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ON THE IDEALS OF ORLICZ TYPE OPERATORS

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Abstract. We establish results on mappings of type ℓ_M defined by approximation numbers, for a given Orlicz function M. We associate the spaces of mappings of type ℓ_M with Köthe sequence spaces.

1. Introduction

Approximation numbers of an operator have a major role to play in studying several important concepts in functional analysis e.g. compactness, eigenvalue problems, nuclearity and in developing the theory of operator ideals. Indeed, these numbers coincide with the eigenvalues of a compact operator in case of Hilbert spaces and form a null sequence. In case, this sequence is a member of ℓ^1, the operator turns out to be nuclear. Possibly motivated by this observation, Pietsch studied in [5] the class of mappings of type ℓ^p, $0 < p < \infty$, on Banach spaces, which are the mappings for which the sequence of approximation numbers belongs to ℓ^p. These mappings are compact and characterize the nuclear spaces. Moreover, all such operators of the same type consitute an ideal of operators in the class of all bounded linear operators. On the other hand, generalizing the spaces ℓ^p, we have the Orlicz sequence spaces ℓ_M defined with the help of an Orlicz function M. These spaces, besides being of independent interest have been found useful in the development of the theory of sequence spaces [2], [3], [4]. However, in the context of this paper it is natural to ask what we can say about mappings of type ℓ_M? Study of such mappings has been carried out in this paper. Indeed, we show such mappings form an operator ideal and a component of this ideal can be associated with a Köthe sequence space through their approximation numbers.

2. Preliminaries and Notations

Throughout this paper, we denote by $\mathbb{N}$, the set of all natural numbers; by $\mathbb{N}_0$, the set $\{0,1,2,\ldots\}$ and by $\mathbb{K}$, the space of all scalars. An *Orlicz function* is a continuous mapping $M : [0\,\infty) \to [0\,\infty)$ which is convex, strictly increasing and satisfies $M(0) = 0$. If $M(x) \neq x$, it always admits the following integral representation

$$M(x) = \int_0^x p(t)\, dt$$

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where p, called the kernel of M is right-continuous for $t \geqslant 0$, $p(0) = 0$, $p(t) > 0$ for $t > 0$, p is increasing and $p(t) \to \infty$ if $t \to \infty$. If we define

$$q(s) = \sup\{t : p(t) \leqslant s\}$$

and consider

$$N(x) = \int_0^x q(s)\, ds$$

then N is also an Orlicz function. M and N are called *mutually complementary* Orlicz functions. An Orlicz sequence space corresponding to M is defined by

$$\ell_M = \left\{ \{x_i\} \in \omega : \sum_{i=0}^{\infty} M\left(\frac{|x_i|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

where ω stands for the space of all scalar sequences. The subspace of ℓ_M which contains those elements of ℓ_M for which $\sum_{i=0}^{\infty} M\left(\frac{|x_i|}{\rho}\right) < \infty$, for each $\rho > 0$ is *denoted by* h_M. M is said to satisfy the $\triangle_2$ *condition for small* x *or at zero* if for each $k > 0$, $\exists\, R_k \geqslant 1$ and $x_k > 0$ such that $M(kx) \leqslant R_k\, M(x)$, $\forall\, x \in (0\, x_k]$. It is well known that if M satisfies the $\triangle_2$ condition at zero then $\ell_M = h_M$ [2]. We also define

$$\ell_{(M)} = \left\{ \{x_i\} \in \omega : \delta(x,M) = \sum_{i=0}^{\infty} M(|x_i|) < \infty \right\}.$$

For arbitrary Banach spaces X and Y, $\mathscr{L}(X,Y)$ stands for the space of all bounded linear operators from X to Y. We write $\mathscr{L}(X)$ for the space $\mathscr{L}(X,X)$ and U_X represents the closed unit ball in X. For $n \in \mathbb{N}_0$, the n^{th} approximation number of $T \in \mathscr{L}(X,Y)$ is defined as

$$a_n(T) = \inf\{\|T - A\| : A \in \mathscr{L}(X,Y),\ \operatorname{rank}(A) \leqslant n\}$$

For a scalar sequence space λ, T is said to be of type λ if $\{a_n(T)\} \in \lambda$. For various properties and results on these numbers one is referred to [1], [4], [5] and [6].

For $x = \{x_i\} \in \omega$, we use the notation $x^{(n)}$ to denote its n^{th} section which is given by

$$\{x_0, x_1, x_2, \ldots, x_n, 0, 0, \ldots\}$$

and $x_i^{(n)}$ will denote the i^{th} co-ordinate of such a sequence. A subset M of a scalar sequence space λ is said to be *normal* if for any $\{x_i\} \in M$ and $\alpha_i \in \mathbb{K}$, with $|\alpha_i| \leqslant 1$, $i \in \mathbb{N}_0$, the sequence $\{\alpha_i x_i\} \in M$. If P is a collection of real sequences $a = \{a_n\}$ satisfying the properties (i) $a_n \geqslant 0$, $\forall\, n \in \mathbb{N}_0$ and each $a \in P$, (ii) for each $n \in \mathbb{N}_0$, $\exists\, a \in P$ with $a_n > 0$ and (iii) for each a, $b \in P$, $\exists\, c \in P$ with $a_n \leqslant c_n$, $b_n \leqslant c_n$, $\forall\, n \in \mathbb{N}_0$, then P is called a *Köthe set or Power set* and the sequence space $\Lambda(P)$ defined as

$$\Lambda(P) = \{x = \{x_i\} \in \omega : p_a(x) = \sum_{n=0}^{\infty} |x_n| a_n < \infty,\ \forall\, a = \{a_n\} \in P\},$$

is called a *Köthe sequence space*. The space $\Lambda(P)$ is equipped with the locally convex topology generated by the seminorms $\{p_a : a \in P\}$ and in this topology $\Lambda(P)$ is complete [5]. $\Lambda(P)$ is said to be of infinite type or a $G_\infty-$ space (resp. of finite type or a G_1 space) if P satisfies (i) for each $a = \{a_n\} \in P$, $0 < a_n \leqslant a_{n+1}$ (resp. $0 < a_{n+1} \leqslant a_n$), $\forall\ n \in \mathbb{N}_0$. (ii) for each $a = \{a_n\} \in P\ \exists\ b = \{b_n\} \in P$ such that $a_n^2 \leqslant b_n$ (resp. $a_n \leqslant b_n^2$), $\forall\, n \in \mathbb{N}_0$.

3. The space $\ell_M(X,Y)$

For an Orlicz function M and Banach spaces X, Y, let us introduce

$$\ell_M(X,Y) = \{T \in \mathscr{L}(X,Y) : \{a_n(T)\} \in \ell_M\},$$

and for $T \in \ell_M(X,Y)$,

$$\|T\|_M = \inf\left\{\rho > 0 : \sum_{i=0}^{\infty} M\left(\frac{a_i(T)}{\rho}\right) \leqslant 1\right\}.$$

Also, we define

$$h_M(X,Y) = \{T \in \mathscr{L}(X,Y) : \{a_n(T)\} \in h_M\}.$$

We prove

PROPOSITION 3.1. *$(\ell_M(X,Y), \|.\|_M)$ is a quasi-Banach space with norm satisfying the condition,*

$$\sum_{i=0}^{\infty} M\left(\frac{a_i(T)}{\|T\|_M}\right) \leqslant 1,$$

for $T \in \ell_M(X,Y)$.

Proof. For T, $S \in \ell_M(X,Y)$ if $\mu_1 > 0$ and $\mu_2 > 0$ are such that

$$\sum_{i=0}^{\infty} M\left(\frac{a_i(T)}{\mu_1}\right) < \infty \qquad \text{and} \qquad \sum_{i=0}^{\infty} M\left(\frac{a_i(S)}{\mu_2}\right) < \infty,$$

then for $\delta = 2\max(\mu_1, \mu_2)$,

$$\sum_{i=0}^{\infty} M\left(\frac{a_i(T+S)}{\delta}\right) \leqslant \sum_{i=0}^{\infty} M\left(\frac{a_i(T)}{\mu_1}\right) + \sum_{i=0}^{\infty} M\left(\frac{a_i(S)}{\mu_2}\right) < \infty,$$

by using the inequalities

$$M(a_{2i}(T+S)) + M(a_{2i+1}(T+S)) \leqslant 2M(a_{2i}(T+S))$$

and

$$a_{2i}(T+S)) \leqslant a_i(T) + a_i(S).$$

So, $T+S \in \ell_M(X,Y)$. Similarly, for $\alpha \in \mathbb{K}$, one can easily check that $\alpha T \in \ell_M(X,Y)$.

For showing that $\|.\|_M$ is a quasi-norm on $\ell_M(X,Y)$, we first show that $\|T\|_M = 0 \Rightarrow T=0$. If $\|T\|_M = 0$, then $\forall\, \varepsilon > 0$, we have

$$\sum_{i=0}^{\infty} M\left(\frac{a_i(T)}{\varepsilon}\right) \leqslant 1.$$

$$\Rightarrow \left\{\frac{a_i(T)}{\varepsilon},\ i=0,1,2,\ldots\right\}$$

is bounded in ℓ^∞ [2] and so $a_0(T) \leqslant \varepsilon\, K$, for some $K>0$. As $\varepsilon > 0$ is arbitray, we get $T=0$.

It can be easily verified that $\|\alpha T\|_M = |\alpha|\|T\|_M$, for each $\alpha \in \mathbb{K}$.

For triangle inequality note that for $T,\ S \in \ell_M(X,Y)$ and $\delta = 2\ (\rho+\eta)$, where $\rho > 0,\ \eta > 0$, we have

$$\sum_{i=0}^{\infty} M\left(\frac{a_i(T+S)}{\delta}\right) \leqslant \frac{\rho}{\rho+\eta}\sum_{i=0}^{\infty} M\left(\frac{a_i(T)}{\rho}\right) + \frac{\eta}{\rho+\eta}\sum_{i=0}^{\infty} M\left(\frac{a_i(S)}{\eta}\right).$$

If ρ and η are such that

$$\sum_{i=0}^{\infty} M\left(\frac{a_i(T)}{\rho}\right) \leqslant 1, \quad 0<\rho<\|T\|_M + \frac{\varepsilon}{2}$$

and

$$\sum_{i=0}^{\infty} M\left(\frac{a_i(S)}{\eta}\right) \leqslant 1, \quad 0<\eta<\|S\|_M + \frac{\varepsilon}{2},$$

for each $\varepsilon > 0$, we get

$$\|T+S\|_M \leqslant 2\ (\|T\|_M + \|S\|_M).$$

The fact

$$\sum_{i=0}^{\infty} M\left(\frac{a_i(T)}{\|T\|_M}\right) \leqslant 1,$$

clearly holds.

Let $\{T_n\}$ be a Cauchy sequence in $\ell_M(X,Y)$. Then $\forall\, \varepsilon > 0$, $\exists\, p \in \mathbb{N}$ such that

$$\sum_{i=0}^{\infty} M\left(\frac{a_i(T_n - T_m)}{\varepsilon}\right) \leqslant 1,\ \forall\, n,\ m \geqslant p. \tag{3.1}$$

$$\Rightarrow\ \frac{a_i(T_n - T_m)}{\varepsilon} \leqslant K,\ \ \forall\, n,\ m \geqslant p,$$

for each $i \in \mathbb{N}_0$ and some constant $K > 0$. In particular when $i=0$, we conclude that $\{T_n\}$ is a Cauchy sequence in $\mathscr{L}(X,Y)$ and thus there exists $T \in \mathscr{L}(X,Y)$ such that

$T_n \to T$ as $n \to \infty$, in the operator norm. Hence $\lim_{n\to\infty} a_i(T_m - T_n) = a_i(T_m - T)$, [5] By taking the limit in (3.1) as $n \to \infty$, we get $T \in \ell_M(X,Y)$ and

$$\|T - T_m\|_M < \varepsilon \ , \ \forall\, m \geqslant p.$$

This completes the proof. $\square$

Next we have

PROPOSITION 3.2. *$\ell_M(X,Y)$ is a subspace of $\mathscr{L}(X,Y)$ such that the inclusion map from $(\ell_M(X,Y), \|.\|_M)$ to $(\mathscr{L}(X,Y), \|.\|)$ is continuous.*

Proof. Let $T_n \to T$ in $\ell_M(X,Y)$. Then for $\varepsilon > 0$, we can find $n_0 \in \mathbb{N}_0$ such that $\|T_n - T\| \leqslant \varepsilon\, K$, for some $K > 0$ and all $n \geqslant n_0$ as in the proof of Proposition 3.1. Hence the result follows. $\square$

PROPOSITION 3.3. (i) *For $T \in \mathscr{L}(X,Y)$ and $S \in \ell_M(Y,Z)$, $ST \in \ell_M(X,Z)$. Moreover, $\|ST\|_M \leqslant \|S\|_M \|T\|$.*

(ii) *If $T \in \ell_M(X,Y)$, $S \in \mathscr{L}(Y,Z)$, then $ST \in \ell_M(X,Z)$ and $\|ST\|_M \leqslant \|S\| \|T\|_M$.*

Proof. (i) Since

$$\sum_{i=0}^{\infty} M\left(\frac{a_i(ST)}{\rho}\right) \leqslant \sum_{i=0}^{\infty} M\left(\frac{a_i(S)\|T\|}{\rho}\right),$$

for any $\rho > 0$ and so for $\rho = \mu\|T\|$, where $\mu > 0$ satisfies

$$\sum_{i=0}^{\infty} M\left(\frac{a_i(S)}{\mu}\right) < \infty,$$

it follows that $ST \in \ell_M(X,Z)$. Also

$$\sum_{i=0}^{\infty} M\left(\frac{a_i(ST)}{\|S\|_M\|T\|}\right) \leqslant \sum_{i=0}^{\infty} M\left(\frac{a_i(S)}{\|S\|_M}\right) \leqslant 1.$$

$$\Rightarrow \ \|ST\|_M \leqslant \|S\|_M \|T\|.$$

(ii) Omitted as it is analogous to the proof of (i). $\square$

REMARK 3.4. It follows from the above proposition that the space $\mathscr{L}^M$ of all bounded linear operators T between arbitrary Banach spaces such that $\{a_n(T)\} \in \ell_M$, for a fixed Orlicz function M, is an operator ideal whose components are given by $\ell_M(X,Y)$.

PROPOSITION 3.5. *$h_M(X,Y)$ is a closed subspace of $\ell_M(X,Y)$.*

Proof. Clearly $h_M(X,Y)$ is a subspace of $\ell_M(X,Y)$. For showing that it is closed in $\ell_M(X,Y)$, consider $T \in \overline{h_M(X,Y)}$, when closure is being considered in $\ell_M(X,Y)$. Then there exists a sequence $\{T_n\}$ in $h_M(X,Y)$ such that

$$\lim_{n\to\infty} \|T_n - T\|_M = 0.$$

For $\varepsilon > 0$, choose $p \in \mathbb{N}_0$ such that

$$\|T_n - T\|_M < \frac{\varepsilon}{2},\ \forall\, n \geqslant p$$

Now for each $k \in \mathbb{N}_0$

$$\begin{aligned}\sum_{i=0}^{k} M\left(\frac{a_i(T)}{\varepsilon}\right) &\leqslant 2 \sum_{i=0}^{k} M\left(\frac{a_i(T-T_p)+a_i(T_p)}{\varepsilon}\right)\\ &\leqslant \sum_{i=0}^{\infty} M\left(\frac{a_i(T-T_p)}{\|T-T_p\|_M}\right) + \sum_{i=0}^{\infty} M\left(\frac{a_i(T_p)}{\frac{\varepsilon}{2}}\right) < \infty,\end{aligned}$$

as $T_p \in h_M(X,Y)$ and $\sum_{i=0}^{\infty} M\left(\frac{a_i(S)}{\|S\|_M}\right) \leqslant 1$, for any $S \in \ell_M(X,Y)$. Hence $T \in h_M(X,Y)$. □

THEOREM 3.6. *The space $\mathscr{A}(X,Y)$ of all finite rank operators from X to Y is dense in $h_M(X,Y)$.*

Proof. Clearly $\mathscr{A}(X,Y) \subseteq h_M(X,Y)$. For showing that $\mathscr{A}(X,Y)$ is dense in $h_M(X,Y)$, consider $T \in h_M(X,Y)$. Then for $\varepsilon > 0$, in particular for $0 < \varepsilon < 1$, we have

$$\sum_{i=0}^{\infty} M\left(\frac{a_i(T)}{\varepsilon^2}\right) < \infty.$$

$\Rightarrow$ For $0 < \delta < \frac{1}{6}$, $\exists\ k \in \mathbb{N}$ such that

$$k\, M\left(\frac{a_{2k}(T)}{\varepsilon^2}\right) \leqslant \sum_{i=k+1}^{2k} M\left(\frac{a_i(T)}{\varepsilon^2}\right) \leqslant \sum_{i=k}^{\infty} M\left(\frac{a_i(T)}{\varepsilon^2}\right) < \delta,$$

as approximation numbers are decreasing and M is increasing in nature.

Since $\frac{a_{2k}(T)}{\varepsilon} > a_{2k}(T)$, there exists $A \in \mathscr{L}(X,Y)$ of rank at most $2k-1$ such that

$$\|T - A\| < \frac{a_{2k}(T)}{\varepsilon}.$$

$$\begin{aligned}&\Rightarrow\ M\left(\frac{\|T-A\|}{\varepsilon}\right) < M\left(\frac{a_{2k}(T)}{\varepsilon^2}\right).\\ &\Rightarrow k\, M\left(\frac{\|T-A\|}{\varepsilon}\right) < \delta.\end{aligned}$$

Further, we note that for any $n \in \mathbb{N}_0$, we have $a_{n+2k}(T-A) \leqslant a_n(T)$ and so

$$\begin{aligned}\sum_{i=0}^{\infty} M\left(\frac{a_i(T-A)}{\varepsilon}\right) &= \sum_{i=0}^{3k-1} M\left(\frac{a_i(T-A)}{\varepsilon}\right) + \sum_{i=3k}^{\infty} M\left(\frac{a_i(T-A)}{\varepsilon}\right) \\ &\leqslant 3k\, M\left(\frac{\|T-A\|}{\varepsilon}\right) + \sum_{i=k}^{\infty} M\left(\frac{a_i(T)}{\varepsilon}\right) \\ &< \frac{1}{2} + \frac{1}{6} < 1.\end{aligned}$$

Thus we get

$$\|T-A\|_M < \varepsilon.$$

Hence $\mathscr{A}(X,Y)$ is dense in $h_M(X,Y)$. □

COROLLARY 3.7. *Every operator $T \in h_M(X,Y)$ is precompact.*

Proof. Follows from Proposition 3.2 and the previous result. □

From Proposition 3.5, we conclude that $\mathscr{A}(X,Y)$ may not be dense in $\ell_M(X,Y)$ until we impose certain restrictions on M. In this context, we prove

PROPOSITION 3.8. *If the Orlicz function M satisfies the $\triangle_2$ condition at zero then $\mathscr{A}(X,Y)$ is dense in $\ell_M(X,Y)$.*

Proof. Obvious from the fact that $\ell_M = h_M$ in case M satisfies the $\triangle_2$ condition at zero. □

4. $\ell_M(X,Y)$ as Köthe sequence space of linear operators

In this section, we assume throughout that M and N are mutually complementary Orlicz functions and show that the space $\ell_M(X,Y)$ studied in the previous section can be identified with a Köthe Power space $\Lambda_M(X,Y)$ of linear operators defined with the help of a Köthe set $\mathscr{P}_N$. Indeed, let us introduce

$$\mathscr{P} = \{\{x_i\} \in \omega : x_i \geqslant 0, \forall\, i \in \mathbb{N} \text{ and } \{x_i\} \text{ is decreasing}\}$$

$$\mathscr{P}_N = \{\{x_i\} \in \mathscr{P} : \delta(x,N) < \infty\}.$$

Then we have

PROPOSITION 4.1. *$\mathscr{P}$ and $\mathscr{P}_N$ are Power sets.*

Proof. Obvious. □

Let us now define the Köthe Power space of linear operators

$$\Lambda_M(X,Y) = \{T \in \mathscr{L}(X,Y) : \sum_{i=0}^{\infty} a_i(T) y_i < \infty,\ \forall\, y = \{y_i\} \in \mathscr{P}_N\}.$$

THEOREM 4.2. *$\Lambda_M(X,Y)$ is a quasi-Banach space.*

We split the proof of the above theorem in the following four propositions.

PROPOSITION 4.3. *$\Lambda_M(X,Y)$ is a vector space.*

Proof. Note that for $T,\ S \in \Lambda_M(X,\ Y)$ and $y = \{y_n\} \in \mathscr{P}_N$ we have

$$\sum_{n=0}^{\infty} a_n(T+S)y_n \leqslant 2 \sum_{n=0}^{\infty} (a_n(T)+a_n(S))\, y_n < \infty$$

and

$$\sum_{n=0}^{\infty} a_n(\alpha T)y_n \;=\; |\alpha| \sum_{n=0}^{\infty} a_n(T)y_n < \infty,$$

for any $\alpha \in \mathbb{K}$. Thus $\Lambda_M(X,\ Y)$ is a linear space. $\square$

PROPOSITION 4.4. *For each $T \in \Lambda_M(X,Y)$,*

$$\sup\{\sum_{i=0}^{\infty} a_i(T)y_i : y = \{y_i\} \in \mathscr{P}_N,\ \delta(y,N) \leqslant 1\} < \infty. \tag{4.1}$$

Proof. Let $\sup\{\sum_{i=0}^{\infty} a_i(T)y_i : y = \{y_i\} \in \mathscr{P}_N,\ \delta(y,N) \leqslant 1\} = \infty$. Then for each $n \in \mathbb{N}_0$, $\exists\, y^n = \{y_i^n\} \in \mathscr{P}_N$ with $\delta(y^n,N) \leqslant 1$ and

$$\sum_{i=0}^{\infty} a_i(T)y_i^n > 2^n.$$

Define

$$x_i = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}\, y_i^{n+1}.$$

Note that the sequence $\{x_i\}$ is well defined and $x_i \geqslant x_{i+1},\ \forall\ i \in \mathbb{N}_0$. Further, by the convexity of N,

$$N(\sum_{n=0}^{m-1} \frac{1}{2^{n+1}}\, y_i^{n+1}) \leqslant \sum_{n=0}^{m-1} \frac{1}{2^{n+1}}\, N(y_i^{n+1}),$$

for each $m \geqslant 1$ and so

$$\delta(x,N) = \sum_{i=0}^{\infty} N(x_i) \leqslant \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{i=0}^{\infty} N(y_i^{n+1}) \leqslant 1,$$

as N is continuous. Thus $x \in \mathscr{P}_N$.

On the other hand

$$\sum_{i=0}^{\infty} a_i(T)x_i \geqslant \sum_{i=0}^{\infty} a_i(T) \sum_{n=0}^{m-1} \frac{1}{2^{n+1}}\, y_i^{n+1} \geqslant m$$

for each $m \in \mathbb{N}$. This contradicts that $T \in \Lambda_M(X,Y)$. Hence the result follows. □

The finiteness of the expression in (4.1) leads us to define

$$\|T\|'_M = \sup\{\sum_{i=0}^{\infty} a_i(T)y_i : y = \{y_i\} \in \mathscr{P}_N,\ \delta(y,N) \leqslant 1\},$$

for $T \in \Lambda_M(X,Y)$.

PROPOSITION 4.5. *$\|.\|'_M$ defines a quasi-norm on $\Lambda_M(X,Y)$ and the inclusion map from $(\Lambda_M(X,Y), \|.\|'_M)$ to $(\mathscr{L}(X,Y), \|.\|)$ is continuous.*

Proof. Clearly for any $T \in \Lambda_M(X,\ Y)$, $\|T\|'_M \geqslant 0$ and $\|T\|'_M = 0$ if $T = 0$. If $\|T\|'_M = 0$, then $a_0(T)\ y_0 = 0$, $\forall\ y = \{y_i\} \in \mathscr{P}_N$, with $\delta(y,N) \leqslant 1$. If $\{y_i\}$ is a non-zero sequence then $a_0(T) = \|T\| = 0 \Rightarrow T = 0$. Clearly, for any $\alpha \in \mathbb{K}$, $\|\alpha\ T\|'_M = |\alpha|\|T\|'_M$. If $S \in \Lambda_M(X,Y)$, then for every $y = \{y_i\} \in \mathscr{P}_N$ with $\delta(y,N) \leqslant 1$,

$$\sum_{n=0}^{\infty} a_n(T+S)y_n \leqslant 2\ \sum_{n=0}^{\infty}(a_n(T)+a_n(S))\ y_n \leqslant 2\ (\|T\|'_M + \|S\|'_M).$$

$$\Rightarrow\ \|T+S\|'_M \leqslant\ 2(\|T\|'_M + \|S\|'_M).$$

The fact that the inclusion map from $\Lambda_M(X,Y)$ to $\mathscr{L}(X,Y)$ is continuous, can be easily verified. □

PROPOSITION 4.6. *$(\Lambda_M(X,\ Y),\ \|.\|'_M)$ is a quasi-Banach space.*

Proof. Let $\{T_n\}$ be a Cauchy sequence in $\Lambda_M(X,\ Y)$ and $y = \{y_i\} \in \mathscr{P}_N$ be a non-zero sequence with $\delta(y,N) \leqslant 1$. Then for any $\varepsilon > 0$ we can find $k \in \mathbb{N}_0$ such that

$$\sum_{i=0}^{\infty} a_i(T_n - T_m)\ y_i < \varepsilon,\ \ \forall\ n,m \geqslant k. \tag{4.2}$$

$\Rightarrow$ $\{T_n\}$ is a Cauchy sequence in $\mathscr{L}(X,Y)$. Thus there exists a $T \in L(X,Y)$ such that

$$\lim_{n\to\infty} \|T_n - T\| = 0.$$

Since

$$|a_i(T-T_m) - a_i(T_m - T_n)| \leqslant \|T - T_n\|,$$

we have

$$\lim_{m\to\infty} a_i(T_m - T_n) = a_i(T - T_n),\ \forall\ n \in \mathbb{N}_0$$

Taking the limit in (4.2) as $m \to \infty$ we get

$$\sum_{i=0}^{\infty} a_i(T - T_n)\ y_i \leqslant \varepsilon,\ \ \forall\ n \geqslant k,\ \forall\ y = \{y_i\} \in \mathscr{P}_N,\ \delta(y,N) \leqslant 1.$$

Hence $T - T_k$ and thus $T \in \Lambda_M(X, Y)$. Further

$$\sup\{\sum_{i=0}^{\infty} a_i(T - T_n)\, y_i : y = \{y_i\} \in \mathscr{P}_N,\ \delta(y,N) \leqslant 1\} \leqslant \varepsilon,\ \ \forall\, n \geqslant\ k.$$

$$\Rightarrow \lim_{n\to\infty} \|T - T_n\|'_M = 0.$$

Thus $\Lambda_M(X, Y)$ is a quasi-Banach space. □

By combining the Propositions 4.3, 4.4, 4.5 and 4.6, we get Theorem 4.2.

We now proceed proving the equality between the spaces $\ell_M(X,Y)$ and $\Lambda_M(X,Y)$. To begin with, let us introduce the notations

$$\ell_{(M)}(X,Y) = \{T \in \mathscr{L}(X,Y) : \{a_n(T)\} \in \ell_{(M)}\}$$

and

$$\delta(T,M) = \sum_{n=0}^{\infty} M(a_n(T)),$$

for $T \in \ell_{(M)}(X,Y)$.

Using Young's inequality [2], we immediately get the following

PROPOSITION 4.7. $\ell_{(M)}(X,Y) \subseteq \Lambda_M(X,Y)$.

For proving the other inclusion we make use of the following results, proofs of which are being omitted as they are similar to the ones given in [2] pp. 301–303.

PROPOSITION 4.8. *For each $T \in \Lambda_M(X,Y)$ with $\|T\|'_M \leqslant 1$, the sequence $\phi = \{\phi_i\} \in \mathscr{P}_N$ with $\delta(\phi,N) \leqslant 1$, where $\phi_i = p(a_i(T))$, $\forall\, i \in \mathbb{N}_0$.*

PROPOSITION 4.9. *If $T \in \Lambda_M(X,Y)$ with $\|T\|'_M \leqslant 1$, then $T \in \ell_{(M)}(X,Y)$ and $\delta(T,M) \leqslant \|T\|'_M$.*

We also have

PROPOSITION 4.10. *$\Lambda_M(X,Y) \subseteq \ell_M(X,Y)$ and $\|T\|_M \leqslant \|T\|'_M$, for $T \in \Lambda_M(X\,Y)$.*

PROPOSITION 4.11. *$\ell_M(X,Y) \subseteq \Lambda_M(X,Y)$ and $\|T\|'_M \leqslant 2\,\|T\|_M$.*

Combining the above two propositions we conclude

THEOREM 4.12. *$\ell_M(X,Y) = \Lambda_M(X,Y)$ and $\|T\|_M \leqslant \|T\|'_M \leqslant 2\,\|T\|_M$.*

COROLLARY 4.13. *$T \in \mathscr{L}(X,Y)$ is of type ℓ_M if and only if $\{a_n(T)\} \in \Lambda(\mathscr{P}_N)$.*

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