_∞). The conjugate of p is denoted by p', i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. For $n \in \mathbb{N}$ we denote the n-dimensional ℓ_p space by $\ell_p(n)$ and its unit vector basis by $(e_{(p,n,j)} : j = 1, 2, \ldots, n)$. The usual norm on ℓ_p or $\ell_p(n)$, $n \in \mathbb{N}$ is denoted by $\|\cdot\|_p$. If X_n is a Banach space for $n \in \mathbb{N}$, the ℓ_p*-sum of* X_n, $n \in \mathbb{N}$, is the space of all sequences $(x_n : n \in \mathbb{N})$, with $x_n \in X_n$, for $n \in \mathbb{N}$, and

$$\|(x_n)_{n \in \mathbb{N}}\|_p = \Big(\sum_{n \in \mathbb{N}} \|x_n\|^p\Big)^{1/p} < \infty, \text{ if } p < \infty.$$

We denote the ℓ_p-sum of (X_n) by $(\oplus_{n=1}^{\infty} X_n)_p$. If $p=\infty$ we denote by $(\oplus_{n=1}^{\infty} X_n)_\infty$ the c_0-*sum,* the space of all sequences (x_n), with $x_n \in X_n$, for $n \in \mathbb{N}$, for which $\lim_{n\to\infty} \|x_n\| = 0$.

The sphere and the unit ball of a Banach space are denoted by S_X and B_X, respectively. For simplicity all our Banach spaces are defined over the real field $\mathbb{R}$. It is easy to see how our results can be extended to Banach spaces over the complex field $\mathbb{C}$.

2. Review of the known results on the closed subideals of $L(\ell_p \oplus \ell_q)$ and $L(\ell_p, \ell_q)$

We will now review the known results on the lattice structure of subideals of $L(\ell_p \oplus \ell_q)$. We will assume from now on that $1 < p < q < \infty$ and later that $1<p<2<q<\infty$.

Every operator $T = \ell_p \oplus \ell_q \to \ell_p \oplus \ell_q$, consists of four operators $T_{(1,1)} \in L(\ell_p)$, $T_{(1,2)} \in L(\ell_q, \ell_p)$ and $T_{(2,1)} \in L(\ell_p, \ell_q)$, and $T_{(2,2)} \in L(\ell_p, \ell_p)$, and acts as a 2 by 2 matrix on the elements of $\ell_p \oplus \ell_q$

$$T = \begin{pmatrix} T_{(1,1)} & T_{(1,2)} \\ T_{(2,1)} & T_{(2,2)} \end{pmatrix} : \ell_p \oplus \ell_q \to \ell_p \oplus \ell_q,$$
$$(x,y) \mapsto \big(T_{(1,1)}(x) + T_{(1,2)}(y), T_{(2,1)}(x) + T_{(2,2)}(y)\big).$$

By the above cited result from [13], the operators $T_{(1,1)}$ and $T_{(2,2)}$ are either compact or the identity on ℓ_p, respectively ℓ_q, factors through them. By Pitt's Theorem (c.f. [11, Proposition 6.25]), $T_{(1,2)}$ is compact, and since every infinite dimensional subspace of ℓ_p contains a subspace isomorphic to ℓ_p, and since ℓ_p and ℓ_q are incomparable, we conclude that $T_{(2,1)}$ must be strictly singular. So, if $\mathscr{J}$ is a closed subideal of $L(\ell_p \oplus \ell_q)$ which contains an operator T for which $T_{(1,1)}$ and $T_{(2,2)}$ are not compact, we conclude that the identity on $\ell_p \oplus \ell_q$ factors through T and thus $\mathscr{J} = L(\ell_p \oplus \ell_q)$. If $\mathscr{J}$ contains an operator for which $T_{(1,1)}$ is not compact, but for all elements $U \in \mathscr{J}$, $U_{(2,2)}$ is compact, then the identity on ℓ_p factors through T, but not the identity on ℓ_q, and we therefore deduce that J must be the closure of the operators factoring through ℓ_p, which must therefore be a maximal proper subideal of $L(\ell_p \oplus \ell_q)$ (for more details see [24, Theorem 5.3.2]). Similarly we conclude that the closure of all operators factoring through ℓ_q is a maximal proper subideal of $L(\ell_p \oplus \ell_q)$.

For all other closed proper subideals $\mathscr{J} \subset L(\ell_p \oplus \ell_q)$, and all $T \in \mathscr{J}$ it therefore follows that $T_{(1,1)}$, $T_{(1,2)}$ and $T_{(2,2)}$ are compact, and can therefore be approximated by finite rank operators which factor through ℓ_p as well as ℓ_q. Of course $T_{(2,1)}$ also factors through ℓ_p as well as ℓ_q, and we deduce that all other closed ideals are subideals of $\mathscr{J}^{\ell_p}(\ell_p \oplus \ell_q) \cap \mathscr{J}^{\ell_q}(\ell_p \oplus \ell_q)$, and thus not maximal proper closed ideals.

Assume now that $\mathscr{J} \subset \mathscr{J}^{\ell_p}(\ell_p \oplus \ell_q) \cap \mathscr{J}^{\ell_p}(\ell_p \oplus \ell_q)$ is a closed ideal in $L(\ell_p \oplus \ell_q)$ An easy computation yields that $\tilde{\mathscr{J}} := \{T_{(2,1)} : T \in \mathscr{J}\}$ is a closed subideal of $L(\ell_p, \ell_q)$, and that for two different ideals $\mathscr{J}_1, \mathscr{J}_2 \subset \mathscr{J}^{\ell_p}(\ell_p \oplus \ell_q) \cap \mathscr{J}^{\ell_p}(\ell_p \oplus \ell_q)$ the ideals $\tilde{\mathscr{J}}_1$ and $\tilde{\mathscr{J}}_2$ are different. Conversely if $\mathscr{J}$ is a closed subideal of $L(\ell_p, \ell_q)$

then

$$\mathscr{J}' = \left\{ \begin{pmatrix} T_{(1,1)} & T_{(1,2)} \\ T_{(2,1)} & T_{(2,2)} \end{pmatrix} : T_{(2,10} \in \mathscr{J} \text{ and } T_{(1,1)} \in \mathscr{K}(\ell_p),\, T_{(1,2)} \in \mathscr{K}(\ell_q, \ell_p), \right.$$
$$\left. \text{and } T_{(2,2)} \in \mathscr{K}(\ell_q) \right\}$$

is a closed subideal of $L(\ell_p \oplus \ell_q)$ and for two different closed subideals $J_1, J_2 \subset L(\ell_p, \ell_q)$, $\mathscr{J}_1'$ and $\mathscr{J}_2'$ are different. Thus there is a bijection between the set of all closed subideals of $L(\ell_p, \ell_q)$ and the non maximal closed subideals of $L(\ell_p \oplus \ell_q)$, which preserves the lattice structure with respect to inclusions.

Let us summarize the observations we just made in the following proposition.

PROPOSITION 2.1. *For $1 < p < q < \infty$, the space $L(\ell_p \oplus \ell_q)$ has exactly two maximal proper closed subideals, namely $\mathscr{J}^{\ell_p}(\ell_p \oplus \ell_q)$ and $\mathscr{J}^{\ell_q}(\ell_p \oplus \ell_q)$.*

All other closed subideals of $L(\ell_p \oplus \ell_q)$, are subideals of $\mathscr{J}^{\ell_p}(\ell_p \oplus \ell_q) \cap \mathscr{J}^{\ell_q}(\ell_p \oplus \ell_q)$, and there is a bijection between the closed subideals of $\mathscr{J}^{\ell_p}(\ell_p \oplus \ell_q) \cap \mathscr{J}^{\ell_q}(\ell_p \oplus \ell_q)$ and closed subideals of $L(\ell_p, \ell_q)$ which preserves the lattice structure.

We are therefore interested in the closed subideals of $L(\ell_p, \ell_q)$. Instead of writing $\mathscr{K}(\ell_p, \ell_q)$, $\mathscr{FSS}(\ell_p, \ell_q)$, or $\mathscr{J}^S(X, Y)$ etc. we will from now on simply write $\mathscr{K}$, $\mathscr{FSS}$ or $\mathscr{J}^S$ etc.

The following diagram summarizes the results established in [22] and [25], under the assumption that $1 < p < 2 < q < \infty$.

$$\begin{tikzcd}
 & & & \boxed{\mathscr{FSS}} \arrow[dr, Rightarrow] & & \\
\boxed{\mathscr{K}} \arrow[r, Rightarrow] & \boxed{\mathscr{J}}^{I(p,q)} \arrow[r, dotted] & \boxed{\mathscr{FSS} \cap \mathscr{J}^{\ell_2}} \arrow[ur, dashed] \arrow[dr] & \nparallel & \boxed{\mathscr{FSS} \vee \mathscr{J}^{\ell_2}} \arrow[r, dotted] & \boxed{L(\ell_p, \ell_q)} \\
 & & & \boxed{\mathscr{J}^{\ell_2}} \arrow[ur, dashed] & &
\end{tikzcd}$$

Here arrows stand for inclusions. A solid arrow ($\Rightarrow$ or $\longrightarrow$) between two ideals means that there are no other ideals sitting properly between the two, while a double arrow coming out of an ideal indicates the only immediate successor. A hyphenated arrow ($-->$) indicates a proper inclusion, while a dotted one indicates that we do not know whether or not the inclusion is proper. In particular, the closed ideals in $L(\ell_p, \ell_q)$ are not totally ordered.

Let us explain the diagram "from the left to the right" (for a more detailed explanation we refer the reader to [25]):

If $T : \ell_p \to \ell_q$ is not compact, then there is a normalized block sequence (x_n) in ℓ_p whose image $(y_n) = (T(x_n)$ is equivalent to $(e_{(q,j)} : j \in \mathbb{N})$ (the unit vector basis in ℓ_q) and so that $\operatorname{span}(y_n : n \in \mathbb{N})$ is complemented in ℓ_p. It follows that $I(p,q)$ factors through T, and that therefore $\mathscr{J}^{I(p,q)}$ is the only successor of $\mathscr{K}$.

It is clear that $\mathscr{J}^{I(p,q)} \subset \mathscr{J}^{\ell_2}$ (recall that we assume that $p < 2 < q$). The fact that $\mathscr{J}^{I(p,q)} \subset \mathscr{FSS}$ follows from the following result in [22] (see also [25, Proposition 3.3]).

PROPOSITION 2.2. *For any choices of $1 \leqslant p < q \leqslant \infty$ is the formal identity $I(p,q)$ is a finitely strictly singular operator.*

The way to verify Proposition 2.2 is to show first (see [22] or [25, Lemma 3.4]) by induction on $n \in \mathbb{N}$, that in every n-dimensional subspace E of c_0 there is $x \in E$ which attains its sup-norm on at least n coordinates. In order to see then, that $I(p,q)$ is finitely strictly singular, let $\varepsilon > 0$ and pick $n \in \mathbb{N}$ with $n^{-(q-p)/q} < \varepsilon$. If E is any subspace of ℓ_p of dimension n we can find $x \in E$, $\|x\|_p = 1$, so that $\|x\|_\infty \leqslant n^{-1/p}$ (since the maximum is attained on at least n coordinates), and thus $\|x\|_q^q = \sum_{i=1}^\infty |x(i)|^{q-p}|x(i)|^p \leqslant \|x\|_\infty^{q-p}\|x\|_p^p \leqslant n^{p-q}$ and thus $\|x\|_q \leqslant n^{-(q-p)/q} \leqslant \varepsilon$. We therefore established that $\mathscr{J}^{I(p,q)} \subset \mathscr{FSS} \cap \mathscr{J}^{\ell_2}$. In Section 2 we will show that this inclusion is strict. More precisely, we will show that the ideals $\mathscr{J}^{I(p,2)}$ and $\mathscr{J}^{I(2,q)}$ are two distinct closed ideals which lie between $\mathscr{J}^{I(p,q)}$ and $\mathscr{FSS} \cap \mathscr{J}^{\ell_2}$.

In order to show that $\mathscr{FSS} \cap \mathscr{J}^{\ell_2}$ is not all of $L(\ell_p, \ell_q)$ Milman [22] used the fact that ℓ_p (and ℓ_q)is isomorphic the ℓ_p-sum (respectively the ℓ_q sum) of $\ell_2(n)$, $n \in \mathbb{N}$ (see [19, page 73]). Letting $U : \ell_p \to (\oplus_{n\in\mathbb{N}}\ell_2(n))_p$ and $V : \ell_q \to (\oplus_{n\in\mathbb{N}}\ell_2(n))_q$ be isomorphisms and letting $I'(p,q)$ be the formal identity

$$I'(p,q) : (\oplus_{n\in\mathbb{N}}\ell_2(n))_p \to (\oplus_{n\in\mathbb{N}}\ell_2(n))_q, \qquad (x_n) \mapsto (x_n),$$

we define $T(p,q) = V \circ I'(p,q) \circ U$. $T(p,q)$ depends on the choice of the isomorphisms U and V, nevertheless it is easy to see that for any other isomorphisms $\tilde{U} : \ell_p \to (\oplus_{n\in\mathbb{N}}\ell_2(n))_p$ and $\tilde{V} : \ell_q \to (\oplus_{n\in\mathbb{N}}\ell_2(n))_q$, the operator $\tilde{T}(p,q) = \tilde{V} \circ I'(p,q) \circ \tilde{U}$, factors through $T(p,q)$ and vice versa, and thus $\mathscr{J}^{T(p,q)} = \mathscr{J}^{\tilde{T}(p,q)}$. Clearly $T(p,q) \notin \mathscr{FSS}$, and thus $\mathscr{FSS}$ is a proper closed subideal of $L(\ell_p, \ell_q)$.

It is clear that $\mathscr{J}^{T(p,q)} \subset \mathscr{J}^{\ell_2}$. Conversely, Theorem 4.7 in [25] shows that every operator $S : \ell_p \to \ell_q$, which factors through ℓ_2, belongs to $\mathscr{J}^{T(p,q)}$, thus we deduce that $\mathscr{J}^{T_{(p,q)}} = \mathscr{J}^{\ell_2}$. Moreover, if $S \in L(\ell_p, \ell_q)$ is not in $\mathscr{FSS}$, it follows from Khintchine's theorem (for more detail see Theorem 3.2 in Section 3 and the remarks thereafter) that for some $c > 0$ there are c-complemented subspaces $F_n \subset \ell_p$, which are c-isomorphic to $\ell_2(n)$, for $n \in \mathbb{N}$, on which S is a c-isomorphism. After perturbing S we can find a sequence $(k_n) \subset \mathbb{N}$, so that if we write ℓ_p as an ℓ_p-sum of $\ell_p(k_n)$ and ℓ_q as the ℓ_q-sum of $\ell_q(k_n)$, we can assume that $F_n \subset \ell_p(k_n) \subset \ell_p$ and $S(F_n) \subset \ell_q(k_n) \subset \ell_q$. From this (see [25, Theorem 4.13]) it follows that $T(p,q)$ factors through S. We deduce therefore that the ideal $\mathscr{J}^{\ell_2} \vee \mathscr{FSS} = \mathscr{J}^{T(p,q)} \vee \mathscr{FSS}$ (the closed ideal generated by the elements of $\mathscr{FSS}$ and $\mathscr{J}^{\ell_2}$) is the only successor of $\mathscr{FSS}$.

Finally we need to construct an operator $U : \ell_p \to \ell_q$ which is in $\mathscr{FSS}$ but cannot be approximated by operators which factor through ℓ_2. This will show that $\mathscr{FSS}$ and $\mathscr{J}^{\ell_2}$ are incomparable, they both strictly contain $\mathscr{FSS} \cap \mathscr{J}^{\ell_2}$ and are properly contained in $\mathscr{J}^{\ell_2} \vee \mathscr{FSS}$.

To do that we write ℓ_p as ℓ_p sum of $\ell_p(2^n)$, $n \in \mathbb{N}$, and ℓ_q as ℓ_q-sum of $\ell_q(2^n)$, $n \in \mathbb{N}$. For $n \in \mathbb{N} \cup \{0\}$ let H_n be the n-th Hadamard matrix. This is an 2^n by 2^n matrix with entries which are either 1 or -1, and can be defined by induction as follows; $H_0 = (1)$, and assuming that H_n has been defined one puts $H_{n+1} = \begin{pmatrix} H_n & H_n \\ H_n & -H_n \end{pmatrix}$.

It is easy to see that H_n as operator from $\ell_1(2^n) \to \ell_\infty(2^n)$ is of norm 1, and that $2^{-n/2}H_n$ is a unitary matrix (i.e., an isometry on $\ell_2(2^n)$). It follows therefore from the Riesz Thorin Interpolation Theorem (c.f. [4]) that $U_n = 2^{-n\frac{1}{\min(p',q)}} H_n$ is of norm at most 1 as an operator in $L(\ell_p(2^n), \ell_q(2^n))$.

We define

$$U : \ell_p = \left(\oplus_{n=1}^\infty \ell_p(2^n)\right)_p \to \left(\oplus_{n=1}^\infty \ell_p(2^n)\right)_q, \qquad (x_n) \mapsto (U_n(x_n)).$$

The fact that U can not be approximated by operators which factor though ℓ_2 can be obtained from the following Corollary of Theorem 9.13 in [9] (see also [25, Theorem]).

PROPOSITION 2.3. cf. [25, Corollary] *Let $m \in \mathbb{N}$, $C > 1$, and $r > 1$, and assume that V is an invertible m by m matrix. Let $\delta = \|V^{-1}\|_{L(\ell_r', \ell_{r'})}$. Then $\|B\|_{L(\ell_p,\ell_r)} \cdot \|A\|_{L(\ell_r,\ell_q)} \geqslant \delta^{-1}$ for any factorization $V = AB$. Moreover, if $\widetilde{V}$ is another m by m matrix with*

$$\|\widetilde{V} - V\|_{L(\ell_p,\ell_q)} \leqslant \left(2 \max_{1\leqslant i\leqslant m} \|V^{-1}e_i\|_p\right)^{-1}, \tag{3}$$

then it follows that for any factorization $\widetilde{V} = AB$ we have $\|B\|_{L(\ell_p,\ell_r)} \cdot \|A\|_{L(\ell_r,\ell_q)} \geqslant (2\delta)^{-1}$.

If $q \neq p'$ then it is easy to see that U is finitely strictly singular. Indeed if $p' < q$, it follows that $U_n = 2^{-n/p'}H_n$, and we deduce again form the Riesz Thorin Interpolation Theorem that U_n is as operator between $\ell_p(2^n)$ and $\ell_{p'}(2^n)$ of norm not larger than 1, and thus $U \in L(\ell_p, \ell_{p'})$. But this implies that U (as element in $L(\ell_p, \ell_q)$) factors through $I(p', q)$, which is finitely strictly singular by Proposition 2.3. A similar argument shows that if $p' > q$, and thus $p < q'$, then U factors through $I(p, q')$.

The hard case is the case $q = p' \neq 2$, in which the previous factorization argument does not work. In this case it is better to see $\ell_p(n)$ as the space $L_p(n)$, the space of all p- integrable functions on $\{1, 2 \dots n\}$ with the normalized counting measure (i.e. $\|x\|_{L_p} = \frac{1}{n^{/1p}}\|x\|_p$)). Using interpolation between Schatten p-classes one can prove the following result

THEOREM 2.4. [25, Theorem 6.5] *Suppose that $T : L_p(N) \to \ell_{p'}(N)$. Let E be a k-dimensional subspace of $L_p(N)$, and C_1, C_2, and C_3 be positive constants such that*

1. $\|T\|_{L(L_2(N),\ell_2(N))} \leqslant 1$ *and* $\|T\|_{L(L_1(N),\ell_\infty(N)))} \leqslant 1$*;*
2. *E is C_1-isomorphic to ℓ_2^k;*
3. *$F = T(E)$ is C_2-complemented in $\ell_{p'}^N$; and*
4. *$T_{|E}$ is invertible and $\left\|(T_{|E})^{-1}\right\| \leqslant C_3$.*

Then $k \leqslant \left(C_1^3 C_2 C_3^2 K_G^2\right)^{p'}$. Here K_G denotes the Grothendieck constant.

Now, if $q = p'$, then we apply for $n \in \mathbb{N}$ Theorem 2.4 to $N = 2^n$ and $T_n = \frac{1}{n^{1/p}} U_n = \frac{1}{n} H_n$ (note T_n satisfies (1) of Theorem 2.4). If U where not finitely strictly singular, we could find constants C_1, C_2 and C_3 and for any $k\in\mathbb{N}$ we would could find $n=n_k\in\mathbb{N}$ large enough so that (2) and (3) of Theorem 2.4 are satisfied (using again Theorem 3.2 in Section 2) for $T=T_n$. But this contradicts the conclusion of Theorem 2.4.

3. Two new closed ideals of $L(\ell_p,\ell_q)$

We now state our main result, which exhibits two new closed subideals of $L(\ell_p,\ell_q)$, and shows that $\mathscr{J}^{I(p,q)} \subsetneq \mathscr{FSS}\cap\mathscr{J}^{\ell_2}$ and increases the count of the known closed proper and non trivial subideals of $L(\ell_p,\ell_q)$ to 7.

THEOREM 3.1. *Assume that $1<p<2<q<\infty$. Then the two ideals $\mathscr{J}^{I(p,2)}$ and $\mathscr{J}^{I(2,q)}$ are two incomparable closed subideals of $\mathscr{FSS}\cap\mathscr{J}^{\ell_2}$.*

We assume from now on that $1<p<2<q<\infty$. It is clear that $\mathscr{J}^{I(p,q)}\subset\mathscr{J}^{I(p,2)}$ and that by Proposition 2.2 $\mathscr{J}^{I(p,2)}\subset\mathscr{FSS}\cap\mathscr{J}_2^{\ell}$ and similarly $\mathscr{J}^{I(p,q)}\subset\mathscr{J}^{I(2,q)}\subset\mathscr{FSS}\cap\mathscr{J}^{\ell_2}$. We can therefore extend the diagram of Section 2 to the following diagram.

$$\begin{tikzcd}
 & & \boxed{\mathscr{J}^{I(p,2)}} \arrow[dr, dashed] & & \boxed{\mathscr{FSS}} \arrow[dr, Rightarrow] & & \\
\boxed{\mathscr{K}} \arrow[r, Rightarrow] & \boxed{\mathscr{J}}^{I(p,q)} \arrow[ur, dashed] \arrow[dr, dashed] & \nparallel & \boxed{\mathscr{FSS}\cap\mathscr{J}^{\ell_2}} \arrow[ur, dashed] \arrow[dr] & \nparallel & \boxed{\mathscr{FSS}\vee\mathscr{J}^{\ell_2}} \arrow[r, dotted] & \boxed{L(\ell_p,\ell_q)} \\
 & & \boxed{\mathscr{J}^{I(2,q)}} \arrow[ur, dashed] & & \boxed{\mathscr{J}^{\ell_2}} \arrow[ur, dashed] & &
\end{tikzcd}$$

This solves Question (i) in [25] and shows that $\mathscr{J}^{I(p,q)}$ is different from $\mathscr{FSS}\cap\mathscr{J}^{\ell_2}$, and that the two (different) closed subideals $\mathscr{J}^{I(p,2)}$ and $\mathscr{J}^{I(2,q))}$ lie between them.

In order to show Theorem 3.1 we need to find two operators T and S in $\mathscr{FSS}\cap\mathscr{J}^{\ell_2}$, so that $T\in\mathscr{J}^{I(p,2)}\setminus\mathscr{J}^{I(2,q)}$ and $S\in\mathscr{J}^{I(2,q)}\setminus\mathscr{J}^{I(p,2)}$. We will first need the following result.

THEOREM 3.2. *For every $1<r<\infty$ there exists a constant $K=K(r)>0$ and for all $n\in\mathbb{N}$ a number $N=N(n,r)\in\mathbb{N}$, such that every N–dimensional subspace $F\subset\ell_r$ contains an n–dimensional subspace E which is K–complemented in ℓ_r and K–isomorphic to $\ell_2(n)$.*

REMARK 3.3. Theorem 3.2 follows from the finite dimensional version of Khintichin's Theorem (see [11, Theorem 6.28]). Better estimates on $N(n,r)$ and $K(r)$ can be obtained by applying simultaneously Dvoretzky's theorem both to a subspace $F\subset\ell_r$ and to its dual F^* (see e.g., [23]). This gives the result with $N=Cn^{r/2}$ and $K=C'\sqrt{\max\{r,r'\}}$, where $C,C'>0$ are absolute constants. This theorem can also be viewed, for example, as a special case of results in [12].

Proof of Theorem 3.1. We will now construct the operators $T\in\mathscr{J}^{I(p,2)}\setminus\mathscr{J}^{I(2,q)}$ and $S\in\mathscr{J}^{I(2,q)}\setminus\mathscr{J}^{I(p,2)}$.

Put $C = \max(K(p), K(q))$ and for $n \in \mathbb{N}$ let $k_n = \max(N(p,n), N(q,n))$, where $K(p)$, $K(q)$, $N(p,n)$ and $N(q,n)$ are chosen as in Theorem 3.2. Using that result we can find for every $n \in \mathbb{N}$ a sequence $(x_{(n,i)})_{i=1}^n$ in $CB_{\ell_p(k_n)}$ so that

$$(x_{(n,i)})_{i=1}^n \text{ is } C\text{-equivalent to the unit vector basis of } \ell_2(n) \text{ and} \tag{4}$$

$$\text{there is a projection } P_n \text{ from } \ell_p(k_n) \text{ onto } \mathrm{span}(x_{(n,i)} : i = 1,2,\dots n) \text{ with } \|P_n\| \leqslant C. \tag{5}$$

For $n \in \mathbb{N}$ we define $I_n : \mathrm{span}(x_i^{(n)} : i = 1,2\dots,n) \to \ell_2(n)$, by $I_n(x_{(n,i)}) = e_{(2,n,i)}$, $i = 1,\dots n$. I_n is thus a C-isomorphism. Writing ℓ_p as ℓ_p-sum of $\ell_p(k_n)$ and ℓ_2 as ℓ_2-sum of $\ell_2(n)$, $n \in \mathbb{N}$, we define $\tilde{S}$ as follows

$$\tilde{S} : \big(\oplus_{n=1}^{\infty} \ell_p(k_n)\big)_p \to \big(\oplus_{n=1}^{\infty} \ell_2(n)\big)_2, \quad (x_n) \mapsto \big(I_n \circ P_n(x_n) : n \in \mathbb{N}\big).$$

It follows that $\|\tilde{S}\| \leqslant C^2$. Finally we let $S := I(2,q) \circ \tilde{S} \in \mathscr{J}^{I(2,q)}$.

The construction of $T : \ell_p \to \ell_q$ is similar. Using again Theorem 3.2 we find for each $n \in \mathbb{N}$ vectors $(y_{(n,i)} : i = 1,2\dots n)$ in $CB_{\ell_q(k_n)}$ so that

$$(y_{(n,i)})_{i=1}^n \text{ is } C\text{-equivalent to the unit vector basis of } \ell_2(n), \text{ and} \tag{6}$$

$$\text{there is a projection } Q_n \text{ from } \ell_q(k_n) \text{ onto } \mathrm{span}(y_{(n,i)} : i = 1,2,\dots n) \text{ with } \|Q_n\| \leqslant C. \tag{7}$$

Let $J_n : \ell_2(n) \to \ell_q(k_n)$, be the linear map which assigns to $e_{(2,n,i)}$ the vector $y_{(n,i)}$, $i = 1,2\dots n$, then J_n is a C-isomorphism onto its image, and by writing again ℓ_2 as ℓ_2-sum of $\ell_2(n)$ and ℓ_q as ℓ_q-sum of $\ell_q(k_n)$, $n \in \mathbb{N}$, we define $\tilde{T}$ as

$$\tilde{T} : \big(\oplus_{n=1}^{\infty} \ell_2(n)\big)_2 \to \big(\oplus_{n=1}^{\infty} \ell_q(k_n)\big)_q, \quad (x_n) \mapsto \big(J_n(x_n) : n \in \mathbb{N}\big).$$

Thus $\tilde{T}$ is a bounded operator with $\|\tilde{T}\| \leqslant C$ and $T := \tilde{T} \circ I(p,2) \in \mathscr{J}^{I(p,2)}$.

In order to show that $S \notin \mathscr{J}^{I(p,2)}$ and $T \notin \mathscr{J}^{I(2,q)}$ we will find two functionals Φ and Ψ in $L^*(\ell_p, \ell_q)$ so that $\Phi(S) = 1$ and $\Phi|_{\mathscr{J}^{I(p,2)}} \equiv 0$, and, conversely $\Psi(T) = 1$ and $\Psi|_{\mathscr{J}^{I(2,q)}} \equiv 0$.

Let q' be the conjugate of q (i.e. $\frac{1}{q} + \frac{1}{q'} = 1$). For $n \in \mathbb{N}$ we define

$$\tilde{\Phi}_n : L(\ell_p(k_n), \ell_q(n)) \to \mathbb{R}, \quad \text{with } \tilde{\Phi}_n(V) = \frac{1}{n} \sum_{i=1}^n \langle e_{(q',n,i)}, V(x_{(n_i)}) \rangle.$$

Since by choice $\|x_{(n,i)}\| \leqslant C$, for $i = 1,\dots,n$, it follows that $\|\tilde{\Phi}_n\| \leqslant C$. We extend $\tilde{\Phi}_n$ in the canonical way to a functional in $L^*(\ell_p, \ell_q)$, i.e let $E_n : \ell_p(k_n) \to \ell_p = (\oplus_{n=1}^{\infty} \ell_p(k_n))$ be the canonical embedding to the n-component and let $F_n : \ell_q = (\oplus_{j=1}^{\infty} \ell_q(j)) \to \ell_q(n)$ be the projection onto the n-th component, for $n \in \mathbb{N}$ and put $\Phi_n(U) = \tilde{\Phi}_n(F_n \circ U \circ E_n)$ for $U \in L(\ell_p, \ell_q)$. Then also $\|\Phi_n\| \leqslant C$ and we let $\Phi \in L^*(\ell_p, \ell_q)$ be a w* accumulation point of the sequence (Φ_n) in $L^*(\ell_p, \ell_q)$. Since $F_n \circ S \circ E_n(x_{(n,i)})$ is the i-th unit vector in $\ell_q(n)$ it follows that $\Phi(S) = \lim_{n\to\infty} \Phi_n(S) = 1$.

The definition of $\Psi \in L^*(\ell_p,\ell_q)$ is as follows. Since $(y_{(n,i)} : i = 1,2,\dots,n)$ is C-isomorphic to $(e_{(2,n,i)} : i = 1,2\dots,n)$ and its linear span is C-complemented in $\ell_q(k_n)$, we can find a sequence $(y^*_{(n,i)} : i = 1,2\dots,n) \subset \ell_{q'}(k_n)$, which is C-isomorphic to $(e_{(2,n,i)} : i = 1,2\dots,n)$, and satisfies $\langle y^*_{(n,i)}, y_{(n,j)}\rangle = \delta_{(i,j)}$ for $1 \leqslant i,j \leqslant n$.

For $n\in\mathbb{N}$ we can then write the projection $Q_n : \ell_q(k_n) \to \mathrm{span}(y_{(n,i)} : i = 1,2,\dots,n)$ (which was introduced in (7)) as

$$Q_n = \sum_{i=1}^n y_{(n,i)} \otimes y^*_{(n,i)} : \ell_q(k_n) \to \mathrm{span}(y_{(n,i)} : i = 1,2,\dots,n), \quad z \mapsto \sum_{i=1}^n y_{(n,i)}\langle y^*_{(n,i)}, z\rangle.$$

Then we define for $n \in \mathbb{N}$

$$\tilde\Psi_n : L(\ell_p(n),\ell_q(k_n)) \to \mathbb{R} \qquad \text{by } \tilde\Psi(U) = \frac{1}{n}\sum_{i=1}^n \langle y^*_{(n,i)}, U(e_{(p,n.i)})\rangle.$$

We let Ψ_n be the canonical extension of $\tilde\Psi$ to a functional in $L^*(\ell_p,\ell_q)$, i.e. for $U \in L(\ell_p,\ell_q)$ we let $\Psi_n(U) = \tilde\Psi(F'_n \circ U \circ E'_n)$, where $E'_n : \ell_p(n) \to \ell_p = \big(\oplus_{j\in\mathbb{N}} \ell_p(j)\big)_p$, is the canonical embedding into the n-th component, and $F'_n : \big(\oplus_{j\in\mathbb{N}} \ell_q(k_j)\big)_q \to \ell_q(k_n)$ is the projection onto the n-th component. Since $\|y^*_{(n,i)}\|_{q'} \leqslant C$, for $i = 1,2\dots n$, it follows that $\|\Psi_n\| \leqslant C$ and we let $\Psi \in L^*(\ell_p,\ell_q)$ be a w*-accumulation point of (Ψ_n). Since $T(e_{(p,n,i)}) = y_{(n,i)}$ for $i = 1,2\dots,n$, it follows that $\Psi(T) = \lim_{n\to\infty}\langle \Psi_n, T\rangle = 1$.

It is left to show that $\mathscr{J}^{I(p,2)} \subset \ker(\Phi)$ and that $\mathscr{J}^{I(2,q)} \subset \ker(\Psi)$. To do so, we need a result which is of independent interest and will therefore be stated separately and more generally than needed. $\square$

DEFINITION 3.4. Let X be a finite or infinite dimensional Banach space with a normalized basis (e_i). If X is infinite dimensional put for $j \in \mathbb{N}$,

$$n_X(j) = \min\Big\{\Big\|\sum_{i\in I} e_i\Big\| : I \subset \mathbb{N}, \#I = j\Big\}, \text{ and } N_X(j) = \max\Big\{\Big\|\sum_{i\in I} e_i\Big\| : I \subset \mathbb{N}, \#I = j\Big\},$$

and if $j \leqslant \dim(X) < \infty$, then put

$$n_X(j) = \min\Big\{\Big\|\sum_{i\in I} e_i\Big\| : I \subset \{1,2,\dots\dim(X)\}, \#I = j\Big\} \text{ and}$$
$$N_X(j) = \max\Big\{\Big\|\sum_{i\in I} e_i\Big\| : I \subset \{1,2,\dots\dim(X)\}, \#I = j\Big\}.$$

LEMMA 3.5. *Assume that E and F are two finite dimensional spaces, both having C_u-unconditional and normalized bases $(e_i : i = 1,2\dots m)$ and $(f_j : j = 1,\dots n)$, respectively.*

Assume further that there are $1 < t < s < \infty$ and positive constants c_1, and c_2, so that for all $\ell \in \mathbb{N}$

$$N_E(\ell) \leqslant c_1 \ell^{1/s} \text{ and } n_F(\ell) \geqslant c_2 \ell^{1/t}. \tag{8}$$

Then there exists a number $c > 0$, depending only on s, t, c_u, c_1, and c_2, so that for every linear operator $T : E \to F$ and any $\rho > 0$

$$\left|\left\{i \leqslant m : ||T(e_i)||_\infty = \max_{j\leqslant n} |f_j^*(T(e_i))| \geqslant \|T\|\rho\right\}\right| \leqslant c\rho^{\frac{-s^2}{(s-1)(s-t)}}, \tag{9}$$

where (f_j^*) are the coordinate functionals to (f_j). Moreover, if $c_u = c_1 = c_2 = 1$, then we can choose $c = 1$.

COROLLARY 3.6. *Under the assumptions of Lemma 3.5, it follows that*

$$\frac{1}{m}\sum_{i=1}^{m} \|T(e_i)\|_\infty \leqslant \|T\|(1+c)m^{-r(s,t)}, \tag{10}$$

where

$$r(s,t) = \frac{(s-1)(s-t)}{(s-1)(s-t)+s^2},$$

for $s > t \geqslant 1$.

Proof. First note that for any $\rho > 0$ Lemma 3.5 yields

$$\begin{aligned}\frac{1}{m}\sum_{i=1}^{m} \|T(e_i)\|_\infty &= \frac{1}{m}\sum_{i=1,\|T(e_i)\|_\infty\leqslant\rho\|T\|}^{m} \|T(e_i)\|_\infty + \frac{1}{m}\sum_{i=1,\|T(e_i)\|_\infty>\rho\|T\|}^{m} \|T(e_i)\|_\infty \\ &\leqslant \|T\|\rho + c\|T\|\frac{\rho^{\frac{-s^2}{(s-1)(s-t)}}}{m}.\end{aligned}$$

Then we let

$$\rho = m^{-\frac{(s-1)(s-t)}{(s-1)(s-t)+s^2}},$$

which implies that

$$\begin{aligned}\frac{1}{m}\sum_{i=1}^{m} \|T(e_i)\|_\infty &\leqslant \|T\|m^{-\frac{(s-1)(s-t)}{(s-1)(s-t)+s^2}} + c\|T\|m^{-1}m^{\frac{(s-1)(s-t)}{(s-1)(s-t)+s^2}\frac{s^2}{(s-1)(s-t)}} \\ &= \|T\|m^{-\frac{(s-1)(s-t)}{(s-1)(s-t)+s^2}} + c\|T\|m^{-\frac{(s-1)(s-t)}{(s-1)(s-t)+s^2}} = (1+c)\|T\|m^{-r(s,t)}. \qquad \square\end{aligned}$$

Proof of Lemma 3.5. For the sake of a better readability we will assume that $c_1 = c_2 = c_u = 1$. The general case follows in the same way. We can also assume that $\|T\| = 1$.

Let $T : E \to F$ and write $y_i = T(e_i)$ as $y_i = \sum_{j=1}^{n} \beta(i,j) f_j$. Let $\rho > 0$ and put

$$A = A_\rho = \left\{i \in \{1,2,\ldots,m\} : \max|\beta(i,j)| \geqslant \rho\right\}.$$

For $i \in A$ choose $j_i \in \{1,2\dots,n\}$ so that $|\beta(i,j_i)| \geqslant \rho$. Let $\tilde{A} = \{j_i : i \in A\}$ and for $j \in \tilde{A}$ let $A_j = \{i \in A : j_i = j\}$. In order to estimate $|A_j|$ and then $\tilde{A}$ we compute

$$\begin{aligned}
|A_j|^{1/s} &\geqslant N_E(|A_j|) \quad \text{(By (8))}\\
&\geqslant \Big\| \sum_{i \in A_j} \operatorname{sign}(\beta(i,j)) e_j \Big\|_E\\
&\geqslant \Big\| T\Big(\sum_{i \in A_j} \operatorname{sign}(\beta(i,j)) e_j \Big) \Big\|_F \quad \text{(Since } \|T\| = 1\text{)}\\
&\geqslant \Big\langle f_j^*, \sum_{i \in A_j} T\Big(\sum_{i \in A_j} \operatorname{sign}(\beta(i,j)) e_j \Big) \Big\rangle\\
&= \sum_{i \in A_j} |\beta(i,j)| \geqslant |A_j| \rho
\end{aligned}$$

which yields $|A_j|^{1-\frac{1}{s}} \leqslant \rho^{-1}$, and thus

$$|A_j| \leqslant \rho^{-1/(1-\frac{1}{s})} = \rho^{-\frac{s}{s-1}}.$$

Since $|A| = \sum_{j \in \tilde{A}} |A_j| \leqslant |\tilde{A}| \cdot \rho^{-\frac{s}{s-1}}$, we obtain

$$|\tilde{A}| \geqslant |A| \rho^{\frac{s}{s-1}}. \tag{11}$$

Let $(r_j)_{j=1}^m$ be a Rademacher sequence on some probability space $(\Omega, \Sigma, \mathbb{P})$, this means that $r_1, r_2, \dots r_m$ are independent and $\{\pm 1\}$-valued, with $\mathbb{P}(\{r_j = 1\}) = \mathbb{P}(\{r_j = -1\}) = 1/2$ for $j = 1,2\dots n$. We compute

$$\begin{aligned}
|A|^{1/s} &\geqslant N_E(|A|) \quad \text{(By (8))}\\
&\geqslant \mathbb{E}\Big(\Big\| \sum_{i \in A} r_i e_i \Big\|_E \Big)\\
&\geqslant \mathbb{E}\Big(\Big\| \sum_{i \in A} \sum_{j=1}^n r_i \beta(i,j) f_j \Big\|_F \Big) \quad \text{(Since } \|T\| \leqslant 1\text{)}\\
&= \mathbb{E}\Big(\Big\| \sum_{j=1}^n f_j \Big| \sum_{i \in A} r_i \beta(i,j) \Big| \Big\|_F \Big) \quad \text{(By 1-unconditionality of } (f_j)\text{)}.
\end{aligned}$$

Applying the multidimensional version of Jensen's inequality (c.f [8, 10.2.6, page 348]) to the convex function $\mathbb{R}^n \ni z \to \| \sum_{j=1}^n z_j f_j \|_F$ and the $\mathbb{R}^n$ valued random vector $Z = \big(| \sum_{i \in A} r_i \beta(i,j) | : j \leqslant n \big)$ we obtain

$$\begin{aligned}
|A|^{1/s} &\geqslant \Big\| \sum_{j=1}^n f_j \mathbb{E}\Big(\Big| \sum_{i \in A} r_i \beta(i,j) \Big| \Big) \Big\|_F\\
&\geqslant \Big\| \sum_{j \in \tilde{A}} f_j \mathbb{E}\Big(\Big| \sum_{i \in A} r_i \beta(i,j) \Big| \Big) \Big\|_F \quad \text{(By 1-uncondtionality of } (f_j)\text{)}.
\end{aligned}$$

For each $j \in \tilde{A}$ there is an $i_j \in A$ so that $|\beta(i,j)| \geqslant \rho$. Let r be anther ± 1 random variable with $\mathbb{P}(r=1) = \mathbb{P}(r=-1) = 1/2$, which is independent to $(r_j : j = 1, \dots m)$ then

$$\begin{aligned}
\mathbb{E}\Big(\Big|\sum_{i\in A} r_i\beta(i,j)\Big|\Big) &= \mathbb{E}\Big(\Big|r_{i_j}\beta(i_j,j) + r\sum_{i\in A\setminus\{i_j\}} r_i\beta(i,j)\Big|\Big)\\
&= \mathbb{E}\left(\frac{1}{2}\Big|r_{i_j}\beta(i_j,j) + \sum_{i\in A\setminus\{i_j\}} r_i\beta(i,j)\Big| + \frac{1}{2}\Big|r_{i_j}\beta(i_j,j) - \sum_{i\in A\setminus\{i_j\}} r_i\beta(i,j)\Big|\right)\\
&\geqslant \mathbb{E}(|r_{i_j}\beta(i_j,j)|) \geqslant \rho \qquad (\text{Since } |a+b|+|a-b| \geqslant 2|a|).
\end{aligned}$$

Using again the 1-unconditionality of $(f_j : j = 1,2\dots n)$ we deduce therefore that

$$|A^{1/s}| \geqslant \Big\|\sum_{j\in\tilde{A}} f_j \mathbb{E}\Big(\Big|\sum_{i\in A} r_i\beta(i,j)\Big|\Big)\Big\|_F \geqslant \rho\Big\|\sum_{j\in\tilde{A}} f_j\Big\|_F \geqslant n_F(|\tilde{A}|),$$

and thus by our assumption (8) and by (11) we obtain

$$|A|^{1/s} \geqslant n_F(|\tilde{A}|) \geqslant |\tilde{A}|^{1/t} \geqslant |A|^{1/t}\rho^{\frac{s}{ts-t}}.$$

Solving for $|A|$ yields

$$|A| \leqslant \rho^{-\frac{s}{ts-t}\frac{st}{s-t}} = \rho^{\frac{-s^2}{(s-1)(s-t)}},$$

which proves our claim. $\square$

Continuation of Proof of Theorem 3.1. In order to show that $\mathscr{J}^{I(p,2)} \subset \ker(\Phi)$ we let $A \in L_2(\ell_2, \ell_q)$ and $B \in L(\ell_p)$. We need to show that $\Phi(A \circ I(p,2) \circ B) = 0$. W.lo.g. we assume that $\|A\|, \|B\| \leqslant 1$.

Consider $B'_n : \ell_2(n) \to \ell_p(k_n)$ with $B'(e_{(2,n,i)}) = B(x_{(n,i)})$, where we consider $\ell_p(k_n)$ canonically embedded into $\ell_p = \big(\oplus_{j=1}^\infty \ell(k_j)\big)$. Then $\|B'_n\| \leqslant C$ and applying therefore Corollary 3.6 to B', $s = 2$ and $t = p$, we obtain

$$\frac{1}{n}\sum_{i=1}^n \|B(x_{(n,i)})\|_\infty = \frac{1}{n}\sum_{i=1}^n \|B'_n(e_{(2,n,i)})\|_\infty \leqslant 2Cn^{-r(2,p)}.$$

which by the concavity of $[0,\infty) \ni \xi \mapsto \xi^{(2-p)/2}$ implies that

$$\frac{1}{n}\sum_{i=1}^n \|B(x_{(n,i)})\|_\infty^{(2-p)/2} \leqslant \Big(\frac{1}{n}\sum_{i=1}^n \|B(x_{(n,i)})\|_\infty\Big)^{(2-p)/2} \leqslant (2C)^{(2-p)/2} n^{-r(2,p)(2-p)/2}. \tag{12}$$

Secondly we observe that for any $i = 1,2\dots n$

$$\begin{aligned}
\|I(p,2)(B(x_{(n,i)}))\|_2 &= \Big(\sum_{j=1}^{k_n} |B(x_{(n,i)})(j)|^2\Big)^{1/2} \qquad (13)\\
&= \Big(\sum_{j=1}^{k_n} |B(x_{(n,i)})(j)|^p |B(x_{(n,i)})(j)|^{2-p}\Big)^{1/2}\\
&\leqslant \|B(x_{(n,i)})\|_\infty^{(2-p)/2} \cdot \|B(x_{(n,i)})\|_p^{p/2} \leqslant C^{p/2}\|B(x_{(n,i)})\|_\infty^{(2-p)/2}.
\end{aligned}$$

It follows therefore that

$$
\begin{aligned}
\big|\Phi_n(A\circ I(p,2)\circ B)\big| &= \frac{1}{n}\Big|\sum_{i=1}^{n}\langle e_{(q',n,i)}, A\circ I_{(p,2)}\circ B(x_{(n,i)})\rangle\Big|\\
&= \frac{1}{n}\Big|\sum_{i=1}^{n}\langle A^*(e_{(q',n,i)}), I_{(p,2)}\circ B(x_{(n,i)})\rangle\Big|\\
&\leqslant \frac{1}{n}\sum_{i=1}^{n}\|A^*(e_{(q',n,i)})\|_2\|I_{(p,2)}\circ B(x_{(n,i)}\|_2\\
&\leqslant \|A^*\|C^{p/2}\frac{1}{n}\sum_{i=1}^{n}\|x_{(n,i)}\|_\infty^{(2-p)/2} \quad \text{(By (13))}\\
&\leqslant C^{p/2}(2C)^{(2-p)/2}n^{-r(2,p)(2-p)/2}\to_{n\to\infty} 0 \quad \text{(By (12))}.
\end{aligned}
$$

This implies that $\mathscr{J}^{I(p,2)}\subset\ker(\Phi)$.

In order to show that $\mathscr{J}^{I(2,q)}\subset\ker(\Psi)$, let $B\in L(\ell_p,\ell_2)$ and $A\in L(\ell_q)$ with $\|B\|,\|A\|\leqslant 1$. We need to show that $\Psi(A\circ I_{(2,q)}\circ B)=0$.

Let $A'_n:\ell_2(n)\to\ell_{q'}(k_n)$, defined by $A'_n(e^{(2,n,i)})=A^*(y^*_{(n,i)})$, $i=1,2\dots n$ (we consider $\ell_{q'}(k_n)$ in the canonical way as subspace of $\ell_{q'}=(\oplus_{j=1}^\infty\ell_{q'}(k_n))_q$). It follows from the choice of $(y^*_{(n,i)}:i=1,2\dots n)$ that $\|A'_n\|\leqslant C$ and from Corollary 3.6 (with $s=2$ and $t=q'$) we deduce that

$$\frac{1}{n}\sum_{i=1}^{n}\|A^*(y^*_{(n,i)})\|_\infty=\frac{1}{n}\sum_{i=1}^{n}\|A'(e_{(2,n,i)})\|_\infty\leqslant 2Cn^{-r(2,q')}.$$

Using the concavity of the function $[0,\infty)\ni\xi\to\xi^{(2-q')/2}$ we deduce

$$\frac{1}{n}\sum_{i=1}^{n}\|A^*(y^*_{(n,i)})\|_\infty^{(2-q')/2}=\Big(\frac{1}{n}\sum_{i=1}^{n}\|A^*(y^*_{(n,i)})\|_\infty\Big)^{(2-q')/2}\leqslant(2C)^{(2-q')/2}n^{-r(2,q')(2-q')/2}. \tag{14}$$

It is easy to see that $I_{(q',2)}$ is the adjoint of $I_{(2,q)}$ and we compute for $i=1,2\dots n$

$$
\begin{aligned}
\|I(q',2)\circ A^*(y^*(n,i))\|_2 &= \Big(\sum_{j=1}^{k_n}\big(A^*(y^*(n,i))(j)\big)^2\Big)^{1/2} \qquad (15)\\
&= \Big(\sum_{j=1}^{k_n}\big|A^*(y^*(n,i))(j)\big|^{q'}\big|A^*(y^*(n,i))(j)\big|^{2-q'}\Big)^{1/2}\\
&\leqslant \|A^*(y^*(n,i))\|_\infty^{(2-q')/2}\|y_{(n,i)}\|_{q'}^{q'/2}\\
&\leqslant C^{q'/2}\|A^*(y^*(n,i))\|_\infty^{(2-q')/2}.
\end{aligned}
$$

Therefore it follows

$$\begin{aligned}
|\langle \psi_n, U\rangle| &= \frac{1}{n}\Big|\sum_{i=1}^{n}\langle A\circ I_{(2,q)}\circ B(e_{(p,n,i)}), y^*(n,i)\rangle\Big| \\
&= \frac{1}{n}\Big|\sum_{i=1}^{n}\langle B(e_{(p,n,i)}), I_{(q',2)}\circ A^*(y^*(n,i))\rangle\Big| \\
&\leqslant \frac{1}{n}\sum_{i=1}^{n}\|B(e_{(p,n,i)})\|_2\cdot\|I_{(q',2)}\circ A^*(y^*_{(n,i)})\|_2 \\
&\leqslant \|B\|C^{q'/2}\frac{1}{n}\sum_{i=1}^{n}\|A^*(y^*(n,i))\|_\infty^{(2-q')/2} \ \text{(By (15))} \\
&\leqslant C^{q'/2}(C+1)^{(2-q')/2}n^{-r(2,q')(2-q')/2} \to_{n\to\infty} 0 \ \text{(By (14))}.
\end{aligned}$$

which implies our claim, and finishes the proof or Theorem 3.1. □

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