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MATRIX FORMULATION FOR INFINITE–RANK OPERATORS

BALMOHAN V. LIMAYE

Dedicated to the memory
of Alain Largillier

Abstract. Every finite-rank operator on a linear space X is the composition of an operator from X to a finite dimensional Euclidean space and of an operator from that Euclidean space to X. We consider operators which are the sum of a finite-rank operator and another infinite-rank operator which satisfies an invariance condition with respect to one of the two 'components' of the finite-rank operator. A canonical procedure is given to reduce operator equations, eigenvalue problems and spectral subspace problems involving such operators to corresponding problems for finite matrices.

1. Introduction

Finite-rank operators have long been used in classical methods such as the collocation method, the Galerkin method, the degenerate kernel method or the Nyström method for approximating integral operators. They yield approximate solutions of operator equations and of eigenvalue problems as well as spectral subspace problems involving the given integral operator. (See [3] and [1].) The corresponding problem for a finite-rank operator S is then reduced to a problem for a finite dimensional operator A, that is, to a matrix problem, by a variety of procedures depending on the nature of the finite-rank operator. In 1986, Whitley [7, Lemma 1] first proposed a canonical reduction procedure which unified all known classical procedures. His result was improved, and it was extended to eigenvalue problems and spectral subspace problems in [4, Lemma 1] and [5, Lemma 3.1].

Every finite-rank operator on a linear space X can be written as LK, where K is a linear map from X to a finite dimensional Euclidean space and L is a linear map from that Euclidean space to X. The purpose of the present article is to obtain a canonical reduction procedure for an operator $T := S+U$ or $T := S+V$, where S is any finite-rank operator on X, the operator U on X is such that the range of the transpose of K is invariant under the transpose of U, and the operator V on X is such that the range of L is invariant under V. An operator equation, an eigenvalue problem or a spectral subspace problem involving the operator $T = S+U$ is reduced to a question involving a finite dimensional operator $B := A+C$. This generalizes the results of [2], where the finite-rank operator S is given by evaluation of a continuous function at certain

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nodes, and U is a multiplication operator on the space of continuous functions. This case arises in the singularity subtraction technique for integral operators with weakly singular kernels. (See also [1, §5.1.1].) Our framework also includes a case treated recently by Majidian and Babolian [6], where the operator S arises from a degenerate kernel method with piecewise constant interpolation in the second variable of a smooth kernel, and the operator U arises again from a multiplication by a piecewise constant function. In both cases, the matrix corresponding to the finite dimensional operator C is a diagonal matrix, since the operator U is a multiplication operator. On the other hand, our results include cases where this matrix is diagonal, cross-diagonal, subdiagonal, lower-triangular etc. In the process of this generalization, we have simplified many of the arguments given in [2] and have made the results more elaborate.

Although this paper is of a theoretical nature, issues regarding the actual implementation of the results proved here are kept in view (as in comments after the proofs of Propositions 3.4 and 5.1). The paper is organized as follows. In Section 2, we relate solution of an operator equation and of an eigenvalue problem (including considerations of geometric multiplicity and ascent) involving an operator of the form $T = S+U$ to that of a matrix of the form $B = A+C$ in the framework of a linear space X (without mention of any norm on it). We give several specific examples which include the ones treated in [2] and [6]. In Section 3, we consider a complete norm on X and take up spectral considerations for $T = S+U$, such as finding a basis for a spectral subspace or the algebraic multiplicity of a spectral value. In Section 4, we deal with stability considerations for linear systems arising from operator equations on the lines of [7]. In Section 5, we consider an operator of the form $T = S+V$ mentioned earlier. Here the reduction procedure is not of much use for solving operator equations, but it is useful in dealing with eigenvalue problems and spectral subspace problems.

2. Operator Equation and Eigenvalue Problem for $T = S+U$

Let X be a linear space over $\mathbb{C}$ and $S : X \to X$ be a linear operator. Then the rank of S is at most $n \in \mathbb{N}$ if and only if there are linear maps $K : X \to \mathbb{C}^{n\times 1}$ and $L : \mathbb{C}^{n\times 1} \to X$ such that $LK = S$. In fact, there are elements $x_1, \dots, x_n$ in X and linear functionals $f_1, \dots, f_n$ on X such that

$$Sx = \sum_{j=1}^{n} f_j(x)x_j \quad \text{for all } x \in X.$$

Define $Kx := [f_1(x), \dots, f_n(x)]^t$ for $x \in X$ and $Lu := u(1)x_1 + \cdots + u(n)x_n$ for $u := [u(1), \dots, u(n)]^t \in \mathbb{C}^{n\times 1}$, where the superscript t denotes transpose. Then $S = LK$. We emphasize that neither the elements $x_1, \dots, x_n$ in X nor the linear functionals $f_1, \dots, f_n$ on X need be linearly independent.

Let S be of finite rank, and $S = LK$ as above. Define $A : \mathbb{C}^{n\times 1} \to \mathbb{C}^{n\times 1}$ by $A := KL$. Then $AK = KS$.

Consider a linear map $U : X \to X$ such that $f_i \circ U$ belongs to the linear span of $f_1, \dots, f_n$ for each $i = 1, \dots, n$. Then there are complex numbers $c_{i,j}$ such that $f_i \circ U = \sum_{j=1}^{n} c_{i,j} f_j$ for each $i = 1, \dots, n$. Define a linear map $C : \mathbb{C}^{n\times 1} \to \mathbb{C}^{n\times 1}$ by $Cu :=$

$\left[\sum_{j=1}^n c_{1,j}u(j),\ldots,\sum_{j=1}^n c_{n,j}u(j)\right]^t$ for $u := [u(1),\ldots,u(n)]^t$ in $\mathbb{C}^{n\times 1}$. Then $CK = KU$. Let

$$T := S+U \quad \text{and} \quad B := A+C.$$

It follows that $BK = KT$. We shall reduce problems involving the linear map T to problems involving the finite dimensional linear map B. Before doing so, we give several examples of functionals $f_1,\ldots,f_n$ and linear maps U which satisfy the invariance condition mentioned above.

EXAMPLES 2.1. (i) Let J be a set and X be a subspace of the linear space of all complex-valued functions defined on J. Let $\ell \in \mathbb{N}$, and consider $\xi_k \in X$ and $\alpha_k : J \to J$ such that $\xi_k x$ and $x\circ\alpha_k$ belong to X for each $k = 1,\ldots,\ell$ and all $x \in X$. For $x \in X$, define $Ux := \xi_1(x\circ\alpha_1)+\cdots+\xi_\ell(x\circ\alpha_\ell)$. Then U is a linear map from X to X. In the following two items, we point out situations where $f_i\circ U \in \operatorname{span}\{f_1,\ldots,f_n\}$ for $i = 1,\ldots,n$.

- Let J be a topological space and X denote the linear space of all complex-valued continuous functions on J. Consider distinct points $t_1,\ldots,t_n$ in J, and let $f_i(x) := x(t_i)$ for $i = 1,\ldots,n$ and $x \in X$. Assume that α_k is continuous and $\alpha_k(t_i) \in \{t_1,\ldots,t_n\}$ for each $k = 1,\ldots,\ell$ and $i = 1,\ldots,n$. Then for $i = 1,\ldots,n$ and $x \in X$,

$$(Ux)(t_i) = \sum_{k=1}^{\ell}\xi_k(t_i)x\big(\alpha_k(t_i)\big) = \sum_{j=1}^{n}\left(\sum_{\alpha_k(t_i)=t_j}\xi_k(t_i)\right)x(t_j).$$

Thus $f_i\circ U = \sum_{j=1}^{n} c_{i,j}f_j$, where $c_{i,j} := \sum_{\alpha_k(t_i)=t_j}\xi_k(t_i)$, $1 \leqslant i,j \leqslant n$.

- Let $J := [a,b]$, an interval in $\mathbb{R}$, and let X be the linear space of all complex-valued bounded Lebesgue measurable functions on $[a,b]$. Consider disjoint subintervals $E_1,\ldots,E_n$ of $[a,b]$ of positive lengths and let $f_i(x) := \frac{1}{m(E_i)}\int_{E_i} x\,dm$ for $i = 1,\ldots,n$ and $x \in X$. Assume that α_k is Lebesgue measurable and $\alpha_k(E_i) \in \{E_1,\ldots,E_n\}$ for each $k = 1,\ldots,\ell$ and $i = 1,\ldots,n$. Suppose every ξ_k is constant and every α_k is affine on each E_i. Thus there are complex numbers $p_{i,k}$ and $q_{i,k}$ such that $\xi_k = p_{i,k}$ and $\alpha_k' := q_{i,k}$ on E_i for $k = 1,\ldots,\ell$ and $i = 1,\ldots,n$. Then for $i = 1,\ldots,n$ and $x \in X$,

$$\begin{aligned}\int_{E_i}(Ux)dm &= \sum_{k=1}^{\ell}\int_{E_i}\xi_k(x\circ\alpha_k)dm = \sum_{k=1}^{\ell}\frac{p_{i,k}}{|q_{i,k}|}\int_{\alpha_k(E_i)} x\,dm \\ &= \sum_{j=1}^{n}\left(\sum_{\alpha_k(E_i)=E_j}\frac{p_{i,k}}{|q_{i,k}|}\right)\int_{E_j} x\,dm.\end{aligned}$$

Thus $f_i\circ U = \sum_{j=1}^n c_{i,j}f_j$, where

$$c_{i,j} := \frac{m(E_j)}{m(E_i)}\sum_{\alpha_k(E_i)=E_j}\frac{p_{i,k}}{|q_{i,k}|}, \quad 1 \leqslant i,j \leqslant n.$$

Let us give specific instances of the two items mentioned above. Let $J=[0,1]=[a,b]$, $n\geqslant 2$, $t_i:=(i-1)/(n-1)$ and $E_i:=((i-1)/n,i/n)$ for $i=1,\ldots,n$. First, let $\ell=1$ and $\alpha_1(t):=t$ for $t\in[0,1]$. In the case of the first item, $\alpha_1(t_i)=t_i$ for $i=1,\ldots,n$, so that $c_{i,j}=\xi_1(t_i)$ if $i=j$ and $c_{i,j}=0$ if $i\neq j$ for $i,j=1,\ldots,n$. Similarly, in the case of the second item, $\alpha_1(E_i)=E_i$, so that $c_{i,j}=\frac{1}{m(E_i)}\int_{E_i}\xi_1\,dm$ if $i=j$, and $c_{i,j}=0$ if $i\neq j$ for $i,j=1,\ldots,n$. In both items, the linear map C is defined by a diagonal matrix. (Compare [2] and [6].) Next, let $\ell=2$, $\alpha_1(t):=t$ and $\alpha_2(t):=1-t$ for $t\in[0,1]$. In the case of the first item, $\alpha_1(t_i)=t_i$ and $\alpha_2(t_i)=t_{n-i+1}$ for $i=1,\ldots,n$, so that

$$c_{i,j}=\begin{cases}\xi_1(t_i) & \text{if } i=j \text{ but } i\neq n-j+1,\\ \xi_2(t_i) & \text{if } i=n-j+1 \text{ but } i\neq j,\\ (\xi_1+\xi_2)(t_i) & \text{if } i=j=n-j+1,\\ 0 & \text{otherwise}\end{cases}$$

for $i,j=1,\ldots,n$. Similarly, in the case of the second item, $\alpha_1(E_i)=E_i$, $\alpha_2(E_i)=E_{n-i+1}$ and $m(E_i)=m(E_{n-i+1})$ for $i=1,\ldots,n$, so that

$$c_{i,j}=\begin{cases}\frac{1}{m(E_i)}\int_{E_i}\xi_1\,dm & \text{if } i=j \text{ but } i\neq n-j+1,\\ \frac{1}{m(E_i)}\int_{E_i}\xi_2\,dm & \text{if } i=n-j+1 \text{ but } i\neq j,\\ \frac{1}{m(E_i)}\int_{E_i}(\xi_1+\xi_2)dm & \text{if } i=j=n-j+1,\\ 0 & \text{otherwise}\end{cases}$$

for $i,j=1,\ldots,n$. In both items, the linear map C is defined by a matrix which has nonzero entries only along the diagonal and the cross diagonal.

(ii) Let X be a subspace of the linear space of all complex sequences. Let $f_i(x):=x(i)$ for $i=1,\ldots,n$ and $x:=(x(1),x(2),\ldots)\in X$. Let a lower-triangular infinite matrix $[u_{i,j}]$ define a linear map $U:X\to X$. Note that $u_{i,j}:=0$ for all $i,j\in\mathbb{N}$ with $j>i$. Then $(Ux)(i)=\sum_{j=1}^{i}u_{i,j}x(j)$ for $i=1,\ldots,n$ and $x\in X$. Thus $f_i\circ U=\sum_{j=1}^{n}c_{i,j}f_j$, where $c_{i,j}:=u_{i,j}$ if $j\leqslant i$ and $c_{i,j}:=0$ if $j>i$ for $1\leqslant i,j\leqslant n$. To obtain a specific example, let $w_1,w_2,\ldots$ be complex numbers such that $Ux:=(0,w_1x(1),w_2x(2),\ldots)\in X$ for each $x:=(x_1,x_2,\ldots)\in X$. Then $f_1\circ U=0$ and $f_i\circ U=w_{i-1}f_{i-1}$ for $i=2,\ldots,n$. For this weighted right-shift operator U on X, the linear map C is defined by a matrix which has nonzero entries only along the subdiagonal.

We now turn to a simple-minded but crucial result.

PROPOSITION 2.2. *Let $\zeta\in\mathbb{C}$, $y\in X$, and define $v:=Ky$.*

(i) *Let $x\in X$ satisfy $\zeta x-Tx=y$ and define $u:=Kx$. Then $\zeta u-Bu=v$ and $\zeta x-Ux=Lu+y$.*

(ii) *Suppose ζ is not an eigenvalue of C. Let $u\in\mathbb{C}^{n\times 1}$ satisfy $\zeta u-Bu=v$ and let $x\in X$ satisfy $\zeta x-Ux=Lu+y$. Then $\zeta x-Tx=y$ and $Kx=u$.*

Proof. Let I denote the identity operator on X as well as on $\mathbb{C}^{n\times 1}$. Consider the diagrams

$$
\begin{array}{ccc}
X & \xrightarrow{\zeta I - T} & X \\
{\scriptstyle K}\big\downarrow & & \big\downarrow{\scriptstyle K} \\
\mathbb{C}^{n\times 1} & \xrightarrow[\zeta I - B]{} & \mathbb{C}^{n\times 1}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X & \xrightarrow{\zeta I - U} & X \\
{\scriptstyle K}\big\downarrow & & \big\downarrow{\scriptstyle K} \\
\mathbb{C}^{n\times 1} & \xrightarrow[\zeta I - C]{} & \mathbb{C}^{n\times 1}
\end{array}.
$$

(i) Since $BK = KT$, the first diagram above is commutative. Hence $\zeta u - Bu = v$. Also, $Lu + y = LKx + y = Sx + y = Tx - Ux + y = \zeta x - Ux$.

(ii) Since $CK = KU$, the second diagram above is commutative. Hence $(\zeta I - C)Kx = K(\zeta I - U)x = K(Lu + y) = Au + v = Bu - Cu + v = (\zeta I - C)u$. Since ζ is not an eigenvalue of C, it follows that $Kx = u$. In turn, $\zeta x - Tx = (\zeta x - Ux) - Sx = (\zeta x - Ux) - LKx = Lu + y - Lu = y$. $\square$

Given $\zeta \in \mathbb{C}$, which is not an eigenvalue of C, any $y \in X$ and any $u \in \mathbb{C}^{n\times 1}$, Proposition 2.2 gives a prescription for finding $x \in X$ which satisfies the operator equation $\zeta x - Tx = y$ along with the boundary condition $Kx = u$ as follows. Let $v := Ky$. Check whether u satisfies the finite linear system $\zeta u - Bu = v$. If so, part (ii) says that any $x \in X$ satisfying $\zeta x - Ux = Lu + y$ is a desired solution, and part (i) says that all desired solutions are obtained in this manner. Of course, for this procedure to work, the operator $\zeta I - U$ would have to be surjective.

We now relate the eigenvalue problem for the operator T to the eigenvalue problem for the finite dimensional operator B.

COROLLARY 2.3. *Let $\lambda \in \mathbb{C}$.*

(i) *Suppose $\lambda I - U$ is injective and λ is an eigenvalue of T. Then λ is an eigenvalue of B; if x is an eigenvector of T corresponding to λ, then Kx is an eigenvector of B corresponding to λ. The geometric multiplicity of λ as an eigenvalue of T is less than or equal to the geometric multiplicity of λ as an eigenvalue of B.*

(ii) *Suppose $\lambda I - U$ is surjective and λ is an eigenvalue of B, but not of C. Then λ is an eigenvalue of T; if u is an eigenvector of B corresponding to λ, and $x \in X$ is such that $\lambda x - Ux = Lu$, then x is an eigenvector of T corresponding to λ. The geometric multiplicity of λ as an eigenvalue of T is greater than or equal to the geometric multiplicity of λ as an eigenvalue of B.*

(iii) *Suppose $\lambda I - U$ is bijective and λ is not an eigenvalue of C. Then λ is an eigenvalue of T if and only if λ is an eigenvalue of B. The geometric multiplicity of λ as an eigenvalue of T is equal to the geometric multiplicity of λ as an eigenvalue of B. Let g be this geometric multiplicity, $\{u_1, \dots, u_g\}$ be a basis of the eigenspace $N(\lambda I - B)$ of B, and let $x_i := (\lambda I - U)^{-1} L u_i$ for $i = 1, \dots, g$. Then $\{x_1, \dots, x_g\}$ is a basis of the eigenspace $N(\lambda I - T)$ of T, and $K(x_i) = u_i$ for $i = 1, \dots, g$.*

Proof. Let $\zeta := \lambda$ and $y := 0$ in Proposition 2.2. Then $v := Ky = 0$.

(i) Let nonzero $x \in X$ be such that $Tx = \lambda x$, and let $u := Kx$. Then by part (i) of Proposition 2.2, $Bu = \lambda u$ and $Lu = \lambda x - Ux$. Since λ is not an eigenvalue of U, we see that $Lu \neq 0$, and hence $u \neq 0$, so that u is an eigenvector of B corresponding to λ. This also shows that K maps $N(\lambda I - T)$ to $N(\lambda I - B)$ injectively. Hence the dimension of $N(\lambda I - T)$ is less than or equal to the dimension of $N(\lambda I - B)$.

(ii) Let nonzero $u \in \mathbb{C}^{n\times 1}$ be such that $Bu = \lambda u$. Since $\lambda I - U$ is surjective, there is $x \in X$ such that $\lambda x - Ux = Lu$. Then by part (ii) of Proposition 2.2, $Tx = \lambda x$ and $Kx = u$. Also, $x \neq 0$, since $u \neq 0$, so that x is an eigenvector of T corresponding to λ. This also shows that K maps $N(\lambda I - T)$ to $N(\lambda I - B)$ surjectively. Hence the dimension of $N(\lambda I - T)$ is greater than or equal to the dimension of $N(\lambda I - B)$.

(iii) It follows from (i) and (ii) above that K maps $N(\lambda I - T)$ to $N(\lambda I - B)$ bijectively. Also, $Tx_i = \lambda x_i$ and $Kx_i = u_i$ for $i = 1, \ldots, g$, as in (ii) above. Further, since $\{u_1, \ldots, u_g\}$ is a linearly independent subset of $\mathbb{C}^{n\times 1}$, it follows that $\{x_1, \ldots, x_g\}$ is a linearly independent subset of X. $\square$

Next, we relate the ascent of an eigenvalue λ of T to the ascent of λ as an eigenvalue of B.

PROPOSITION 2.4. *Let $\lambda \in \mathbb{C}$. For $j \in \mathbb{N}$, let $Y_j := N((\lambda I - T)^j)$ and $V_j := N((\lambda I - B)^j)$. Then K maps Y_j to V_j for every $j \in \mathbb{N}$.*

(i) *Suppose $\lambda I - U$ is injective. Then the map K restricted to Y_j is injective for every $j \in \mathbb{N}$.*

(ii) *Suppose $\lambda I - U$ is surjective and λ is not an eigenvalue of C. Then the map K from Y_j to V_j is surjective for every $j \in \mathbb{N}$.*

(iii) *Suppose $\lambda I - U$ is bijective and λ is not an eigenvalue of C. Then the ascent of λ as an eigenvalue of T is equal to the ascent of λ as an eigenvalue of B.*

Proof. Let $j \in \mathbb{N}$ and $x \in Y_j$. Since $BK = KT$, we see that $(\lambda I - B)^j Kx = K(\lambda I - T)^j x = K(0) = 0$. Thus $Kx \in V_j$.

(i) Let $j \in \mathbb{N}$ and $x \in Y_j$ be such that $Kx = 0$. Then $Sx = LKx = L(0) = 0$, and hence $(\lambda I - U)^j x = (\lambda I - T + S)^j x = (\lambda I - T)^j x = 0$. Also, since $(\lambda I - U)^j$ is injective, we see that $x = 0$. Thus the restriction of K to Y_j is injective.

(ii) We prove this part by induction on $j \in \mathbb{N}$. The proof of part (ii) of Corollary 2.3 shows that the result holds for $j = 1$. Assume that the result holds for $j \in \mathbb{N}$. Let $u \in V_{j+1}$, and define $v := \lambda u - Bu$. Then $v \in V_j$. By the inductive assumption, there is $y \in Y_j$ such that $Ky = v$. Since $\lambda I - U$ is surjective, there is $x \in X$ such that $\lambda x - Ux = Lu + y$. By part (ii) of Proposition 2.2, we see that $\lambda x - Tx = y$ and $Kx = u$. Hence $(\lambda I - T)^{j+1} x = (\lambda I - T)^j y = 0$. Thus $x \in Y_{j+1}$ and $Kx = u$, and so the result holds for $j + 1$. Thus the proof by induction is complete.

(iii) It follows from (i) and (ii) above that K maps Y_j to V_j bijectively, so that $\dim Y_j = \dim V_j < \infty$, and hence $Y_{j+1} = Y_j$ if and only if $V_{j+1} = V_j$ for any $j \in \mathbb{N}$. Consequently, the smallest $\ell \in \mathbb{N}$ for which $V_{\ell+1} = V_\ell$ is also the smallest $\ell \in \mathbb{N}$ for which $Y_{\ell+1} = Y_\ell$. Thus the ascent of λ as an eigenvalue of T is equal to the ascent of λ as an eigenvalue of B. $\square$

3. Spectral Subspace problem for $T = S + U$

In this section, we shall relate the spectrum of T to the spectrum of B. For the development of spectral theory, it is convenient to use the framework of a Banach algebra with identity. We consider a complete norm $\|\cdot\|$ on the linear space X over $\mathbb{C}$, and $BL(X)$ denote the Banach algebra of all bounded linear maps on X. We assume that the finite-rank operator S and the operator U are in $BL(X)$. We can (and we shall) assume that the functionals $f_1,\ldots,f_n$ on X appearing in the presentation of S are continuous, that is, they belong to the normed dual X' of X. We specify a norm $\|\cdot\|_0$ on $\mathbb{C}^{n\times 1}$, and let $K' : (\mathbb{C}^{n\times 1})' \to X'$ denote the transpose of the continuous linear map $K : X \to \mathbb{C}^{n\times 1}$. The invariance condition on the map $U : X \to X$ says that the range of K' is invariant under the transpose $U' : X' \to X'$ of U. Let us denote the spectrum $\{\lambda \in \mathbb{C} : \lambda I - T$ is not invertible in $BL(X)\}$ of $T \in BL(X)$ by $\mathrm{sp}(T)$.

PROPOSITION 3.1. *Suppose $\lambda \in \mathbb{C}$ is not an eigenvalue of C, and $\lambda \notin \mathrm{sp}(U)$. Then $\lambda \in \mathrm{sp}(T)$ if and only if $\lambda \in \mathrm{sp}(B)$.*

Proof. Suppose $\lambda \in \mathrm{sp}(B)$. Since λ is not an eigenvalue of C, and $\lambda I - U$ is surjective, part (ii) of Corollary 2.3 shows that λ is an eigenvalue of T, and hence a spectral value of T. Conversely, suppose $\lambda \notin \mathrm{sp}(B)$. We show that $\lambda \notin \mathrm{sp}(T)$. Since $\lambda I - U$ is injective, part (i) of Corollary 2.3 shows that $\lambda I - T$ is injective. To show that $\lambda I - T$ is also surjective, we argue as follows. Let $y \in X$ and define $v := Ky$. Since $\lambda I - B$ is surjective, there is $u \in \mathbb{C}^{n\times 1}$ such that $\lambda u - Bu = v$. Also, since $\lambda I - U$ is surjective, there is $x \in X$ such that $\lambda x - Ux = Lu + y$. Part (ii) of Proposition 2.2 shows that $\lambda x - Tx = y$. This completes the proof. $\square$

PROPOSITION 3.2. *Let $E := \mathrm{sp}(C) \cup \mathrm{sp}(U)$ and $\Lambda \subseteq \mathrm{sp}(T) \setminus E$. Then Λ is a finite set; each $\lambda \in \Lambda$ is an eigenvalue of T and it is an isolated point of $\mathrm{sp}(T)$. Further, K maps the spectral subspace $M(T,\Lambda)$ associated with T and Λ into the spectral subspace $M(B,\Lambda)$ associated with B and Λ injectively.*

Proof. By Proposition 3.1, $\Lambda \subseteq \mathrm{sp}(T) \setminus E = \mathrm{sp}(B) \setminus E \subseteq \mathrm{sp}(B)$. Hence Λ is a finite set. Also, $\mathrm{sp}(T) \setminus \Lambda$ is a closed set since it is the union of the closed sets $\mathrm{sp}(T) \cap E$ and $(\mathrm{sp}(T) \setminus E) \setminus \Lambda$. This shows that each $\lambda \in \Lambda$ is an isolated point of $\mathrm{sp}(T)$. The proof of Proposition 3.1 shows that each $\lambda \in \Lambda$ is an eigenvalue of T. Let Γ denote a Cauchy contour whose interior contains Λ, whose exterior contains $(\mathrm{sp}(T) \setminus \Lambda) \cup E$, and each point of which is in the resolvent sets of T, B and U. (See, for example, Corollary 1.22 of [1].) The spectral projections associated with T and Λ, and with B and Λ are given respectively by

$$P := \frac{1}{2\pi i} \int_\Gamma (\zeta I - T)^{-1} d\zeta \quad \text{and} \quad Q := \frac{1}{2\pi i} \int_\Gamma (\zeta I - B)^{-1} d\zeta.$$

Since $BK = KT$, we see that

$$QK = \frac{1}{2\pi i} \int_\Gamma (\zeta I - B)^{-1} K d\zeta = \frac{1}{2\pi i} \int_\Gamma K(\zeta I - T)^{-1} d\zeta = KP.$$

Hence K maps the range of P into the range of Q, that is, K maps $M(T,\Lambda)$ into $M(B,\Lambda)$. Finally, we show that this map is injective. For ζ on Γ and $y \in X$, let $x :=$

$(\zeta I - T)^{-1}y$, so that $\zeta x - Tx = y$. Also, let $u := Kx$. Then by part (i) of Proposition 2.2, $\zeta u - Bu = Ky$ and $\zeta x - Ux = Lu + y$, and so

$$(\zeta I - T)^{-1}y = x = (\zeta I - U)^{-1}(Lu + y) = (\zeta I - U)^{-1}\left(L(\zeta I - B)^{-1}Ky + y\right).$$

Thus $(\zeta I - T)^{-1} = (\zeta I - U)^{-1}\left(L(\zeta I - B)^{-1}K + I\right)$. Hence

$$P = \left(\frac{1}{2\pi i}\int_\Gamma (\zeta I - U)^{-1}L(\zeta I - B)^{-1}d\zeta\right)K + \frac{1}{2\pi i}\int_\Gamma (\zeta I - U)^{-1}d\zeta.$$

But since no spectral values of U is inside Γ, the second term on the right is zero. Now if x is in the range of P and $Kx = 0$, then

$$x = Px = \left(\frac{1}{2\pi i}\int_\Gamma (\zeta I - U)^{-1}L(\zeta I - B)^{-1}d\zeta\right)(0) = 0.$$

This completes the proof. $\square$

The above result shows that if $\Lambda \subseteq \mathrm{sp}(T) \setminus (\mathrm{sp}(C) \cup \mathrm{sp}(U))$, then the dimension of the spectral subspace $M(T, \Lambda)$ associated with T and Λ is less than or equal to the dimension of the spectral subspace $M(B, \Lambda)$ associated with B and Λ. In particular, if $\lambda \in \mathrm{sp}(T)$, but $\lambda \notin \mathrm{sp}(C) \cup \mathrm{sp}(U)$, then the algebraic multiplicity of λ as an eigenvalue of T is less than or equal to the algebraic multiplicity of λ as an eigenvalue of B. In order to prove the equality of the dimensions mentioned above, we need to generalize Proposition 2.2 to product spaces. This will also yield a method for constructing an ordered basis of $M(T, \Lambda)$ starting with an ordered basis of $M(B, \Lambda)$, just as part (ii) of Corollary 2.3 yields a method for constructing an eigenvector of T corresponding to an eigenvalue λ starting with an eigenvector of B.

Let $m \in \mathbb{N}$. Given a linear map F from a linear space X to a linear space Y, the linear map $\underline{F}$ from the Cartesian product $X^{1\times m} := \{[x_1, \ldots, x_m] : x_j \in X \text{ for } 1 \leqslant j \leqslant m\}$ to $Y^{1\times m}$ is defined by $\underline{F}\,\underline{x} := [F(x_1), \ldots, F(x_m)]$ for $\underline{x} := [x_1, \ldots, x_m] \in X^{1\times m}$. Also, for $\underline{x} := [x_1, \ldots, x_m] \in X^{1\times m}$ and $\mathsf{Z} := [\zeta_{i,j}] \in \mathbb{C}^{m\times m}$, let $\underline{x}\,\mathsf{Z} := \left[\sum_{i=1}^m \zeta_{i,1}x_i, \ldots, \sum_{i=1}^m \zeta_{i,m}x_i\right]$. It is easy to see that $\underline{F}(\underline{x}\,\mathsf{Z}) = (\underline{F}\,\underline{x})\,\mathsf{Z}$ for all $\underline{x} \in X^{1\times m}$ and $\mathsf{Z} \in \mathbb{C}^{m\times m}$.

PROPOSITION 3.3. *Let $m \in \mathbb{N}$, $\mathsf{Z} \in \mathbb{C}^{m\times m}$, $\underline{y} \in X^{1\times m}$, and define $\underline{v} := \underline{K}\,\underline{y}$.*

(i) *Let $\underline{x} \in X^{1\times m}$ satisfy $\underline{x}\,\mathsf{Z} - \underline{T}\,\underline{x} = \underline{y}$ and define $\underline{u} := \underline{K}\,\underline{x}$. Then $\underline{u}\,\mathsf{Z} - \underline{B}\,\underline{u} = \underline{v}$ and $\underline{x}\,\mathsf{Z} - \underline{U}\,\underline{x} = \underline{L}\,\underline{u} + \underline{y}$.*

(ii) *Suppose Z and C have no common eigenvalues. Let $\underline{u} \in \mathbb{C}^{n\times m}$ satisfy $\underline{u}\,\mathsf{Z} - \underline{B}\,\underline{u} = \underline{v}$ and let $\underline{x} \in X^{1\times m}$ satisfy $\underline{x}\,\mathsf{Z} - \underline{U}\,\underline{x} = \underline{L}\,\underline{u} + \underline{y}$. Then $\underline{x}\,\mathsf{Z} - \underline{T}\,\underline{x} = \underline{y}$ and $\underline{K}\,\underline{x} = \underline{u}$.*

Proof. The proof of Proposition 2.2 carries over in a straightforward manner except for the following point. In part (ii), we obtain $(\underline{K}\,\underline{x})\mathsf{Z} - \underline{C}(\underline{K}\,\underline{x}) = \underline{u}\,\mathsf{Z} - \underline{C}\,\underline{u}$. Then $\underline{K}\,\underline{x} = \underline{u}$, since the map $\underline{\alpha} \longmapsto \underline{\alpha}\,\mathsf{Z} - \underline{C}\,\underline{\alpha}$ from $\mathbb{C}^{n\times 1}$ to $\mathbb{C}^{n\times 1}$ is injective. We give here a simple proof of the injectivity of this map instead of referring to a general result. Let $\zeta_1, \ldots, \zeta_m$ denote the (possibly repeated) eigenvalues of Z, and let $\mathsf{Q} \in \mathbb{C}^{m\times m}$ be a unitary matrix such that $\mathsf{W} := \mathsf{Q}^*\mathsf{Z}\mathsf{Q}$ is upper-triangular with $\mathrm{diag}\,\mathsf{W} = (\zeta_1, \ldots, \zeta_m)$.

Let $\underline{\alpha} \in \mathbb{C}^{n\times m}$ be such that $\underline{\alpha}\mathsf{Z} - \underline{C}\,\underline{\alpha} = \underline{0}$. Then $\underline{\alpha}\mathsf{QW} - \underline{C}\,\underline{\alpha}\mathsf{Q} = \underline{0}$. Let $\underline{\beta} := \underline{\alpha}\mathsf{Q}$. If $\underline{\beta} := [\beta_1,\dots,\beta_m]$ and $\mathsf{W} := [w_{i,j}]$, then $(\zeta_1 I - C)\beta_1 = 0$, $w_{1,2}\beta_1 + (\zeta_2 I - C)\beta_2 = 0,\dots,\sum_{i=1}^{m-1} w_{i,m}\beta_i + (\zeta_m I - C)\beta_m = 0$. Since none of $\zeta_1,\dots,\zeta_m$ is an eigenvalue of C, we see that $\beta_1 = 0, \beta_2 = 0,\dots,\beta_m = 0$, that is, $\underline{\beta} = \underline{0}$, and so $\underline{\alpha} = \underline{0}$, establishing the injectivity. $\square$

PROPOSITION 3.4. *Let $E := \mathrm{sp}(C)\cup\mathrm{sp}(U)$ and $\Lambda \subseteq \mathrm{sp}(T)\setminus E$. Then $M(T,\Lambda)$ and $M(B,\Lambda)$ have the same dimension. Let this dimension be m, and let $\underline{u} \in \mathbb{C}^{n\times m}$ form an ordered basis of $M(B,\Lambda)$. If Θ in $\mathbb{C}^{m\times m}$ satisfies $\underline{B}\,\underline{u} = \underline{u}\Theta$, and if $\underline{x}$ is the unique solution of the Sylvester equation $\underline{x}\Theta - \underline{U}\,\underline{x} = \underline{L}\,\underline{u}$, then $\underline{x}$ forms an ordered basis of $M(T,\Lambda)$, and $\underline{K}\underline{x} = \underline{u}$.*

Proof. Let $\underline{u} \in \mathbb{C}^{n\times m}$ form an ordered basis of $M(B,\Lambda)$. Since $M(B,\Lambda)$ is an invariant subspace for B, there is $\Theta \in \mathbb{C}^{m\times m}$ such that $\underline{B}\,\underline{u} = \underline{u}\Theta$ and $\mathrm{sp}(\Theta) = \Lambda$. Since $\mathrm{sp}(\Theta)\cap\mathrm{sp}(U) = \emptyset$, there is unique $\underline{x} \in X^{1\times m}$ such that $\underline{x}\Theta - \underline{U}\,\underline{x} = \underline{L}\,\underline{u}$. (See Proposition 1.50 of [1].) Again, since Θ and C have no common eigenvalues, part (ii) of Proposition 3.3 (with $\underline{y} = \underline{0}$ and $\mathsf{Z} = \Theta$) shows that $\underline{x}\Theta - \underline{T}\,\underline{x} = \underline{0}$ and $\underline{K}\,\underline{x} = \underline{u}$. Let $\underline{x} := [x_1,\dots,x_m]$. We first show that the set $\{x_1,\dots,x_m\}$ is linearly independent in X. Let $\mathsf{c} \in \mathbb{C}^{m\times 1}$ be such that $\underline{x}\mathsf{c} = 0$. Then $\underline{u}\mathsf{c} = (\underline{K}\,\underline{x})\mathsf{c} = K(\underline{x}\mathsf{c}) = K(0) = 0$. But since $\underline{u} := [u_1,\dots,u_m] \in \mathbb{C}^{m\times 1}$ forms a basis of $M(B,\Lambda)$, we must have $\mathsf{c} = 0$, showing that the set $\{x_1,\dots,x_m\}$ is linearly independent in X. Let $Y := \mathrm{span}\{x_1,\dots,x_m\}$. The m-dimensional subspace Y is closed in X. Since $\underline{T}\,\underline{x} = \underline{x}\Theta$, the matrix Θ represents the operator $T_{|Y} : Y \to Y$ and $\mathrm{sp}(T_{|Y}) = \mathrm{sp}(\Theta) = \Lambda$. Hence Y is contained in $M(T,\Lambda)$. (See Proposition 1.28 of [1].) Thus we see that the dimension of $M(T,\Lambda)$ is greater than or equal to the dimension m of $M(B,\Lambda)$. On the other hand, Proposition 3.2 shows that dimension of $M(T,\Lambda)$ is less than or equal to the dimension of $M(B,\Lambda)$. Thus the two dimensions are equal. $\square$

If $\lambda \in \mathrm{sp}(T)\setminus\big(\mathrm{sp}(C)\cup\mathrm{sp}(U)\big)$, then the above result shows that the algebraic multiplicity of λ as an eigenvalue of T is equal to the algebraic multiplicity of λ as an eigenvalue of B. Since any subset of $\mathrm{sp}(T)$ which does not intersect $\mathrm{sp}(C)\cup\mathrm{sp}(U)$ consists of a finite number of such λ's, we may as well have found an ordered basis for the spectral subspace associated with T and each such λ, and considered their union to obtain an ordered basis for $M(T,\Lambda)$. However, since some of these λ's may be close each other, the construction of an ordered basis for each such λ can be computationally unstable. For this reason, we have considered Λ as a cluster of spectral values of T. If $\Theta \in \mathbb{C}^{m\times m}$ and $\mathrm{sp}(\Theta)\cap\mathrm{sp}(U) = \emptyset$, then for every $\underline{y} \in X^{1\times m}$, there is unique $\underline{x} \in X^{1\times m}$ such that $\underline{x}\Theta - \underline{U}\,\underline{x} = \underline{y}$. If we let $\underline{x} := R(\underline{U},\Theta)\,\underline{y}$, then the operator $R(\underline{U},\Theta) \in BL(X^{1\times m})$ is known as the block resolvent of $\underline{U}$ at Θ.

4. A Bound on the condition number of $\zeta I - B$

Proposition 2.2 reduces the problem of solving the operator equation $\zeta x - Tx = y$ to solving the finite linear system $\zeta u - Bu = v$ in $\mathbb{C}^{n\times 1}$. Let us assume that $\zeta I - T$

and $\zeta I - B$ are invertible in $BL(X)$ and $BL(\mathbb{C}^{n\times 1})$ respectively. We now consider the stability of the solution of this linear system vis-a-vis the stability of the given operator equation in X.

In this section we shall assume that the functionals $f_1,\ldots,f_n$ appearing in the presentation of the finite-rank operator S are linearly independent. Then there are $e_1,\ldots,e_n$ in X such that $f_i(e_j) = \delta_{i,j}$ for $i,j = 1,\ldots,n$. Consequently, the continuous linear map $K : X \to \mathbb{C}^{n\times 1}$ is surjective.

Let λ be an eigenvalue of C. Then it is an eigenvalue of C' as well. Since $CK = KU$, we see that $U'K' = K'C'$. Since $f_1,\ldots,f_n$ are linearly independent, it follows that λ is an eigenvalue of U'. But $\mathrm{sp}(U') = \mathrm{sp}(U)$. Thus $\mathrm{sp}(C) \subseteq \mathrm{sp}(U)$, and the condition '$\lambda$ is not an eigenvalue of C' stated in the last sentences of Corollary 2.3 and Proposition 2.4 as well as in Proposition 3.1 is superfluous. Also, the set $E = \mathrm{sp}(C) \cup \mathrm{sp}(U)$ introduced in Propositions 3.2 and 3.4 reduces to $\mathrm{sp}(U)$.

Let $\|\cdot\|_0$ be a specified norm on $\mathbb{C}^{n\times 1}$, and let $\|\cdot\|_0'$ denote the corresponding dual norm on $\mathbb{C}^{n\times 1}$. Then the normed dual of $(\mathbb{C}^{n\times 1}, \|\cdot\|_0)$ can be identified with $(\mathbb{C}^{n\times 1}, \|\cdot\|_0')$. Note that the range of K' is equal to $\mathrm{span}\{f_1,\ldots,f_n\}$. The map $K' : (\mathbb{C}^{n\times 1})' \to \mathrm{span}\{f_1,\ldots,f_n\}$ can be identified with the map $\sigma : (\mathbb{C}^{n\times 1}, \|\cdot\|_0') \to \mathrm{span}\{f_1,\ldots,f_n\}$ given by $\sigma(v) := v(1)f_1 + \cdots + v(n)f_n$ for $v := [v(1),\ldots,v(n)]^t \in \mathbb{C}^{n\times 1}$. Then the map σ is linear, continuous and bijective, and so is $\sigma^{-1} : \mathrm{span}\{f_1,\ldots,f_n\} \to (\mathbb{C}^{n\times 1}, \|\cdot\|_0')$. If $\|\sigma\|$ and $\|\sigma^{-1}\|$ denote their norms, then $\mathrm{cond}(\sigma) := \|\sigma\|\,\|\sigma^{-1}\|$ is known as the condition number of σ. Note that $\mathrm{cond}(\sigma) \geqslant 1$ always.

Let $\zeta \in \mathbb{C}$. Since $BK = KT$, we see that $(\zeta I - B)K = K(\zeta I - T)$, and so $K'(\zeta I - B') = (\zeta I - T')K'$. The last equality can be written as $\sigma(\zeta I - B') = (\zeta I - T')\sigma$, that is, $\zeta I - B' = \sigma^{-1}(\zeta I - T')\sigma$. Hence

$$\|\zeta I - B\| = \|\zeta I - B'\| \leqslant \mathrm{cond}(\sigma)\|\zeta I - T'\| = \mathrm{cond}(\sigma)\|\zeta I - T\|.$$

Let ζ be in the resolvent sets of B and T. Then $\zeta I - B'$ and $\zeta I - T'$ are invertible and $(\zeta I - B')^{-1} = \sigma^{-1}(\zeta I - T')^{-1}\sigma$. As before,

$$\|(\zeta I - B)^{-1}\| \leqslant \mathrm{cond}(\sigma)\|(\zeta I - T)^{-1}\|.$$

Thus

$$\begin{aligned}\mathrm{cond}(\zeta I - B) &= \|(\zeta I - B)\|\,\|(\zeta I - B)^{-1}\| \\ &\leqslant [\mathrm{cond}(\sigma)]^2\|(\zeta I - T)\|\,\|(\zeta I - T)^{-1}\| \\ &= [\mathrm{cond}(\sigma)]^2\mathrm{cond}(\zeta I - T).\end{aligned}$$

The relative error in the solution of the operator equation $\zeta x - Tx = y$ is bounded by $\mathrm{cond}(\zeta I - T)$, and the relative error in the solution of the linear system $\zeta u - Bu = v$ is bounded by $\mathrm{cond}(\zeta I - B)$. We have shown that the latter bound is at most $[\mathrm{cond}(\sigma)]^2$ times the former bound. Thus if $\mathrm{cond}(\sigma)$ is not too big, then the reduction procedure involved in passing from the operator equation $\zeta x - Tx = y$ to the linear system $\zeta u - Bu = v$ is fairly stable. Of course, the best possible situation occurs $\mathrm{cond}(\sigma) = 1$. We now give three examples (which were considered in Section 1 without any mention of norms) wherein this holds.

EXAMPLES 4.1. (i) Let J be a compact Hausdorff topological space. Consider the sup norm $\|\cdot\|_\infty$ on $X := C(J)$, the space of all complex-valued continuous functions defined on J. Given distinct points $t_1,\ldots,t_n$ in J, let $f_i(x) := x(t_i)$ for $i = 1,\ldots,n$ and $x \in X$. By Urysohn's lemma, there are $e_1,\ldots,e_n$ in X such that $f_i(e_j) = e_j(t_i) = \delta_{i,j}$ for $i,j = 1,\ldots,n$. Hence the set $\{f_1,\ldots,f_n\}$ is linearly independent in X'. Also, it is clear that $\|f_i\| = 1$ for each $i = 1,\ldots,n$. If the norm $\|\cdot\|_0$ on $\mathbb{C}^{n\times 1}$ is taken to be the norm $\|\cdot\|_\infty$ also, then the dual norm $\|\cdot\|_0'$ on $\mathbb{C}^{n\times 1}$ is the norm $\|\cdot\|_1$. For $v := [v(1),\ldots,v(n)]^t \in \mathbb{C}^{n\times 1}$ and $x \in X$, we have $|\sigma(v)(x)| = |v(1)x(t_1) + \cdots + v(n)x(t_n)| \leqslant \|x\|_\infty(|v(1)| + \cdots + |v(n)| = \|x\|_\infty\|v\|_1$, so that $\|\sigma(v)\| \leqslant \|v\|_1$. On the other hand, given $v \in \mathbb{C}^{n\times 1}$, we can find $x \in X$ such that $x(t_i) = \operatorname{sgn} v(i)$ for each $i = 1,\ldots,n$ and $\|x\|_\infty \leqslant 1$ by Tietze's extension theorem, and then $\sigma(v)(x) = |v(1)| + \cdots + |v(n)| = \|v\|_1$. This shows that $\|\sigma(v)\| = \|v\|_1$ for each $v \in \mathbb{C}^{n\times 1}$. Thus σ is an isometry, and so $\operatorname{cond}(\sigma) = 1$.

(ii) Let $J := [a,b]$, an interval in $\mathbb{R}$. Consider the essential-sup norm $\|\cdot\|_\infty$ on $X := L^\infty([a,b])$, the space of all equivalence classes of complex-valued bounded Lebesgue measurable functions defined on J. Given disjoint subintervals $E_1,\ldots,E_n$ of positive lengths in J, let $f_i(x) := \frac{1}{m(E_i)}\int_{E_i} x\,dm$ for $i = 1,\ldots,n$ and $x \in X$. Let χ_i denote the characteristic function of the set E_i for $i = 1,\ldots,n$. Then $f_i(\chi_j) = \delta_{i,j}$ for $i,j = 1,\ldots,n$. Proceeding exactly as in (i) above, we see that $\|\sigma(v)\| \leqslant \|v\|_1$. Also, given $v \in \mathbb{C}^{n\times 1}$, let $x := \operatorname{sgn} v(1)\chi_1 + \cdots + \operatorname{sgn} v(n)\chi_n$. Then $\|x\|_\infty \leqslant 1$ and $\sigma(v)(x) = \|v\|_1$. This shows that $\|\sigma(v)\| = \|v\|_1$ for each $v \in \mathbb{C}^{n\times 1}$. Thus σ is an isometry, and so $\operatorname{cond}(\sigma) = 1$.

(iii) Let $1 \leqslant p < \infty$ and $X := \ell^p$, the linear space of all p-summable complex sequences. Let $f_i(x) := x(i)$ for $i = 1,\ldots,n$ and $x := (x(1),x(2),\ldots)$ in X. Clearly, the set $\{f_1,\ldots,f_n\}$ is linearly independent in X' and $\|f_i\| = 1$ for each $i = 1,\ldots,n$. If the norm $\|\cdot\|_0$ on $\mathbb{C}^{n\times 1}$ is taken to be the norm $\|\cdot\|_p$ also, then the dual norm $\|\cdot\|_0'$ on $\mathbb{C}^{n\times 1}$ is the norm $\|\cdot\|_q$, where $(1/p) + (1/q) = 1$. It is routine to verify that $\|\sigma(v)\| = \|v\|_q$ for each $v \in \mathbb{C}^{n\times 1}$. Thus σ is an isometry, and so $\operatorname{cond}(\sigma) = 1$.

5. The case $T = S + V$

As in Section 2, we consider a finite-rank linear operator S on a linear space X over $\mathbb{C}$ presented as follows: $Sx = \sum_{j=1}^n f_j(x)x_j$ for all $x \in X$, where $x_1,\ldots,x_n$ are in X and $f_1,\ldots,f_n$ are linear functionals on X. Consider a linear map $V : X \to X$ such that $Vx_j \in \operatorname{span}\{x_1,\ldots,x_n\}$ for each $j = 1,\ldots,n$. Then there are complex numbers $d_{i,j}$ such that $Vx_j = \sum_{i=1}^n d_{i,j}x_i$ for each $j = 1,\ldots,n$. Define a linear map $D : \mathbb{C}^{n\times 1} \to \mathbb{C}^{n\times 1}$ by $Du := \left[\sum_{j=1}^n d_{1,j}u(j),\ldots,\sum_{j=1}^n d_{n,j}u(j)\right]^t$ for $u := [u(1),\ldots,u(n)]^t \in \mathbb{C}^{n\times 1}$. Recalling the maps $K : X \to \mathbb{C}^{n\times 1}$, $L : \mathbb{C}^{n\times 1} \to X$ and $A : \mathbb{C}^{n\times 1} \to \mathbb{C}^{n\times 1}$ of Section 2, we see that $S = LK$, $A = KL$, $LA = SL$ and $LD = VL$. Let

$$T := S + V \quad \text{and} \quad B := A + D.$$

It follows that $LB = TL$. As before, we wish to reduce problems involving the linear map T to problems involving the finite dimensional linear map B. The analogue of

Proposition 2.2 is as follows.

PROPOSITION 5.1. *Let* $\zeta \in \mathbb{C}$, $v \in \mathbb{C}^{n\times 1}$, *and define* $y := Lv$.

(i) *Let* $u \in \mathbb{C}^{n\times 1}$ *satisfy* $\zeta u - Bu = v$ *and define* $x := Lu$. *Then* $\zeta x - Tx = y$ *and* $Kx = \zeta u - Du - v$.

(ii) *Suppose* $\zeta I - V$ *is injective. Let* $x \in X$ *satisfy* $\zeta x - Tx = y$ *and let* $u \in \mathbb{C}^{n\times 1}$ *satisfy* $Kx = \zeta u - Du - v$. *Then* $\zeta u - Bu = v$ *and* $Lu = x$.

Proof.

(i) Since $L(\zeta I - B) = (\zeta I - T)L$, we have $\zeta x - Tx = y$. Also, $Kx = KLu = Au = (B - D)u = \zeta u - Du - v$.

(ii) Since $LD = VL$, we have $(\zeta I - V)Lu = L(\zeta I - D)u = L(Kx + v) = Sx + y = Tx - Vx + y = (\zeta I - V)x$. Since $\zeta I - V$ is injective, it follows that $Lu = x$. In turn, $\zeta u - Bu = (\zeta u - Du) - Au = (\zeta u - Du) - KLu = (\zeta u - Du) - Kx = v$. □

We remark that, unlike Proposition 2.2, the above result is not suitable for reducing the operator equation $\zeta x - Tx = y$ in X to a linear system $\zeta u - Bu = v$ in $\mathbb{C}^{n\times 1}$. This is because of two reasons. First, the right side y of the operator equation in X cannot be an arbitrary element of X; it must be of the form Lv for some $v \in \mathbb{C}^{n\times 1}$. This is too restrictive. Secondly, the boundary condition to be satisfied by the solution x of the operator equation, namely $Kx = \zeta u - Du - v$, is rather involved and also unnatural. For these reasons, we do not dwell any further on reducing a general operator equation in X to a linear system in $\mathbb{C}^{n\times 1}$. However, if we let $v = 0$ in Proposition 5.1, then $y = Lv = 0$, and the eigenvalue problem for T can very well be reduced to the eigenvalue problem for B in the same manner as before.

COROLLARY 5.2. *Suppose* $\lambda \in \mathbb{C}$ *is not an eigenvalue of* D.

(i) *If* λ *is an eigenvalue of* B, *then* λ *is an eigenvalue of* T; *if* u *is an eigenvector of* B *corresponding to* λ, *then* Lu *is an eigenvector of* T *corresponding to* λ. *The geometric multiplicity of* λ *as an eigenvalue of* B *is less than or equal to the geometric multiplicity of* λ *as an eigenvalue of* T.

(ii) *Suppose* $\lambda I - V$ *is injective. Then* λ *is an eigenvalue of* T *if and only if* λ *is an eigenvalue of* B; *if* x *is an eigenvector of* T *corresponding to* λ, *then* $u := (\lambda I - D)^{-1}Kx$ *is an eigenvector of* B *corresponding to* λ. *The geometric multiplicity of* λ *as an eigenvalue of* T *is equal to the geometric multiplicity of* λ *as an eigenvalue of* B.

Proof. The proof is similar to the proof of Corollary 2.3. We merely mention that L maps the the eigenspace $N(\lambda I - B)$ to the eigenspace $N(\lambda I - T)$ injectively in (i), and bijectively in (ii). □

Here is an analogue of Proposition 2.4 for the ascent of an eigenvalue.

PROPOSITION 5.3. *Let* $\lambda \in \mathbb{C}$. *For* $j \in \mathbb{N}$, *let* $Y_j := N((\lambda I - T)^j)$ *and* $V_j := N((\lambda I - B)^j)$. *Then* L *maps* V_j *into* Y_j *for every* $j \in \mathbb{N}$. *Suppose* λ *is not an eigenvalue of* D.

(i) *The map* L *restricted to* V_j *is injective for every* $j \in \mathbb{N}$.

(ii) *Suppose* $\lambda I - V$ *is injective. Then the map* L *from* V_j *to* Y_j *is bijective for every* $j \in \mathbb{N}$. *Consequently, the ascent of* λ *as an eigenvalue of* T *is equal to the ascent of* λ *as an eigenvalue of* B.

Proof. The proof is similar to the proof of Proposition 2.4. $\square$

In order to relate $\mathrm{sp}(T)$ with $\mathrm{sp}(B)$, let us assume, as in Section 3, that X is a Banach space, the operators S and V are in $BL(X)$ and $f_i \in X'$ for $i = 1, \ldots, n$.

PROPOSITION 5.4. *Suppose* $\lambda \in \mathbb{C}$ *is not an eigenvalue of* D, *and* $\lambda \notin \mathrm{sp}(V)$. *Then* $\lambda \in \mathrm{sp}(T)$ *if and only if* $\lambda \in \mathrm{sp}(B)$.

Let $E \subseteq \mathrm{sp}(D) \cup \mathrm{sp}(V)$ *and* $\Lambda \subseteq \mathrm{sp}(T) \setminus E$. *Then* Λ *is a finite set, each* $\lambda \in \Lambda$ *is an eigenvalue of* T *and it is an isolated point of* $\mathrm{sp}(T)$. *Further, the dimension of the spectral subspace associated with* T *and* Λ *is equal to the dimension of the spectral subspace associated with* B *and* Λ.

Proof. Since $Sx = f_1(x)x_1 + \cdots + f_n(x)x_n$ for $x \in X$, the map $S' : X' \to X'$ is given by

$$S'(f) = f(x_1)f_1 + \cdots + f(x_n)f_n \quad \text{for } f \in X'.$$

Let $F_i \in X''$ denote the canonical embedding of the element $x_i \in X$ for $i = 1, \ldots, n$. Then S' is presented as follows: $S'f = F_1(f)f_1 + \cdots + F_n(f)f_n$ for $f \in X'$. Thus S' is a finite-rank operator on X' and $S' = (LK)' = K'L'$. Also, $A' = (KL)' = L'K'$, and so $A'L' = L'S'$. Now $T' = S' + V'$, $B' = A' + D'$, and since $D'L' = (LD)' = (VL)' = L'V'$, we see that $B'L' = L'T'$. Our hypothesis says that λ is not an eigenvalue of D' and $\lambda \notin \mathrm{sp}(V')$. While the map K' is from $(\mathbb{C}^{n\times 1})'$ to X', and the map L' is from X' to $(\mathbb{C}^{n\times 1})'$, we may identify $(\mathbb{C}^{n\times 1})'$ with $\mathbb{C}^{n\times 1}$ in a natural way. Replacing X, U, T, C and B in Proposition 3.1 by X', V', T', D' and B' respectively, we see that $\lambda \in \mathrm{sp}(T')$ if and only if $\lambda \in \mathrm{sp}(B')$, that is, $\lambda \in \mathrm{sp}(T)$ if and only if $\lambda \in \mathrm{sp}(B)$.

Since $\Lambda \subseteq \mathrm{sp}(T) \setminus E$, we see that Λ is a finite set, $\mathrm{sp}(T) \setminus \Lambda$ is a closed set, and so each $\lambda \in \Lambda$ is an isolated point of $\mathrm{sp}(T)$ exactly as in the proof of Proposition 3.2. Let P and Q denote the spectral projections associated with T and Λ, and with B and Λ respectively. Then P' and Q' are the spectral projections associated with T' and Λ, and with B' and Λ respectively. We may replace X, $\mathrm{sp}(C) \cup \mathrm{sp}(U)$, T and B in Proposition 3.4 by X', $\mathrm{sp}(D) \cup \mathrm{sp}(V)$, T' and B' respectively, and obtain $\operatorname{rank} P' = \operatorname{rank} Q'$, that is, $\operatorname{rank} P = \operatorname{rank} Q$, as desired. In particular, each $\lambda \in \Lambda$ is an isolated spectral value of T of finite algebraic multiplicity, and hence it is an eigenvalue of T [1, Proposition 1.31]. $\square$

It follows from the above proposition that if $\lambda \in \mathrm{sp}(T)$, but $\lambda \notin \mathrm{sp}(D) \cup \mathrm{sp}(V)$, then the algebraic multiplicity of λ as an eigenvalue of T is equal to the algebraic multiplicity of λ as an eigenvalue of B. We remark that if the dimension of the spectral subspace associated with B and Λ is m, and if $\underline{u} \in \mathbb{C}^{n\times m}$ forms an ordered basis, then $\underline{x} := \underline{L}\underline{u}$ forms an ordered basis of the spectral subspace associated with T and Λ. However, we refrain from giving details of the proof.

Examples of elements $x_1,\dots,x_n$ in a linear space X and of linear maps V on X such that $Vx_j \in \operatorname{span}\{x_1,\dots,x_n\}$ for each $j=1,\dots,n$ can be obtained by considering the 'dual' situations given in Examples 2.1. For instance, we may consider $x_j := (0,\dots,0,1,0,\dots,0)$, where 1 occurs in the jth place, for $j=1,\dots,n$, and an upper-triangular infinite matrix $[v_{i,j}]$ instead of a lower-triangular infinite matrix $[u_{i,j}]$ in Example 2.1 (ii).

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Balmohan V. Limaye
Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai 400076
India
e-mail: bvl@math.iitb.ac.in