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BANACH ALGEBRAS OF OPERATOR SEQUENCES

M. SEIDEL AND B. SILBERMANN

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Abstract. During the last decades it turned out to be fruitful to apply certain Banach algebra techniques in the theory of approximation of operators by matrix sequences. Here we discuss the case of operator sequences (acting on infinite dimensional Banach spaces and which do not necessarily converge strongly) and we derive analogous results concerning the stability and Fredholm properties of such sequences. For this, the notions of $\mathscr{P}$-Fredholmness and $\mathscr{P}$-strong convergence play an important role and are extensively studied. As an application we consider the finite sections of band-dominated operators on l^p-spaces, including the cases $p \in \{1, \infty\}$.

Introduction

The finite section method for the approximate solution of infinite dimensional operator equations is very well studied for many classes of operators. Roughly speaking, one replaces the "big" equation $Ax = b$ on a Banach space $\mathbf{X}$ with the unknown x by its truncations $P_nAP_nx_n = P_nb$, where (P_n) is a sequence of projections onto certain subspaces of $\mathbf{X}$ and one hopes that these substitutes are solvable and their solutions x_n converge in some sense to the solution x of the big equation. It is well known that the stability of the operator sequence (P_nAP_n) plays a crucial role for this convergence.

Undeniably, the most popular class of operators which appeared in this topic is the set of Toplitz operators. Actually, they have been the engine for the development of several C^*- and Banach algebra tools throughout more than 20 years. These methods have applications also in many other classes of operators, like convolution-type integral operators or general band-dominated operators, and even for other approximation methods, e.g. spline Galerkin and collocation methods for integral equations, or the collocation method for Cauchy singular integral equations (see for instance the books [2], [8], [9], [15], [20] and the paper [11] as well as its successors).

Besides the answer for the stability problem, there are further results which describe the properties of sequences which are not stable, but not far away from being so. More pecisely, there is a Fredholm theory for certain algebras of approximation sequences. More than 10 years ago S. Roch and one of the autors observed the so-called splitting phenomenon for the singular values of the matrices which constitute a sequence in a C^*-algebra of structured matrix sequences. A. Böttcher [1] was able to

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prove that this phenomenon is in force also for finite section sequences of Toeplitz operators with piecewise continuous generating functions on l^p-spaces with $1 < p < \infty$ when the singular values are replaced by the approximation numbers. A more general framework to this question was developed by A. Rogozhin and one of the authors in [32] and [31] under the assumption that the matrix sequences converge $*$-strongly (i.e. the sequences and their adjoints converge strongly). Moreover, this Banach algebraic framework provides a formula which links the indices of all snapshots of such sequences and, by this, provides a tool for the calculation of the index of a Fredholm operator via its so-called limit operators. Notice that this Banach algebraic framework encloses only sequences of matrices (that is, operators on finite dimensional spaces).

Until now, there were only a few attempts to establish a Fredholm theory for approximation sequences (A_n) whose elements A_n are operators acting on infinite dimensional Banach spaces. Such sequences naturally arise when the finite section method is applied to integral equations, for instance. One version was drawn up for the case of C^*-algebras in [8] (the so-called Standard algebras); H. Mascarenhas and one of the authors [17] treated a class of convolution type operators on $L^p(\mathbb{R}^2)$, $1 < p < \infty$; but in both cases there were additional conditions, like $\operatorname{ind} A_n = 0$, involved.

The papers [22] and [21] led, by different methods, to an index formula for a special class of band-dominated operators of the form $I + K$, K being locally compact, on $L^p(\mathbb{R})$, $1 < p < \infty$.

During all that time and in (almost) all modifications of these Banach algebra techniques the $*$-strong convergence of the sequences under consideration towards their snapshots played a crucial role. That's why, for instance, operators on l^∞- or L^∞-spaces stayed out of the focus, because there the classical approximation methods do not converge even strongly.

The present paper has two aims. For one thing, we want to present an approach which completely involves the cases of infinite dimensional operator sequences in an abstract Banach algebraic framework, so that the amenities of a Fredholm theory become available there. It particularly provides results on the stability, the splitting phenomenon, and an index formula. Moreover, it unifies several developments which differ from each other, since it covers and generalizes Standard algebras (i.e. the Hilbert space case), the Banach algebra approach of [32] for (finite dimensional) matrix sequences, as well as the special infinite dimensional version of [17].

For another thing, the present approach shall also cover non-strongly converging approximation methods in such a way that, for instance, the treatment of operators on the spaces l^p or L^p becomes homogeneous for all $1 \leqslant p \leqslant \infty$ (including $p = \infty$!).

The price that we (and the reader) have to pay for this goal is to bid goodbye to the strong convergence and to open up for a more appropriate notion of convergence. What does "more appropriate" mean? Actually, the ideal of compact operators is a corner stone in the former concepts since compactness turns strong convergence into uniform convergence and Fredholmness is invertiblity modulo compact operators. These two observations make a major contribution to large parts of the proofs. Furthermore, in many cases one also has (and exploits) the compactness of the operators P_n. So, this triple (compactness, Fredholmness and strong convergence) dictates to some extend what kinds of approximation methods can be handled. The concept of $\mathscr{P}$-strong convergence now turns the tables: Here, one starts with the approximation method (for instance given by a sequence $\mathscr{P} = (P_n)$ of projections) and introduces a matching triple ($\mathscr{P}$-compactness, $\mathscr{P}$-Fredholmness and $\mathscr{P}$-strong convergence) which then opens the way to translate the classical results. In fact, the coincidences and the achieved flexibility are so amazing, that one might dare to say that this gives a more natural language for the treatment of approximation methods than the classical notions do.

The paper is organized as follows.

The first part is devoted to the introduction and a deeper study of $\mathscr{P}$-strong convergence, the related notions of $\mathscr{P}$-compact and $\mathscr{P}$-Fredholm operators, as well as their properties. Principally, this concept already appeared in [30] and [20], and it attained great attention in the theory of Fredholm band-dominated operators (see e.g. [23], the books [25], [15] and the literature cited there). We show that, under a natural condition, $\mathscr{P}$-Fredholmness coincides with invertibility at infinity, which was an open problem for many years. Furthermore, it takes some work to embed the usual Fredholm property into this $\mathscr{P}$-concept. The key to manage this is the utilization of $\mathscr{P}$-compact projections and the introduction of the so-called $\mathscr{P}$-dichotomy. Finally, as an example, we consider band-dominated operators on $l^p(\mathbb{Z}^K, \mathbf{X})$ (with $\mathbf{X}$ being a Banach space and $p \in [1, \infty]$) and discuss the notion and application of limit operators there. We give some new proofs and add some further pieces of the puzzle in this topic.

The second part deals with the announced Banach algebras of structured operator sequences and extends the idea of [34]. Here, "structured" means that there are homomorphisms which condense a given sequence (A_n) to single operators $W^t(A_n)$ ($t \in T$ and T an index set) which appear as $\mathscr{P}$-strong limits. We call these operators snapshots of the sequence, since each of them captures certain aspects of its asymptotic behavior. The notion of $\mathscr{J}^T$-Fredholmness plays a central role, and the splitting phenomenon for the approximation numbers as well as the index formula are derived. Finally, in this second part we also study a general notion of Fredholm sequences (without any asymptotic structure). This is motivated by the fact that the $\mathscr{J}^T$-Fredholmness (which implies this general Fredholmness under a natural condition) heavily depends on the underlying algebra which seems to be artificial in a sense. Such an abstract notion already appeared in the much more comfortable C^*-algebra setting in [8]. It is further proved that (general) Fredholmness of a sequence is again equivalent to a special behavior of the approximation numbers.

The third part is devoted to applications. Here we consider finite section sequences of band-dominated operators acting on $l^p(\mathbb{Z}, \mathbf{X})$, with $1 \leqslant p \leqslant \infty$ and $\mathbf{X}$ a Banach space, and by a tricky transformation (which we borrow from [23] or [15]) also on $L^p(\mathbb{R})$. It is worth noticing that the finite section sequence of a band-dominated operator is not a structured sequence in the sense of the second part, but the picture changes if one switches to suitable subsequences. This is the way to prove stability, to describe the Fredholm properties of a sequence, and even to provide an index formula for a Fredholm band-dominated operator which covers and extends all formulas known until now. To be a bit more precise, we give a new proof for the known result on the stability of the finite section sequence (P_nAP_n) of a band-dominated operator A, which has grown within lots of papers, e.g. [23], [24], [29], [26], [15], [3], [14] by V. Rabinovich, S. Roch, M. Lindner and one of the authors, and we extend this result to the algebra which is generated by the finite section sequences. Moreover, the Fredholm theory (including splitting and index relations), which has only been considered for the finite sections of band-dominated operators in the l^2-case [29] and the case of matrix sequences [34] until now, is presented for all $p \in [1,\infty]$ and arbitrary Banach spaces $\mathbf{X}$. The formula which expresses the index of a band-dominated operator $A = I + K$ (with K being locally compact) is now available for all band-dominated A. The paper ends with a note on harmonic approximations of Fredholm Toeplitz operators and their indices.

We also mention that the present approach opens the door for the application of localizing techniques which lead to results on the convergence of norms and condition numbers as well as of pseudospectra for sequences (A_n) arising from (non-strongly converging) approximation methods like the finite section method for band-dominated operators. In particular, we now can extend also the results of [17] on convolution-type operators on $L^p(\mathbb{R}^2)$ (and on cones) to the L^1 and L^∞-case. But, this will be part of future work.

1. $\mathscr{P}$-compact and $\mathscr{P}$-Fredholm operators, $\mathscr{P}$-strong convergence

1.1. Basic definitions

DEFINITION 1.1. Let $\mathbf{X}$ be a Banach space and let $\mathscr{P} = (P_n)_{n\in\mathbb{N}}$ be a bounded sequence of operators in $\mathscr{L}(\mathbf{X})$ with the following properties:

- $P_n \neq 0$ and $P_n \neq I$ for all $n \in \mathbb{N}$,
- For every $m \in \mathbb{N}$ there is an $N_m \in \mathbb{N}$ such that $P_nP_m = P_mP_n = P_m$ if $n \geqslant N_m$.

Then $\mathscr{P}$ is referred to as an approximate projection. In all what follows we set $Q_n := I - P_n$ and we further write $m \ll n$ if $P_kQ_l = Q_lP_k = 0$ for all $k \leqslant m$ and all $l \geqslant n$.

$\mathscr{P}$-compactness Let $\mathscr{P}$ be an approximate projection. A bounded linear operator K is called $\mathscr{P}$-compact if $\|KP_n - K\|$ and $\|P_nK - K\|$ tend to zero as $n \to \infty$. By $\mathscr{K}(\mathbf{X},\mathscr{P})$ we denote the set of all $\mathscr{P}$-compact operators on $\mathbf{X}$ and by $\mathscr{L}(\mathbf{X},\mathscr{P})$ the set of all operators $A \in \mathscr{L}(\mathbf{X})$ for which AK and KA are $\mathscr{P}$-compact whenever K is $\mathscr{P}$-compact.

THEOREM 1.2. (see [25], Proposition 1.1.8)
Let $\mathscr{P}$ be an approximate projection on the Banach space $\mathbf{X}$. $\mathscr{L}(\mathbf{X},\mathscr{P})$ is a closed subalgebra of $\mathscr{L}(\mathbf{X})$, it contains the identity operator, and $\mathscr{K}(\mathbf{X},\mathscr{P})$ is a closed ideal of $\mathscr{L}(\mathbf{X},\mathscr{P})$. An operator $A \in \mathscr{L}(\mathbf{X})$ belongs to $\mathscr{L}(\mathbf{X},\mathscr{P})$ if and only if, for every $k \in \mathbb{N}$,

$$\|P_kAQ_n\| \to 0 \text{ and } \|Q_nAP_k\| \to 0 \text{ as } n \to \infty.$$

$\mathscr{P}$-Fredholmness and invertibility at infinity Here are two possible generalizations of Fredholmness based on $\mathscr{P}$-compact operators instead of compact ones.

DEFINITION 1.3. We say that $A \in \mathscr{L}(\mathbf{X})$ is invertible at infinity (with respect to $\mathscr{P}$) if there is an operator $B \in \mathscr{L}(\mathbf{X})$ with $I - AB, I - BA \in \mathscr{K}(\mathbf{X}, \mathscr{P})$. In this case B is referred to as a $\mathscr{P}$-regularizer for A.

An operator $A \in \mathscr{L}(\mathbf{X}, \mathscr{P})$ is said to be $\mathscr{P}$-Fredholm if the coset $A + \mathscr{K}(\mathbf{X}, \mathscr{P})$ is invertible in the quotient algebra $\mathscr{L}(\mathbf{X}, \mathscr{P})/\mathscr{K}(\mathbf{X}, \mathscr{P})$.

Notice that invertibility at infinity is defined in $\mathscr{L}(\mathbf{X})$, whereas for $\mathscr{P}$-Fredholmness we are restricted to $\mathscr{L}(\mathbf{X}, \mathscr{P})$, since we need that $\mathscr{K}(\mathbf{X}, \mathscr{P})$ forms a closed two-sided ideal in $\mathscr{L}(\mathbf{X}, \mathscr{P})$. It has been an open problem for a long time if these notions coincide for $A \in \mathscr{L}(\mathbf{X}, \mathscr{P})$ but we will finally tackle this question in Section 1.2.

Properly $\mathscr{P}$-Fredholm operators Throughout this paper we let $\operatorname{im} A$ and $\ker A$ denote the range and the kernel of the operator $A \in \mathscr{L}(\mathbf{X})$, respectively, as well as $\operatorname{coker} A := \mathbf{X}/\operatorname{im} A$ its cokernel. It is well known that the usual Fredholm property of bounded linear operators can also be described in terms of compact projections, in the sense that there are projections P, P' with $\operatorname{im} P = \ker A$ and $\ker P' = \operatorname{im} A$. On the other hand, a non-Fredholm operator A can be characterized by the following property: For each $l \in \mathbb{N}$ and each $\varepsilon > 0$ there exists a projection $Q \in \mathscr{K}(\mathbf{X})$ (where $\mathscr{K}(\mathbf{X})$ denotes the ideal of compact operators on $\mathbf{X}$) with $\operatorname{rank} Q \geqslant l$ and such that $\|AQ\| < \varepsilon$ or $\|QA\| < \varepsilon$. (see [34], Theorem 3). Before we define an analogon based on $\mathscr{P}$-compact projections, we recall a useful result on generalized invertibility:

PROPOSITION 1.4. *Let $A, B \in \mathscr{L}(\mathbf{X})$. Then the following are equivalent*

- $ABA = A$
- *$I - BA$ is a bounded projection with* $\operatorname{im}(I - BA) = \ker A$
- *$I - AB$ is a bounded projection with* $\ker(I - AB) = \operatorname{im} A$.

Moreover, if $A \in \mathscr{L}(\mathbf{X})$ and $P, P' \in \mathscr{L}(\mathbf{X})$ are projections with $\operatorname{im} P = \ker A$ *and* $\ker P' = \operatorname{im} A$ *then there is an operator $C \in \mathscr{L}(\mathbf{X})$ with $A = ACA$, $C = CAC$ and $P = I - CA$, $P' = I - AC$.*

DEFINITION 1.5. An operator $A \in \mathscr{L}(\mathbf{X})$ is said to be properly $\mathscr{P}$-Fredholm, if there exist projections $P, P' \in \mathscr{K}(\mathbf{X}, \mathscr{P})$ such that $\operatorname{im} P = \ker A$ and $\ker P' = \operatorname{im} A$.

An operator $A \in \mathscr{L}(\mathbf{X})$ is called properly $\mathscr{P}$-deficient from the right (left) if, for each $\varepsilon > 0$ and each $k \in \mathbb{N}$, there is a projection $R \in \mathscr{K}(\mathbf{X}, \mathscr{P})$ of rank at least k such that $\|AR\| < \varepsilon$ ($\|RA\| < \varepsilon$, respectively).

From Proposition 1.4 we conclude

COROLLARY 1.6. *An operator $A \in \mathscr{L}(\mathbf{X})$ is properly $\mathscr{P}$-Fredholm if and only if there is a $\mathscr{P}$-regularizer $B \in \mathscr{L}(\mathbf{X})$ for A which is also a generalized inverse, that is $I - AB, I - BA \in \mathscr{K}(\mathbf{X}, \mathscr{P})$ and $ABA = A$, $BAB = B$.*

The $\mathcal{P}$-dichotomy, or how to grasp Fredholmness in terms of $\mathcal{P}$-compact operators

PROPOSITION 1.7. *Let $\mathcal{P}$ be an approximate projection and let $A \in \mathcal{L}(\mathbf{X})$ be invertible at infinity.*

If A is normally solvable (which means that is $\operatorname{im} A$ is closed) then for every $k \in \mathbb{N}$ with $k \leqslant \dim\ker A$ ($k \leqslant \dim\operatorname{coker} A$) there is a $\mathcal{P}$-compact projection P of the rank k such that $AP = 0$ ($PA = 0$).

If A is not normally solvable then A is properly $\mathcal{P}$-deficient from both sides.

Proof. Let $B \in \mathcal{L}(\mathbf{X})$ be a $\mathcal{P}$-regularizer for A. For every $x \in \ker A$ we find that $Q_n x = Q_n(I-BA)x + Q_nBAx = Q_n(I-BA)x$ tends to zero as $n \to \infty$. Let $\mathbf{X}_1$ be a finite dimensional subspace of $\ker A$. Then we fix $m \in \mathbb{N}$, s.t. $\sup\{\|Q_m x\| : x \in \mathbf{X}_1, \|x\| = 1\}$ is less than $1/2$ and deduce that the operator $P_m : \mathbf{X}_1 \to \mathbf{X}_2 := P_m(\mathbf{X}_1)$ is invertible and its inverse has norm less than 2. Let S denote its inverse and let $\tilde{m} \gg m$. Since $\mathbf{X}_2$ is a finite dimensional subspace of the Banach space $\ker Q_{\tilde{m}}$, there is a bounded projection $R \in \mathcal{L}(\ker Q_{\tilde{m}})$ onto $\mathbf{X}_2$ with $\|R\| \leqslant \dim \mathbf{X}_1$ (see [19], B.4.9). Now, we define $P := SRP_m$ and we easily check that $\operatorname{im} P = \mathbf{X}_1$ and $P^2 = SRP_mP = SP_mP = P$, hence P is a projection onto $\mathbf{X}_1$ which obviously belongs to $\mathcal{K}(\mathbf{X}, \mathcal{P})$.

Assume now that A is not normally solvable and fix $\varepsilon > 0$ and $k \in \mathbb{N}$. Then there is a rank-k-projection K such that $\|AK\| \leqslant \varepsilon$ (see [34], Theorem 3 for a proof). Further, denote by d the finite number $\sup\{\|Q_n\| : n \in \mathbb{N}\}$ and check that

$$\begin{aligned}\|Q_nK\| &\leqslant \|Q_n(I-BA)K\| + \|Q_nBAK\| \\ &\leqslant \|Q_n(I-BA)\|\|K\| + d\|B\|\varepsilon \leqslant 2d\|B\|\varepsilon\end{aligned}$$

for sufficiently large n. Further, for $x \in \operatorname{im} K$,

$$\frac{\|AP_nx\|}{\|P_nx\|} \leqslant \frac{\|Ax\| + \|A\|\|Q_nx\|}{\|x\| - \|Q_nx\|} \leqslant \frac{1+2d\|B\|\|A\|}{1-2d\|B\|\varepsilon}\varepsilon.$$

This shows that for sufficiently small ε and sufficiently large l the space $\mathbf{X}_3 := \operatorname{im} P_lK$ is of the dimension k and it holds that $\|Az\| \leqslant 4d\|B\|\|A\|\varepsilon\|z\|$ for all $z \in \mathbf{X}_3$. Since $\mathbf{X}_3 \subset \ker Q_{\tilde{l}} \subset \ker Q_{\hat{l}}$ for $\hat{l} \gg \tilde{l} \gg l$ we can again choose a projection $R \in \mathcal{L}(\ker Q_{\hat{l}})$ onto $\mathbf{X}_3$ of the norm at most k and define $P := RP_{\tilde{l}}$. Obviously, P is a $\mathcal{P}$-compact projection of rank k and we have $\|AP\| \leqslant 4kd(d+1)\|B\|\|A\|\varepsilon$. Since ε was chosen arbitrarily, this proves the proper $\mathcal{P}$-deficiency from the right.

Further, if A is not normally solvable, then A^* is not normally solvable, too (see [7], §4, Theorem 4.2, for instance), and for every prescribed $\varepsilon > 0$ we find a projection $T \in \mathcal{K}(\mathbf{X}^*, \mathcal{P}^*)$ of the rank k with $\|A^*T\| \leqslant \varepsilon$. We show that there is a projection $P' \in \mathcal{K}(\mathbf{X}, \mathcal{P})$ of the norm less than $2k(k+1)(d+1)$ such that $\operatorname{im}(P')^* = \operatorname{im} T$, and hence

$$\|P'A\| = \|A^*(P')^*\| = \|A^*T(P')^*\| \leqslant 2k(k+1)(d+1)\varepsilon.$$

Then, since ε was chosen arbitrarily, A proves to be properly $\mathcal{P}$-deficient from the left. For this, let l be such that $\|Q_l^*T\| < 1/2$. Then, for every $f \in \operatorname{im} T$ we have

$$\|f \circ P_l\| = \|P_l^*f\| \geqslant \|f\| - \|Q_l^*f\| = \|f\| - \|(Q_l^*T)f\| \geqslant \frac{1}{2}\|f\|,$$

that is $\{f|_{\operatorname{im} P_l} : f \in \operatorname{im} T\}$ forms a linear space of the dimension k. Hence, for $\tilde{l} \gg l$,

$$\mathbf{X}_1 := \bigcap_{f \in \operatorname{im} T} \ker f|_{\ker Q_{\tilde{l}}} \subset \ker Q_{\tilde{l}}$$

has the codimension k in $\ker Q_{\tilde{l}}$. Due to [19], B.4.10 we can choose a projection R parallel to $\mathbf{X}_1$ onto a certain complement $\mathbf{X}_2$ of $\mathbf{X}_1$ in $\ker Q_{\tilde{l}}$ of the norm less than $k+1$. Since the set of all restrictions $g = f|_{\mathbf{X}_2}$ of the functionals $f \in \operatorname{im} T$ to $\mathbf{X}_2$ forms a k-dimensional space we conclude that each functional on $\mathbf{X}_2$ is of the form $g = f|_{\mathbf{X}_2}$ with $f \in \operatorname{im} T$. Auerbachs Lemma [19], B.4.8 provides bases $x_1, \ldots, x_k \in \mathbf{X}_2$ and $f_1, \ldots, f_k \in \operatorname{im} T$ such that $\|x_i\| = \|g_j\| = 1$ and $g_j(x_i) = \delta_{ij}$ (where $g_j := f_j|_{\mathbf{X}_2}$) for all $i, j = 1, \ldots, k$. The norms $\|f_j\|$ can be estimated by

$$2\|f_j \circ P_l\| \leqslant 2\|f_j \circ R \circ P_l\| + 2\|f_j \circ (I-R) \circ P_l\| \leqslant 2\|g_j\|\|R\|\|P_l\| < 2(k+1)(d+1),$$

thus, defining $P'x := \sum_{i=1}^{k} f_i(x)x_i$, we obtain a $\mathscr{P}$-compact projection onto $\mathbf{X}_2$ of the norm less than $2k(k+1)(d+1)$. Since $f_j(P'x) = \sum_{i=1}^{k} f_i(x) f_j(x_i) = f_j(x)$ for all j and x we find that $\operatorname{im}(P')^* = \operatorname{im} T$.

It remains to consider operators A which are normally solvable, hence $\dim \operatorname{coker} A$ equals $\dim \ker A^*$, and to check that for every finite number $k \leqslant \dim \operatorname{coker} A$ there is a $\mathscr{P}$-compact projection P' of the rank k with $P'A = 0$. Since A^* is $\mathscr{P}^*$-Fredholm, we can apply the first part of this proof to find a $\mathscr{P}^*$-compact projection T of the rank k onto a respective subspace of $\ker A^*$. Then simply apply the above argument with $\varepsilon = 0$ to construct P'. □

This justifies the following definition.

DEFINITION 1.8. An operator $A \in \mathscr{L}(\mathbf{X})$ is said to have the $\mathscr{P}$-dichotomy if it is either Fredholm and properly $\mathscr{P}$-Fredholm, or it is properly $\mathscr{P}$-deficient from at least one side.

COROLLARY 1.9. *Let $\mathscr{P}$ be an approximate projection and let $A \in \mathscr{L}(\mathbf{X})$ be invertible at infinity. Then A has the $\mathscr{P}$-dichotomy.*

$\mathscr{P}$-strong convergence Let $\mathscr{P} = (P_n)$ be an approximate projection and, for each $n \in \mathbb{N}$, let $A_n \in \mathscr{L}(\mathbf{X})$. The sequence (A_n) converges $\mathscr{P}$-strongly to $A \in \mathscr{L}(\mathbf{X})$ if, for all $K \in \mathscr{K}(\mathbf{X}, \mathscr{P})$, both $\|(A_n - A)K\|$ and $\|K(A_n - A)\|$ tend to 0 as $n \to \infty$. In this case we write $A_n \to A$ $\mathscr{P}$-strongly or $A = \mathscr{P}\text{-}\lim_{n\to\infty} A_n$.

Proposition 1.1.14 in [25] yields that a bounded sequence (A_n) in $\mathscr{L}(\mathbf{X})$ converges $\mathscr{P}$-strongly to $A \in \mathscr{L}(\mathbf{X})$ iff $\|(A_n - A)P_m\| \to 0$ and $\|P_m(A_n - A)\| \to 0$ for every fixed $P_m \in \mathscr{P}$. Unfortunately, the $\mathscr{P}$-strong limit is not unique in general. Consider, for instance, a projection $P \notin \{0, I\}$ and $\mathscr{P} = (P_n)$ given by $P_n := P$ for all n. Then the sequence (P_n) converges $\mathscr{P}$-strongly to both P and I. Therefore we adopt further conditions on $\mathscr{P}$ from [25] to guarantee the uniqueness.

DEFINITION 1.10. An approximate projection $\mathscr{P}$ is called approximate identity if $\sup_n \|P_n x\| \geqslant \|x\|$ holds for each $x \in \mathbf{X}$. An approximate projection $\mathscr{P}$ is said to be symmetric if $\mathscr{P}^* = (P_n^*)$ is an approximate identity on $\mathbf{X}^*$. An important closed subspace $\mathbf{X}_0$ of $\mathbf{X}$ is given by

$$\mathbf{X}_0 := \{x \in \mathbf{X} : \|Q_n x\| \to 0 \text{ as } n \to \infty\}.$$

Obviously, the approximate projection $\mathscr{P}$ induces an approximate identity on $\mathbf{X}_0$ which tends strongly to the identity. Further, for a functional $f \in (\mathbf{X}_0)^*$ and for $\varepsilon > 0$ let $x \in \mathbf{X}_0$, $\|x\| = 1$ be such that $|f(x)| \geqslant \|f\| - \varepsilon$. Then

$$\|P_n^* f\| \geqslant |(P_n^* f)(x)| = |f(P_n x)| \geqslant |f(x)| - |f(Q_n x)| \geqslant \|f\| - \varepsilon - \|f\| \|Q_n x\|,$$

hence $\sup_n \|P_n^* f\| \geqslant \|f\|$, that is $\mathscr{P}$ is even a symmetric approximate identity on $\mathbf{X}_0$.

REMARK 1.11. If $\mathscr{P}$ is a symmetric approximate projection then it is automatically an approximate identity. Indeed, assume that there is an x with $\sup_n \|P_n x\| < \|x\|$. Then $x \notin \mathbf{X}_0$. Thus, there is a bounded linear functional $f \in \mathbf{X}^*$ with $|f(x)| > 0$ and $f(\mathbf{X}_0) = \{0\}$, that is $f \neq 0$ but $P_n^* f = 0$ for all n. This contradicts the symmetry.

Let us come back to $\mathscr{P}$-strong convergence and, in what follows, let $\mathscr{P}$ be an approximate identity and set $B_{\mathscr{P}} := \sup_n \|P_n\|$. Then no sequence $(A_n) \subset \mathscr{L}(\mathbf{X})$ possesses more than one $\mathscr{P}$-strong limit. Indeed, assume that A, B are $\mathscr{P}$-strong limits of (A_n). Then, for every m,

$$\|P_m(A - B)\| \leqslant \|P_m(A - A_n)\| + \|P_m(B - A_n)\| \to 0 \quad \text{as} \quad n \to \infty,$$

hence $P_m(A - B) = 0$. Assume that $A \neq B$. Then there is an $x \in \mathbf{X}$ with $(A - B)x \neq 0$ and, since $\mathscr{P}$ is an approximate identity, there is also an m such that $P_m(A - B)x \neq 0$, a contradiction.

By $\mathscr{F}(\mathbf{X}, \mathscr{P})$ we denote the set of all bounded sequences $(A_n) \subset \mathscr{L}(\mathbf{X})$, which possess a $\mathscr{P}$-strong limit in $\mathscr{L}(\mathbf{X}, \mathscr{P})$. Notice that this slightly differs from Definition 1.1.18 in [25]. Nevertheless, [25], Proposition 1.1.17 tells us

PROPOSITION 1.12. *Let $\mathscr{P}$ be an approximate identity and $(A_n) \subset \mathscr{L}(\mathbf{X}, \mathscr{P})$ be a sequence converging $\mathscr{P}$-strongly to $A \in \mathscr{L}(\mathbf{X})$. Then A belongs to $\mathscr{L}(\mathbf{X}, \mathscr{P})$ and (A_n) is bounded, that is $(A_n) \in \mathscr{F}(\mathbf{X}, \mathscr{P})$. Moreover, $\|A\| \leqslant B_{\mathscr{P}} \liminf_n \|A_n\|$.*

THEOREM 1.13. *Provided with operations $\alpha(A_n) + \beta(B_n) := (\alpha A_n + \beta B_n)$ and $(A_n)(B_n) := (A_n B_n)$, as well as the norm $\|(A_n)\| := \sup_n \|A_n\| < \infty$, $\mathscr{F}(\mathbf{X}, \mathscr{P})$ becomes a Banach algebra with identity $\mathbb{I} := (I)$. The mapping $\mathscr{F}(\mathbf{X}, \mathscr{P}) \to \mathscr{L}(\mathbf{X}, \mathscr{P})$ which sends (A_n) to $A = \mathscr{P}$-$\lim A_n$ is a unital algebra homomorphism and*

$$\|A\| \leqslant B_{\mathscr{P}}^2 \liminf_{n \to \infty} \|A_n\|. \tag{1.1}$$

This algebra will play a fundamental role in the theory of structured operator sequences in Section 2.

Proof. The proof of $\mathscr{F}(\mathbf{X},\mathscr{P})$ being an algebra is straightforward, and we only note that if $(A_n),(B_n)$ are bounded and converge $\mathscr{P}$-strongly to $A,B \in \mathscr{L}(\mathbf{X},\mathscr{P})$, respectively, then

$$\|K(A_nB_n - AB)\| \leqslant \|K(A_n - A)B_n\| + \|KA(B_n - B)\| \to 0$$

for every $K \in \mathscr{K}(\mathbf{X},\mathscr{P})$, as $n\to\infty$, since $A \in \mathscr{L}(\mathbf{X},\mathscr{P})$ implies $KA \in \mathscr{K}(\mathbf{X},\mathscr{P})$.

For $A \in \mathscr{L}(\mathbf{X},\mathscr{P})$ we have $\|A\| \leqslant B_{\mathscr{P}} \liminf_n \|P_nA\|$, due to Proposition 1.12, hence for given $\varepsilon > 0$ we can fix k such that $\|A\| \leqslant B_{\mathscr{P}}\|P_kA\| + \varepsilon/3$. For the sequence (A_n) with $\mathscr{P}$-strong limit A we choose N such that $\|P_k(A - A_N)\| \leqslant B_{\mathscr{P}}^{-1}\varepsilon/3$ and $\|P_kA_N\| \leqslant \liminf_n \|P_kA_n\| + B_{\mathscr{P}}^{-1}\varepsilon/3$. Then

$$\|P_kA\| \leqslant \|P_kA_N\| + \|P_k(A - A_N)\| \leqslant \liminf_{n\to\infty} \|P_kA_n\| + 2B_{\mathscr{P}}^{-1}\varepsilon/3,$$

hence $\|A\| \leqslant B_{\mathscr{P}}\|P_kA\| + \varepsilon/3 \leqslant B_{\mathscr{P}} \liminf_n \|P_kA_n\| + \varepsilon \leqslant B_{\mathscr{P}}^2 \liminf_n \|A_n\| + \varepsilon$. Since ε is arbitrary, Equation (1.1) follows.

Finally, let $((C_n^m)_n)_m$ be a Cauchy sequence of sequences $(C_n)_n \in \mathscr{F}(\mathbf{X},\mathscr{P})$, where C^m shall denote the $\mathscr{P}$-strong limit of $(C_n^m)_n$, respectively. For every n, $(C_n^m)_m$ converges in $\mathscr{L}(\mathbf{X})$ to an operator D_n, and the sequence (D_n) is bounded. Further, Equation (1.1) yields that $(C^m)_m$ is a Cauchy sequence with a limit $D \in \mathscr{L}(\mathbf{X},\mathscr{P})$. Now one easily checks that (D_n) converges $\mathscr{P}$-strongly to D, thus $\mathscr{F}(\mathbf{X},\mathscr{P})$ is complete. □

1.2. The wonderful world of uniform approximate identities

In this section we answer the question on the coincidence of invertibility at infinity and $\mathscr{P}$-Fredholmness affirmatively under a natural condition on $\mathscr{P}$ which has been in the business since the inverse closedness of $\mathscr{L}(\mathbf{X},\mathscr{P})$ in $\mathscr{L}(\mathbf{X})$ is known, based on a proof of Simonenko [35]. Further some useful criteria for the somewhat mysterious $\mathscr{P}$-dichotomy are derived.

DEFINITION 1.14. Given a Banach space $\mathbf{X}$ with an approximate projection $\mathscr{P} = (P_n)$ we set $S_1 := P_1$ and $S_n := P_n - P_{n-1}$ for $n > 1$. Further, for every bounded subset $U \subset \mathbb{R}$, denote $P_U := \sum_{k\in\mathbb{N}\cap U} S_k$. $\mathscr{P}$ is called uniform if $C_{\mathscr{P}} := \sup\|P_U\| < \infty$, the supremum over all bounded $U \subset \mathbb{R}$.

Two approximate projections $\mathscr{P} = (P_n)$ and $\hat{\mathscr{P}} = (F_n)$ on $\mathbf{X}$ are said to be equivalent if for every $m \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that

$$P_mF_n = F_nP_m = P_m \quad \text{and} \quad F_mP_n = P_nF_m = F_m.$$

In case of equivalent approximate projections $\mathscr{P}$ and $\hat{\mathscr{P}}$, [25], Proposition 1.1.10 shows that $\mathscr{K}(\mathbf{X},\mathscr{P}) = \mathscr{K}(\mathbf{X},\hat{\mathscr{P}})$. Hence, $\mathscr{L}(\mathbf{X},\mathscr{P}) = \mathscr{L}(\mathbf{X},\hat{\mathscr{P}})$ and the notions of $\mathscr{P}$-compactness, $\mathscr{P}$-Fredholmness, and $\mathscr{P}$-strong convergence coincide with the respective $\hat{\mathscr{P}}$-notions.

THEOREM 1.15. *Let $\mathscr{P}$ be a uniform approximate projection on the Banach space $\mathbf{X}$ and $A \in \mathscr{L}(\mathbf{X},\mathscr{P})$. Then there is an equivalent uniform approximate projection $\hat{\mathscr{P}} = (F_n)$ on $\mathbf{X}$ (depending on A) with $C_{\hat{\mathscr{P}}} \leqslant C_{\mathscr{P}}$ such that*

$$\|[F_n,A]\| = \|AF_n - F_nA\| \to 0 \quad as \quad n \to \infty.$$

Proof. Successively choose integers $1 = j_1 \ll i_2 \ll j_2 \ll i_3 \ll j_3 \ll \dots$ such that for every l

$$\|P_sAQ_{j_l}\| \leqslant (2^{l+2}l)^{-1}\ \forall s \leqslant i_l \quad \text{and} \quad \|Q_tAP_{j_l}\| \leqslant (2^{l+5}l)^{-1}\ \forall t \geqslant i_{l+1}. \tag{1.2}$$

Then, set $U_k^n := \{i_{2^n+k-1}+1,\dots,i_{2^n+k}\}$ and $V_k^n := \{j_{2^n+k-2}+1,\dots,j_{2^n+k}\}$ for all $k,n \in \mathbb{N}$, as well as $U_0^n := \{1,\dots,i_{2^n}\}$ and $V_0^n := \{1,\dots,j_{2^n}\}$, and find

$$\|P_{U_k^n}AP_{j_{2^n+k-2}}\| \leqslant (2^{k+2}n)^{-1} \quad \text{and} \quad \|P_{U_k^n}AQ_{j_{2^n+k}}\| \leqslant (2^{k+2}n)^{-1}.$$

Thus

$$\|P_{U_k^n}AQ_{V_k^n}\| \leqslant (2^{k+1}n)^{-1}, \text{ that is } \sum_{k\in\mathbb{Z}_+} \|P_{U_k^n}AQ_{V_k^n}\| \leqslant \frac{1}{n}. \tag{1.3}$$

For $n \in \mathbb{N}$ we set

$$F_n := \sum_{k=0}^{n-1}\left(1-\frac{k}{n}\right)P_{U_k^n}$$

and deduce that $F_nF_{n+1} = F_{n+1}F_n = F_n$ as well as

$$\|F_n\| = \left\|\sum_{k=1}^{n}\frac{k}{n}P_{U_{n-k}^n}\right\| = \frac{1}{n}\left\|\sum_{k=1}^{n}kP_{U_{n-k}^n}\right\| \leqslant \frac{1}{n}\sum_{j=1}^{n}\left\|\sum_{k=j}^{n}P_{U_{n-k}^n}\right\| \leqslant C_{\mathscr{P}}, \tag{1.4}$$

that is $\hat{\mathscr{P}} = (F_n)$ is an approximate projection. Further, $\mathscr{P}$ and $\hat{\mathscr{P}}$ are equivalent, since

$$P_{j_{2^n-1}}F_n = F_nP_{j_{2^n-1}} = P_{j_{2^n-1}} \quad \text{and} \quad F_nP_{j_{2^n+n-1}} = P_{j_{2^n+n-1}}F_n = F_n.$$

For bounded subsets $U \subset \mathbb{R}$ we finally introduce the operators F_U as in Definition 1.14 and easily check that they can be represented in the form

$$F_U = \sum_{k=0}^{N-1}\frac{k}{N}P_{W_k}$$

with certain disjoint bounded sets $W_k \subset \mathbb{R}$, $k = 0,\dots,N-1$. By an estimate similar to (1.4) we deduce that $\hat{\mathscr{P}}$ is uniform with $C_{\hat{\mathscr{P}}} \leqslant C_{\mathscr{P}}$. It remains to consider the commutators of A and F_n. As a start, notice that

$$\begin{aligned} AF_n &= \sum_{k=0}^{n}P_{U_k^n}AF_n + Q_{i_{2^n+n}}AF_n \\ &= \sum_{k=0}^{n}P_{U_k^n}AP_{V_k^n}F_n + \sum_{k=0}^{n}P_{U_k^n}AQ_{V_k^n}F_n + Q_{i_{2^n+n}}AP_{j_{2^n+n-1}}F_n, \end{aligned}$$

where the last term is less than $C_{\mathscr{P}}n^{-1}$ in the norm and the middle term is not greater than $C_{\mathscr{P}}n^{-1}$ as well, due to the above estimate (1.3). The first one equals

$$\begin{aligned}
&\sum_{k=0}^{n} P_{U_k^n} A P_{V_k^n} \left(F_n - \left(1 - \frac{k}{n}\right) I \right) + \sum_{k=0}^{n} \left(1 - \frac{k}{n}\right) P_{U_k^n} A P_{V_k^n} \\
&\quad = \sum_{k=0}^{n} P_{U_k^n} A P_{V_k^n} \frac{1}{n} (P_{U_{k-1}^n} - P_{U_{k+1}^n}) - \sum_{k=0}^{n} \left(1 - \frac{k}{n}\right) P_{U_k^n} A Q_{V_k^n} + F_n A,
\end{aligned}$$

where we redefine $P_{U_{-1}^n} = P_{U_{n+1}^n} := 0$. Since the second item is smaller than n^{-1} in the norm we find

$$\begin{aligned}
\|AF_n - F_n A\| &\leqslant \frac{1}{n} \left\| \sum_{k=0}^{n} P_{U_k^n} A (I - Q_{V_k^n})(P_{U_{k-1}^n} - P_{U_{k+1}^n}) \right\| + \frac{2C_{\mathscr{P}} + 1}{n} \\
&\leqslant \frac{1}{n} \left\| \sum_{k=0}^{n} P_{U_k^n} A (P_{U_{k-1}^n} - P_{U_{k+1}^n}) \right\| + \frac{2C_{\mathscr{P}} n^{-1} + 2C_{\mathscr{P}} + 1}{n}.
\end{aligned}$$

From

$$\begin{aligned}
\left\| \sum_{k=0}^{n} P_{U_k^n} A P_{U_{k-1}^n} \right\| &\leqslant \sum_{l=0}^{2} \left\| \sum_{\substack{k=0 \\ k \equiv l \mod 3}}^{n} P_{U_k^n} A P_{U_{k-1}^n} \right\| \\
&\leqslant \sum_{l=0}^{2} \left[\left\| \sum_{\substack{k=0 \\ k \equiv l(3)}}^{n} P_{U_k^n} A \left(\sum_{\substack{j=0 \\ j \equiv l(3)}}^{n} P_{U_{j-1}^n} \right) \right\| + \sum_{\substack{k=0 \\ k \equiv l(3)}}^{n} \left\| P_{U_k^n} A \left(- \sum_{\substack{j=0, j \neq k \\ j \equiv l(3)}}^{n} P_{U_{j-1}^n} \right) \right\| \right] \\
&\leqslant \sum_{l=0}^{2} \left[\left\| \sum_{\substack{k=0 \\ k \equiv l(3)}}^{n} P_{U_k^n} \right\| \left\| A \left(\sum_{\substack{j=0 \\ j \equiv l(3)}}^{n} P_{U_{j-1}^n} \right) \right\| + \sum_{\substack{k=0 \\ k \equiv l(3)}}^{n} \|P_{U_k^n} A Q_{V_k^n}\| \left\| \sum_{\substack{j=0, j \neq k \\ j \equiv l(3)}}^{n} P_{U_{j-1}^n} \right\| \right]
\end{aligned}$$

we conclude that

$$\left\| \sum_{k=0}^{n} P_{U_k^n} A P_{U_{k-1}^n} \right\| \leqslant \sum_{l=0}^{2} \left[C_{\mathscr{P}} \|A\| C_{\mathscr{P}} + n^{-1} C_{\mathscr{P}} \right] = 3C_{\mathscr{P}} (\|A\| C_{\mathscr{P}} + n^{-1}).$$

With a similar estimate for $\sum P_{U_k^n} A P_{U_{k+1}^n}$ we finally get the assertion by

$$\|AF_n - F_n A\| \leqslant \frac{6C_{\mathscr{P}}(\|A\| C_{\mathscr{P}} + n^{-1}) + 2C_{\mathscr{P}} n^{-1} + 2C_{\mathscr{P}} + 1}{n}. \qquad \square$$

$\mathscr{P}$-Fredholmness and invertibility at infinity coincide

THEOREM 1.16. *Let $\mathscr{P} = (P_n)$ be a uniform approximate projection on $\mathbf{X}$. Then $A \in \mathscr{L}(\mathbf{X},\mathscr{P})$ is $\mathscr{P}$-Fredholm if and only if it is invertible at infinity. In particular, $\mathscr{L}(\mathbf{X},\mathscr{P})$ is inverse closed in $\mathscr{L}(\mathbf{X})$.*

Proof. Obviously, every $\mathscr{P}$-Fredholm operator $A \in \mathscr{L}(\mathbf{X},\mathscr{P})$ is invertible at infinity.

Conversely, let $A \in \mathscr{L}(\mathbf{X},\mathscr{P})$ be invertible at infinity, that is there is an operator $B \in \mathscr{L}(\mathbf{X})$ such that $P := I - BA, P' := I - AB \in \mathscr{K}(\mathbf{X},\mathscr{P})$. Theorem 1.15 yields an equivalent uniform approximate projection $\hat{\mathscr{P}} = (F_n)$ such that $\|[A,F_n]\| \to 0$ as $n \to \infty$. Thus, for $G_n := I - F_n$, we also have $\|[A,G_n]\| \to 0$ as $n \to \infty$. We show that $\|[B,G_n]\| \to 0$ as $n \to \infty$, and conclude, for every k, that $\|F_k B G_n\|$ and $\|G_n B F_k\|$ tend to zero as $n \to \infty$, hence $B \in \mathscr{L}(\mathbf{X},\hat{\mathscr{P}}) = \mathscr{L}(\mathbf{X},\mathscr{P})$ due to Theorem 1.2. For this, notice that $\mathscr{K}(\mathbf{X},\hat{\mathscr{P}}) = \mathscr{K}(\mathbf{X},\mathscr{P})$, fix $\varepsilon > 0$ and choose N such that, for all $n \geqslant N$,

$$\|PG_n\| \leqslant \frac{\varepsilon}{\|B\|}, \quad \|G_n P'\| \leqslant \frac{\varepsilon}{\|B\|}, \quad \text{and} \quad \|[A,G_n]\| \leqslant \frac{\varepsilon}{\|B\|^2}.$$

We write $p \sim_\varepsilon q$ if $\|p - q\| \leqslant \varepsilon$, and obtain

$$BG_n \sim_\varepsilon BG_n(I - P') = BG_nAB \sim_\varepsilon BAG_nB = (I-P)G_nB \sim_\varepsilon G_nB.$$

Counting the $\sim$-signs, we find that $\|[B,G_n]\| \leqslant 3\varepsilon$ for sufficiently large n, which finishes the proof since ε was chosen arbitrarily. □

Moreover, this theorem allows us to give a first improvement of Corollary 1.6 concerning the proper $\mathscr{P}$-Fredholmness for operators $A \in \mathscr{L}(\mathbf{X},\mathscr{P})$.

COROLLARY 1.17. *Let $\mathscr{P}$ be a uniform approximate projection on $\mathbf{X}$. Then $A \in \mathscr{L}(\mathbf{X},\mathscr{P})$ is properly $\mathscr{P}$-Fredholm if and only if A is $\mathscr{P}$-Fredholm and has a generalized inverse in $\mathscr{L}(\mathbf{X},\mathscr{P})$. In this case there is a $B \in \mathscr{L}(\mathbf{X},\mathscr{P})$ which is $\mathscr{P}$-regularizer and generalized inverse at the same time.*

Proof. Let $A \in \mathscr{L}(\mathbf{X},\mathscr{P})$ be properly $\mathscr{P}$-Fredholm. Then there is an operator $B \in \mathscr{L}(\mathbf{X})$ such that $A = ABA$, $B = BAB$ and $I - AB, I - BA \in \mathscr{K}(\mathbf{X},\mathscr{P})$ (see Corollary 1.6). From Theorem 1.16 we obtain that $B \in \mathscr{L}(\mathbf{X},\mathscr{P})$.

Let A be $\mathscr{P}$-Fredholm with $\mathscr{P}$-regularizer $C \in \mathscr{L}(\mathbf{X},\mathscr{P})$ and let $B \in \mathscr{L}(\mathbf{X},\mathscr{P})$ be a generalized inverse for A. The operators $P := I - BA, P' := I - AB$ are contained in $\mathscr{L}(\mathbf{X},\mathscr{P})$. Moreover, the relations $P = (I - CA)P + CAP = (I - CA)P$ and $P' = P'(I - AC) + P'AC = P'(I - AC)$ even show that $P, P' \in \mathscr{K}(\mathbf{X},\mathscr{P})$, that is B is a $\mathscr{P}$-regularizer for A. In view of Corollary 1.6, this yields the proper $\mathscr{P}$-Fredholmness. □

Revisiting the $\mathscr{P}$-dichotomy First, let us prove an auxiliary result.

PROPOSITION 1.18. *1. The restriction $A|_{\mathbf{X}_0}$ of each $A \in \mathscr{L}(\mathbf{X},\mathscr{P})$ to $\mathbf{X}_0$ is contained in $\mathscr{L}(\mathbf{X}_0,\mathscr{P})$ and if $K \in \mathscr{K}(\mathbf{X},\mathscr{P})$ then $K|_{\mathbf{X}_0} \in \mathscr{K}(\mathbf{X}_0,\mathscr{P})$.*

2. Let $\mathscr{P} = (P_n)$ an approximate identity, $A \in \mathscr{L}(\mathbf{X},\mathscr{P})$ and $B_{\mathscr{P}} := \sup_n \|P_n\|$. Then

$$B_{\mathscr{P}}^{-2}\|A\|_{\mathscr{L}(\mathbf{X})} \leqslant \|A|_{\mathbf{X}_0}\|_{\mathscr{L}(\mathbf{X}_0)} \leqslant \|A\|_{\mathscr{L}(\mathbf{X})}, \tag{1.5}$$

and for every $T \in \mathscr{K}(\mathbf{X}_0,\mathscr{P})$ there is a lifting $K \in \mathscr{K}(\mathbf{X},\mathscr{P})$ of T, that is $K|_{\mathbf{X}_0} = T$. The restriction (liftings) of $\mathscr{P}$-compact projections are again projections of the same rank.

3. Let $\mathbf{X}_1 \subset \mathbf{X}_0$ be a finite dimensional subspace. Then there exists a projection $P \in \mathscr{K}(\mathbf{X},\mathscr{P})$ onto $\mathbf{X}_1$ of the norm not greater than $2B_{\mathscr{P}} \dim \mathbf{X}_1$.

Proof. 1. Let $A \in \mathscr{L}(\mathbf{X},\mathscr{P})$ and $x \in \mathbf{X}_0$. Since $\|Q_nAx\| \leqslant \|Q_nAP_l\|\|x\| + \|Q_n\|\|A\|\|Q_lx\|$, where the latter term is smaller than any prescribed $\varepsilon > 0$ if l is large enough, and the first term tends to zero for any fixed l and $n \to \infty$, we find that $Ax \in \mathbf{X}_0$. Hence $A|_{\mathbf{X}_0} \in \mathscr{L}(\mathbf{X}_0,\mathscr{P})$. If $K \in \mathscr{K}(\mathbf{X},\mathscr{P})$ then it is also clear by the definition that $K|_{\mathbf{X}_0} \in \mathscr{K}(\mathbf{X}_0,\mathscr{P})$.

2. For $\varepsilon > 0$ there is an $x \in \mathbf{X}$, $\|x\| = 1$ such that $\|A\| \leqslant \|Ax\| + \varepsilon$. Since $\mathscr{P}$ is an approximate identity, there is an $m \in \mathbb{N}$ such that $\|A\| \leqslant \|P_mAx\| + 2\varepsilon$ and for all sufficiently large n we have $\|P_mAQ_n\| \leqslant \varepsilon$.

Thus, $\|A\| \leqslant \|P_mAP_nx\| + 3\varepsilon \leqslant B_{\mathscr{P}}^2\|A|_{\mathbf{X}_0}\| + 3\varepsilon$, where ε was chosen arbitrarily. Now (1.5) easily follows by the obvious estimate $\|A|_{\mathbf{X}_0}\| \leqslant \|A\|$.

Let $T \in \mathscr{K}(\mathbf{X}_0,\mathscr{P})$. From $\|T - P_nTP_n\|_{\mathscr{L}(\mathbf{X}_0)} \to 0$ as $n \to \infty$ we conclude for the sequence $(P_nTP_n) \subset \mathscr{K}(\mathbf{X},\mathscr{P})$ and n,m large that

$$\begin{aligned}\|P_nTP_n - P_mTP_m\|_{\mathscr{L}(\mathbf{X})} &\leqslant B_{\mathscr{P}}^2\|P_nTP_n - P_mTP_m\|_{\mathscr{L}(\mathbf{X}_0)}\\ &\leqslant B_{\mathscr{P}}^2(\|P_nTP_n - P\|_{\mathscr{L}(\mathbf{X}_0)} + \|P - P_mTP_m\|_{\mathscr{L}(\mathbf{X}_0)}),\end{aligned}$$

which tends to zero as $n,m \to \infty$. Hence $(P_nTP_n) \subset \mathscr{K}(\mathbf{X},\mathscr{P})$ is a Cauchy sequence. For its limit $K \in \mathscr{K}(\mathbf{X},\mathscr{P})$ we easily check that the restriction $K|_{\mathbf{X}_0}$ coincides with T. The rest is easy to prove.

3. Recall the projection $P := SRP_m$ from the proof of Proposition 1.7 which obviously fulfills $\|P\| \leqslant \|S\|\|R\|\|P_m\| \leqslant 2kB_{\mathscr{P}}$. □

COROLLARY 1.19. *Let $\mathscr{P}$ be a uniform approximate projection on $\mathbf{X}$ and let $A \in \mathscr{L}(\mathbf{X},\mathscr{P})$ have the $\mathscr{P}$-dichotomy. Then A is Fredholm if and only if $A|_{\mathbf{X}_0}$ is Fredholm. In this case*

$$\dim\ker A = \dim\ker A|_{\mathbf{X}_0}, \quad \dim\operatorname{coker} A = \dim\operatorname{coker} A|_{\mathbf{X}_0}, \quad \textit{and} \quad \operatorname{ind} A = \operatorname{ind} A|_{\mathbf{X}_0}.$$

Proof. If A is not Fredholm then it is properly $\mathscr{P}$-deficient, which also yields the $\mathscr{P}$-deficiency of $A|_{\mathbf{X}_0} \in \mathscr{L}(\mathbf{X}_0,\mathscr{P})$ (with $\mathscr{P}$ regarded as approximate projection on

$\mathbf{X}_0$) by the previous proposition. Hence $A|_{\mathbf{X}_0}$ can not be Fredholm as well (see [34], Theorem 3).

Vice versa, let A be Fredholm and properly $\mathscr{P}$-Fredholm. Then Corollary 1.17 yields an operator B in $\mathscr{L}(\mathbf{X},\mathscr{P})$ such that $ABA=A$, $BAB=B$ holds and, moreover, $P=I-BA$, $P'=I-AB\in\mathscr{K}(\mathbf{X},\mathscr{P})$. Then P is a projection onto the kernel of A and parallel to the range of B (see Proposition 1.4). Analogously, P' is a projection onto the kernel of B and parallel to the range of A. Thus, $\ker B$ is a complement of $\operatorname{im} A$, $\ker A=\operatorname{im} P=\operatorname{im} P|_{\mathbf{X}_0}=\ker A|_{\mathbf{X}_0}$, and $\ker B=\operatorname{im} P'=\operatorname{im} P'|_{\mathbf{X}_0}=\ker B|_{\mathbf{X}_0}$. This proves $\dim\ker A=\dim\ker A|_{\mathbf{X}_0}$. By the previous proposition we find that the compressions also fulfill $A|_{\mathbf{X}_0}B|_{\mathbf{X}_0}A|_{\mathbf{X}_0}=A|_{\mathbf{X}_0}$, and the projection $P'|_{\mathbf{X}_0}=I-A|_{\mathbf{X}_0}B|_{\mathbf{X}_0}$ is onto $\ker B|_{\mathbf{X}_0}$ and parallel to $\operatorname{im} A|_{\mathbf{X}_0}$. Consequently, $\ker B|_{\mathbf{X}_0}$ is a complement of $\operatorname{im} A|_{\mathbf{X}_0}$ and we conclude that $\dim\operatorname{coker} A=\dim\operatorname{coker} A|_{\mathbf{X}_0}$. The rest now easily follows. □

Now we can state a result which guarantees the $\mathscr{P}$-dichotomy for all situations that are of interest within the present paper.

THEOREM 1.20. *Let $\mathscr{P}$ be a uniform approximate identity. Then every operator $A\in\mathscr{L}(\mathbf{X},\mathscr{P})$ has the $\mathscr{P}$-dichotomy if one of the following conditions is fulfilled:*

- *$\mathscr{P}$ is symmetric.*
- *For every properly $\mathscr{P}$-Fredholm $B\in\mathscr{L}(\mathbf{X}_0,\mathscr{P})$ the sequence $(BP_n)_n$ converges $\mathscr{P}$-strongly in $\mathscr{L}(\mathbf{X})$.*
- *For every bounded sequence $(x_n)_n$ in $\mathbf{X}$ with the property that $(P_kx_n)_n$ is a Cauchy sequence for every k there is a $\mathscr{P}$-strong limit $x\in\mathbf{X}$ of (x_n), that is $\|P_m(x-x_n)\|\to 0$ as $n\to\infty$ for all m.*

In terms of [3] the latter condition means that $\mathbf{X}$ is sequentially complete in the strict topology.

Proof. Let $A\in\mathscr{L}(\mathbf{X},\mathscr{P})$. We prepare the proof with some basic observations in the first and a technical result in the second step.

1st step. Let $\hat{\mathscr{P}}=(F_n)$ be a uniform approximate projection given by Theorem 1.15. Notice first that if $\mathscr{P}$ is an approximate identity then, for every fixed k and sufficiently large n, $\|P_kx\|=\|P_kF_nx\|\leqslant B_{\mathscr{P}}\|F_nx\|\leqslant C_{\mathscr{P}}\|F_nx\|$. Hence, we deduce that

$$\limsup_{n\to\infty}\|F_nx\|\geqslant C_{\mathscr{P}}^{-1}\|x\| \quad \text{for all } x\in\mathbf{X}. \tag{1.6}$$

Moreover, for each $x\in\mathbf{X}$ we have $\|Q_nx\|\to 0$ if and only if $\|(I-F_n)x\|\to 0$ as $n\to\infty$. Further we write $m\ll_{\hat{\mathscr{P}}} n$ if $F_k(I-F_l)=(I-F_l)F_k=0$ for all $k\leqslant m$ and all $l\geqslant n$.

2nd step. Suppose that, for given $\varepsilon,\delta>0$ and $x\in\mathbf{X}$ with $\|x\|=1$, we have $\|Ax\|\leqslant\varepsilon$ and $\limsup_n\|(I-F_n)x\|\geqslant\delta$. Then there is an integer N_0 such that, for all $n,m\geqslant N_0$,

$$\|AF_nx\|\leqslant\|[A,F_n]\|+\|F_n\|\|Ax\|\leqslant 2C_{\mathscr{P}}\varepsilon, \quad \text{hence} \quad \|A(F_n-F_m)x\|\leqslant 4C_{\mathscr{P}}\varepsilon.$$

Choose $n_1 \gg_{\hat{\mathscr{P}}} N_0$ such that $\|(I-F_{n_1})x\| \geqslant \delta/2$ and $m_1 \gg_{\hat{\mathscr{P}}} n_1$ such that, due to (1.6),

$$\|(F_{m_1}-F_{n_1})x\| = \|F_{m_1}(I-F_{n_1})x\| \geqslant (2C_{\mathscr{P}})^{-1}\|(I-F_{n_1})x\| \geqslant (4C_{\mathscr{P}})^{-1}\delta.$$

Set $x_1 := \frac{(F_{m_1}-F_{n_1})x}{\|(F_{m_1}-F_{n_1})x\|}$ and fix $N_1 \gg_{\hat{\mathscr{P}}} m_1$.

We iterate this procedure as follows: Suppose that n_k, m_k, N_k, x_k for $k = 1,\dots,l-1$ are given. Then we analogously choose $n_l \gg_{\hat{\mathscr{P}}} N_{l-1}$ such that $\|(I-F_{n_l})x\| \geqslant \delta/2$, and $m_l \gg_{\hat{\mathscr{P}}} n_l$ such that $\|(F_{m_l}-F_{n_l})x\| \geqslant (4C_{\mathscr{P}})^{-1}\delta$, and set $x_l := \frac{(F_{m_l}-F_{n_l})x}{\|(F_{m_l}-F_{n_l})x\|}$ as well as $N_l \gg_{\hat{\mathscr{P}}} m_l$. By doing this, we obtain a set $\{x_1,\dots,x_l\} \subset \mathbf{X}$ and integers $N_0,\dots,N_l$ such that $(F_{N_k}-F_{N_{k-1}})x_i = \delta_{ik}x_i$. For $y = \sum_{i=1}^{l} \alpha_i x_i$ and every $k = 1,\dots,l$ we find

$$\|y\| = \left\|\sum_{i=1}^{l} \alpha_i x_i\right\| \geqslant \frac{1}{\|F_{N_k}-F_{N_{k-1}}\|}\left\|(F_{N_k}-F_{N_{k-1}})\sum_{i=1}^{l}\alpha_i x_i\right\| \geqslant \frac{1}{C_{\mathscr{P}}}\|\alpha_k x_k\| = \frac{1}{C_{\mathscr{P}}}|\alpha_k|$$

and hence

$$\|Ay\| \leqslant \sum_{i=1}^{l}|\alpha_i|\|Ax_i\| \leqslant \sum_{i=1}^{l} C_{\mathscr{P}}\|y\|\frac{4C_{\mathscr{P}}}{\delta}4C_{\mathscr{P}}\varepsilon = \frac{16C_{\mathscr{P}}^3 l}{\delta}\varepsilon\|y\|.$$

Let $N_{l+1} \gg_{\hat{\mathscr{P}}} N_l$ and choose a projection $R \in \mathscr{L}(\ker(I-F_{N_{l+1}}))$ onto the linear space $\operatorname{span}\{x_1,\dots,x_l\} \subset \ker(I-F_{N_l}) \subset \ker(I-F_{N_{l+1}})$ of the norm at most l, due to [19], B.4.9. Then $P := RF_{N_l}$ is a projection of the rank l and $\|AP\| \leqslant \frac{16C_{\mathscr{P}}^4 l^2}{\delta}\varepsilon$. Moreover, P is $\hat{\mathscr{P}}$-compact, hence $\mathscr{P}$-compact.

3rd step. If there is an $x \in \ker A$ such that the norms $\|Q_n x\|$ do not tend to zero then the second step yields that for every k and every $\gamma > 0$ there is a $\mathscr{P}$-compact projection P of the rank k such that $\|AP\| < \gamma$, hence A is properly $\mathscr{P}$-deficient from the right.

If $\|Q_n x\| \to 0$ as $n \to \infty$ for all $x \in \ker A$, i.e. $\ker A \subset \mathbf{X}_0$ then, due to Proposition 1.18, 3. we conclude that for Fredholm operators A there is always a $\mathscr{P}$-compact projection onto its kernel, and if A has an infinite dimensional kernel then it is properly $\mathscr{P}$-deficient.

4th step. Suppose that $A|_{\mathbf{X}_0}$ is not normally solvable, that is for every fixed $\varepsilon > 0$ and $k \in \mathbb{N}$ there is a subspace $\mathbf{X}_1 \subset \mathbf{X}_0$ of the dimension k s.t. $\|A|_{\mathbf{X}_1}\| \leqslant \varepsilon/(2kB_{\mathscr{P}})$. By Proposition 1.18, 3. we can choose a projection $P \in \mathscr{K}(\mathbf{X},\mathscr{P})$ onto $\mathbf{X}_1$ of the norm not greater than $2kB_{\mathscr{P}}$. Then $\|AP\| \leqslant \|A|_{\mathbf{X}_1}\|\|P\| \leqslant \varepsilon$. Since ε and k are arbitrary, we deduce the proper $\mathscr{P}$-deficiency of A from the right.

Now suppose that $A|_{\mathbf{X}_0}$ is normally solvable and A has a finite dimensional kernel contained in $\mathbf{X}_0$ but A is not normally solvable. Let $\mathbf{X}_2$ denote a complement of $\ker A$. Then the operator $A|_{\mathbf{X}_2} : \mathbf{X}_2 \to \mathbf{X}$ is still not normally solvable, but the compression $A|_{\mathbf{X}_3} : \mathbf{X}_3 \to \mathbf{X}_0$ to the complement $\mathbf{X}_3 := \{x \in \mathbf{X}_2 : \|Q_n x\| \to 0 \text{ as } n \to \infty\}$ of $\ker A$ in $\mathbf{X}_0$ is normally solvable, hence $C := \inf\{\|Ax\| : x \in \mathbf{X}_3, \|x\| = 1\} > 0$. Then for every $x \in \mathbf{X}_2$, $\|x\| = 1$ with $\operatorname{dist}(x,\mathbf{X}_3) < 1/4\min\{1, C/\|A\|\}$ we have $\|Ax\| \geqslant C/2$. Consequently, there is a $\delta > 0$ such that for every $\varepsilon > 0$ there is an $x \in \mathbf{X}_2$ with

$\|x\| = 1$, $\limsup_n \|(I - F_n)x\| \geqslant \delta$ and $\|Ax\| \leqslant \varepsilon$, and the second step again yields the proper $\mathscr{P}$-deficiency from the right.

5th step. It remains to consider operators A which are normally solvable and $\dim\ker A < \infty$. Notice that $\dim\operatorname{coker} A = \dim\ker A^*$, and due to the above $A|_{\mathbf{X}_0}$ is normally solvable, too, hence $\dim\operatorname{coker} A|_{\mathbf{X}_0} = \dim\ker(A|_{\mathbf{X}_0})^*$.

Since $\mathscr{P}$ is a uniform and symmetric approximate identity on $\mathbf{X}_0$, the sequence $\mathscr{P}^* = (P_n^*)$ still forms a uniform approximate identity on $\mathbf{X}_0^*$. Thus, we can apply the previous steps to $(A|_{\mathbf{X}_0})^*$ and find, for every $k \leqslant \dim\operatorname{coker} A|_{\mathbf{X}_0}$, a $\mathscr{P}^*$-compact projection T such that $(A|_{\mathbf{X}_0})^* T = 0$. As in the proof of Proposition 1.7 this yields a $\mathscr{P}$-compact projection P' of the rank k such that $P'A|_{\mathbf{X}_0} = 0$. For the lifting $\tilde{P}'$ which is given by Proposition 1.18 we find that also $\tilde{P}'A = 0$ by the estimate

$$\|\tilde{P}'A\| \leqslant \|\tilde{P}'F_nA\| + \|\tilde{P}'(I - F_n)A\| \leqslant \|P'AF_n\| + \|\tilde{P}'\|\|[F_n, A]\| + \|\tilde{P}'(I - F_n)\|\|A\|, \tag{1.7}$$

where the first term equals zero and the second and last one tend to zero as $n \to \infty$. Thus, if $\dim\operatorname{coker} A|_{\mathbf{X}_0} = \infty$ then $A|_{\mathbf{X}_0}$ and A are properly $\mathscr{P}$-deficient from the left. Moreover, if $\dim\operatorname{coker} A|_{\mathbf{X}_0} < \infty$ then $A|_{\mathbf{X}_0}$ is properly $\mathscr{P}$-Fredholm, and if

$$\dim\operatorname{coker} A = \dim\operatorname{coker} A|_{\mathbf{X}_0}$$

then A is properly $\mathscr{P}$-Fredholm as well.

6th step. Let the second condition of the theorem be fulfilled and let $\dim\ker A < \infty$ as well as $\dim\operatorname{coker} A|_{\mathbf{X}_0} < \infty$. Then $A|_{\mathbf{X}_0}$ is properly $\mathscr{P}$-Fredholm and we can choose a $\mathscr{P}$-regularizer B for $A|_{\mathbf{X}_0}$ such that $P' := I - AB \in \mathscr{K}(\mathbf{X}_0, \mathscr{P})$ is a projection parallel to the range of $A|_{\mathbf{X}_0}$. Then, let $\tilde{B}$ denote the $\mathscr{P}$-strong limit of $(BP_n)_n$ which is in $\mathscr{L}(\mathbf{X}, \mathscr{P})$ (see Proposition 1.12). Further, let $\tilde{P}'$ denote the lifting of P'. By the first step we find for large m and all $x \in \mathbf{X}$, $\|x\| = 1$ that

$$\begin{aligned}
\|(I - \tilde{P}' - A\tilde{B})x\| &\leqslant 2C_{\mathscr{P}} \|F_m(I - \tilde{P}' - A\tilde{B})\| \\
&\leqslant 2C_{\mathscr{P}} \|F_m((I - P')P_n - A\tilde{B})\| + 2C_{\mathscr{P}} \|F_m(I - \tilde{P}')Q_n\| \\
&= 2C_{\mathscr{P}} \|F_mA(BP_n - \tilde{B})\| + 2C_{\mathscr{P}} \|F_m(I - \tilde{P}')Q_n\|.
\end{aligned}$$

Since both items tend to zero as $n \to \infty$ we see that $\tilde{P}' = I - A\tilde{B}$. Together with $\tilde{P}'A = 0$ due to (1.7) this yields that $\tilde{P}'$ is a projection parallel to $\operatorname{im} A$. Hence A is Fredholm and $\mathscr{P}$-Fredholm by step 3 and the assertion of the theorem is proven.

7th step. If $\mathscr{P}$ is a uniform symmetric approximate identity on $\mathbf{X}$ then the proof of the fifth step also works for $\mathbf{X}$ instead of $\mathbf{X}_0$, and the assertion of the theorem is proven as well.

8th step. Let the last condition of the theorem be fulfilled and let $B \in \mathscr{L}(\mathbf{X}_0, \mathscr{P})$ be properly $\mathscr{P}$-Fredholm. Then $(BP_n)_n$ is bounded and $(P_kBP_n)_n$ is a Cauchy sequence in $\mathscr{L}(\mathbf{X}, \mathscr{P})$ for every fixed k, since

$$\|P_kBP_n - P_kBP_m\| \leqslant \|P_kBQ_l\|\|P_n - P_m\|$$

for $n, m \gg l$, and $\|P_kBQ_l\| \to 0$ as $l \to \infty$, hence there are uniform limits $B_k \in \mathscr{L}(\mathbf{X}, \mathscr{P})$ of $(P_kBP_n)_{n\in\mathbb{N}}$ for each k. Moreover, for each $x \in \mathbf{X}$, there is a $\mathscr{P}$-strong limit

$$\tilde{B}x := \mathscr{P}\text{-}\lim_{n\to\infty} BP_nx \quad \text{in} \quad \mathbf{X},$$

and the mapping $x \mapsto \tilde{B}x$ defines a bounded linear operator $\tilde{B}$. Obviously, $\tilde{B}|_{\mathbf{X}_0} = B$ and by

$$\|(P_k\tilde{B} - B_k)x\| \leqslant \|P_k(\tilde{B} - BP_n)x\| + \|P_kBP_n - B_k\|\|x\| \to 0 \text{ as } n \to 0,$$

for every $x \in \mathbf{X}$, we see that $P_k\tilde{B} = B_k$. Hence

$$\|P_k(\tilde{B} - BP_n)\| = \|B_k - P_kBP_n\| \to 0 \text{ as } n \to \infty,$$
$$\|(\tilde{B} - BP_n)P_k\| = \|(\tilde{B} - B)P_k\| = 0 \text{ for } n \gg k.$$

Thus, (BP_n) converges $\mathscr{P}$-strongly to $\tilde{B}$, that is the second condition in the theorem is fulfilled. □

The conditions in this theorem only determine properties of the space $\mathbf{X}$ and the approximate projection $\mathscr{P}$ and if one of them is fulfilled then every operator in $\mathscr{L}(\mathbf{X}, \mathscr{P})$ has the $\mathscr{P}$-dichotomy. That is why one may say that a Banach space $\mathbf{X}$ has the $\mathscr{P}$-dichotomy if each operator $A \in \mathscr{L}(\mathbf{X}, \mathscr{P})$ has the $\mathscr{P}$-dichotomy. So, we see that for uniform approximate identities $\mathscr{P}$ the space $\mathbf{X}$ has the $\mathscr{P}$-dichotomy, if $\mathbf{X}$ is small (in the sense that $\mathscr{P}$ should be symmetric) or large (in the sense of being sequentially complete in the strict topology). Until now it is not clear if there is really a gap between these two extremal cases. The exact formulation of this open question is as follows:

Are there a Banach space $\mathbf{X}$, a uniform approximate identity $\mathscr{P}$ on $\mathbf{X}$ and an operator $A \in \mathscr{L}(\mathbf{X}, \mathscr{P})$ which is normally solvable, and fulfills $\dim\ker A < \infty$ as well as $\dim\operatorname{coker} A|_{\mathbf{X}_0} < \dim\operatorname{coker} A$?

REMARK 1.21. In the Fredholm theory for band-dominated operators on l^∞ the existence of a predual space $\mathbf{Y}$ and preadjoint operators $R_n, B \in \mathscr{L}(\mathbf{Y})$ for (P_n) and A (i.e. $\mathbf{Y}^* = \mathbf{X}$, $R_n^* = P_n$ for all $n \in \mathbb{N}$ and $B^* = A$) played an important role, e.g. in [15] and [3], to ensure that the Fredholm properties of A and $A|_{\mathbf{X}_0}$ coincide (for details see Section 1.4). Now we know from Corollary 1.19 that the $\mathscr{P}$-dichotomy is already sufficient for this goal. Just to complete the picture, we additionally note that the existence of such a predual setting guarantees the $\mathscr{P}$-dichotomy. Indeed, (R_m) is a symmetric uniform approximate projection (and hence a symmetric uniform approximate identity due to Remark 1.11) since $\|(R_n - I)R_m\| = \|P_m(P_n - I)\|$, $\|R_m(R_n - I)\| = \|(P_n - I)P_m\|$, and $\|R_U\| = \|P_U\|$ for all m, n and all bounded $U \subset \mathbb{R}$. Thus, the operator B has the (R_n)-dichotomy by Theorem 1.20 which easily implies the $\mathscr{P}$-dichotomy for A.

1.3. Limit operators

Let $\mathbf{X}$ be a Banach space and $\mathscr{P} = (P_n)$ be an approximate identity. In the following we present an extension of [25], Section 1.2 with a special emphasis on the importance of the new notion "$\mathscr{P}$-dichotomy".

Suppose that K is a positive integer and that there is a bounded family $\mathscr{V} = (V_k)_{k \in \mathbb{Z}^K}$ of operators $V_k \in \mathscr{L}(\mathbf{X})$ such that $V_kV_l = V_{k+l}$ for all $k, l \in \mathbb{Z}^K$ and $V_0 = I$.

Further we assume that $\mathscr{P}$ and $\mathscr{V}$ are related as follows: For every $m,n,l \in \mathbb{N}$ and $r \in \mathbb{Z}^K$

$$\text{there is an } R > 0 \text{ such that } P_m V_k P_n = 0 \text{ for all } |k| > R, \tag{1.8}$$

$$\text{there is an } i_0 \in \mathbb{N} \text{ such that } P_l V_r Q_i = Q_i V_r P_l = 0 \text{ for all } i \geqslant i_0. \tag{1.9}$$

The latter condition ensures that $\mathscr{V} \subset \mathscr{L}(\mathbf{X}, \mathscr{P})$ due to Theorem 1.2.

Finally, $\mathscr{H}$ stands for the set of all sequences $h : \mathbb{N} \to \mathbb{Z}^K$ which tend to infinity in the sense that for every $R > 0$ there is an m_0 such that $|h(m)| > R$ for all $m \geqslant m_0$.

DEFINITION 1.22. Let $A \in \mathscr{L}(\mathbf{X}, \mathscr{P})$, and let $h \in \mathscr{H}$. The operator $A_h \in \mathscr{L}(\mathbf{X})$ is called limit operator of A w.r.t. h if $A_h = \mathscr{P}\text{-}\lim_{m\to\infty} V_{-h(m)} A V_{h(m)}$. The set $\sigma_{\mathrm{op}}(A)$ of all limit operators of A is called the operator spectrum of A.

Notice that the operator spectrum $\sigma_{\mathrm{op}}(A)$ of $A \in \mathscr{L}(\mathbf{X}, \mathscr{P})$ is automatically part of $\mathscr{L}(\mathbf{X}, \mathscr{P})$, that an infinite subsequence g of $h \in \mathscr{H}$ again belongs to $\mathscr{H}$, and if the limit operator A_h of $A \in \mathscr{L}(\mathbf{X}, \mathscr{P})$ exists then A_g exists, too, and equals A_h. Furthermore, the operator spectrum of a $\mathscr{P}$-compact operator always equals $\{0\}$, as [25], Proposition 1.2.6 shows.

PROPOSITION 1.23. (cf. also [25], Proposition 1.2.9)
Let $\mathscr{P}$ be a uniform approximate identity on $\mathbf{X}$ and $A \in \mathscr{L}(\mathbf{X}, \mathscr{P})$ be $\mathscr{P}$-Fredholm.

- *All limit operators of A which have the $\mathscr{P}$-dichotomy are invertible and their inverses are uniformly bounded.*
- *If B is a $\mathscr{P}$-regularizer for A and $A_g \in \sigma_{\mathrm{op}}(A)$ has the $\mathscr{P}$-dichotomy then B_g exists and equals A_g^{-1}.*

Proof. Let $B \in \mathscr{L}(\mathbf{X}, \mathscr{P})$ be a $\mathscr{P}$-regularizer for A and let $T_1, T_2 \in \mathscr{K}(\mathbf{X}, \mathscr{P})$ be such that $BA - I = T_1$ and $AB - I = T_2$. Further fix a sequence $h \in \mathscr{H}$ such that A_h exists. Then, for every $K \in \mathscr{K}(\mathbf{X}, \mathscr{P})$

$$\|K\| = \|V_{-h(n)} I V_{h(n)} K\| \leqslant \|V_{-h(n)} B V_{h(n)}\| \|V_{-h(n)} A V_{h(n)} K\| + \|V_{-h(n)} T_1 V_{h(n)} K\|,$$

and consequently (for a certain constant $D > 0$ independent from h, and $n \to \infty$)

$$\|K\| \leqslant D \|A_h K\| \quad \text{for all} \quad K \in \mathscr{K}(\mathbf{X}, \mathscr{P}).$$

Analogously,

$$\|K\| \leqslant D \|K A_h\| \quad \text{for all} \quad K \in \mathscr{K}(\mathbf{X}, \mathscr{P}).$$

If A_h has the $\mathscr{P}$-dichotomy then these estimates guarantee that A_h is not properly $\mathscr{P}$-deficient, hence Fredholm and properly $\mathscr{P}$-Fredholm. Since we can choose a $\mathscr{P}$-compact projection onto $\ker A$ we conclude again from these estimates that this projection must be the zero operator. Thus A_h is injective. By the same means we find that A_h is surjective as well, that is invertible. Since $A_h \in \mathscr{L}(\mathbf{X}, \mathscr{P})$, Theorem 1.16 yields $A_h^{-1} \in \mathscr{L}(\mathbf{X}, \mathscr{P})$, too.

Fix $\varepsilon > 0$ and choose $x \in \mathbf{X}$, $\|x\| = 1$ such that $\|A_h^{-1}\| \leqslant \|A_h^{-1}x\| + \varepsilon$. Further, due to the fact that $\mathscr{P}$ is an approximate identity, fix m such that $\|A_h^{-1}x\| \leqslant \|P_m A_h^{-1}x\| + \varepsilon$ and let n be such that $\|P_m A_h^{-1} Q_n\| \leqslant \varepsilon$, by Theorem 1.2. Finally notice that for every $y \in \mathbf{X}_0$ there is a projection $R_y \in \mathscr{K}(\mathbf{X}_0, \mathscr{P})$ of the norm 1 onto span$\{y\}$, and therefore the first of the above estimates yields for the operator $A_h^{-1}\tilde{R}_y \in \mathscr{K}(\mathbf{X}, \mathscr{P})$ (where $\tilde{R}_y$ is the lifting of R_y given by Proposition 1.18) that $\|A_h^{-1}\tilde{R}_y\| \leqslant D\|A_h A_h^{-1}\tilde{R}_y\| = B_{\mathscr{P}}^2 D$. Now we conclude

$$\|A_h^{-1}\| \leqslant \|P_m A_h^{-1} P_n x\| + 3\varepsilon \leqslant B_{\mathscr{P}} \|A_h^{-1}\tilde{R}_{P_n x}\| \|P_n x\| + 3\varepsilon \leqslant B_{\mathscr{P}}^4 D + 3\varepsilon.$$

This proves the uniform boundedness. Furthermore, we have

$$\begin{aligned} &V_{-h(n)} B V_{h(n)} - (A_h)^{-1} \\ &\quad = V_{-h(n)} B V_{h(n)} (I - V_{-h(n)} A V_{h(n)} (A_h)^{-1}) + V_{-h(n)} T_1 V_{h(n)} (A_h)^{-1} \\ &\quad = V_{-h(n)} B V_{h(n)} [A_h - V_{-h(n)} A V_{h(n)}] (A_h)^{-1} + V_{-h(n)} T_1 V_{h(n)} (A_h)^{-1} \end{aligned}$$

and hence $\|(V_{-h(n)} B V_{h(n)} - (A_h)^{-1}) J\| \to 0$ for every $J \in \mathscr{K}(\mathbf{X}, \mathscr{P})$. Analogously, we can show that $\|J(V_{-h(n)} B V_{h(n)} - (A_h)^{-1})\| \to 0$ and obtain the $\mathscr{P}$-strong convergence of $V_{-h(n)} B V_{h(n)}$ to $(A_h)^{-1}$. □

COROLLARY 1.24. *Let $\mathscr{P}$ be a uniform approximate identity on $\mathbf{X}$, suppose that $\mathbf{X}$ has the $\mathscr{P}$-dichotomy, and let $A \in \mathscr{L}(\mathbf{X}, \mathscr{P})$. Then $\sigma(B) \subset \sigma_{\mathrm{ess}}(A) \subset \sigma(A)$ for all $B \in \sigma_{\mathrm{op}}(A)$.*

Proof. Let $\lambda \in \mathbb{C}$. Then $B - \lambda I$ belongs to $\sigma_{\mathrm{op}}(A - \lambda I)$ if and only if $B \in \sigma_{\mathrm{op}}(A)$. We only note that each Fredholm operator is $\mathscr{P}$-Fredholm due to the $\mathscr{P}$-dichotomy, hence its limit operators are invertible. □

Rich operators

DEFINITION 1.25. An operator $A \in \mathscr{L}(\mathbf{X}, \mathscr{P})$ is called an operator with rich operator spectrum (or simply a rich operator) if every sequence $h \in \mathscr{H}$ contains an infinite subsequence g such that the limit operator A_g exists.

From Proposition 1.2.6 in [25] we recall that the set $\mathscr{L}^{\$}(\mathbf{X}, \mathscr{P})$ of all rich operators forms a Banach subalgebra of $\mathscr{L}(\mathbf{X}, \mathscr{P})$ and, moreover, $\mathscr{K}(\mathbf{X}, \mathscr{P}) \subset \mathscr{L}^{\$}(\mathbf{X}, \mathscr{P})$.

COROLLARY 1.26. *Let $\mathscr{P}$ be a uniform approximate identity on $\mathbf{X}$ and further suppose that $\mathbf{X}$ has the $\mathscr{P}$-dichotomy. Then every $\mathscr{P}$-regularizer of a rich $\mathscr{P}$-Fredholm operator is rich. In particular, $\mathscr{L}^{\$}(\mathbf{X}, \mathscr{P})$ is inverse closed in $\mathscr{L}(\mathbf{X}, \mathscr{P})$ and $\mathscr{L}(\mathbf{X})$.*

Proof. Let $B \in \mathscr{L}(\mathbf{X})$ be a $\mathscr{P}$-regularizer for $A \in \mathscr{L}^{\$}(\mathbf{X}, \mathscr{P})$. Theorem 1.16 tells us that B belongs to $\mathscr{L}(\mathbf{X}, \mathscr{P})$. Let $h \in \mathscr{H}$ and g be a subsequence of h such that A_g exists. Then Proposition 1.23 provides $B_g = A_g^{-1}$. □

1.4. An Example: l^p-spaces and band-dominated operators

Let X stand for a fixed complex Banach space, $K \in \mathbb{N}$, and let $l^p = l^p(\mathbb{Z}^K, X)$ denote the Banach space of all functions $f : \mathbb{Z}^K \to X$ such that

$$\|f\|_{l^p}^p := \sum_{x \in \mathbb{Z}^K} \|f(x)\|_X^p < \infty.$$

We further introduce $l^\infty = l^\infty(\mathbb{Z}^K, X)$ as the Banach space of all functions f with

$$\|f\|_{l^\infty} := \sup_{x \in \mathbb{Z}^K} \|f(x)\|_X < \infty,$$

and $l^0 = l^0(\mathbb{Z}^K, X)$ as the closed subspace of all functions $f \in l^\infty$ with

$$\lim_{|x| \to \infty} \|f(x)\|_X = 0.$$

Every function $a \in l^\infty(\mathbb{Z}^K, \mathscr{L}(X))$ gives rise to an operator $aI \in \mathscr{L}(l^p)$ (a so-called multiplication operator) via

$$(af)(x) = a(x)f(x), \quad x \in \mathbb{Z}^K.$$

Evidently, $\|aI\|_{\mathscr{L}(l^p)} = \|a\|_\infty$. By this means, the functions in $l^\infty(\mathbb{Z}^K, \mathbb{C})$ induce multiplication operators as well.

Let $\Omega \subset \mathbb{R}^K$ be a compact and convex polytope with vertices in $\mathbb{Z}^K$ and suppose that $0 \in \mathbb{Z}^K$ is an inner point of Ω. Further, let $\hat{\chi}_{m\Omega}$ denote the characteristic function of $m\Omega \cap \mathbb{Z}^K$ and set

$$P_m := \hat{\chi}_{m\Omega} I \quad (m \in \mathbb{N}).$$

Obviously, all $P_m \in \mathscr{L}(l^p)$ are projections with $\|P_m\| = 1$ and $\mathscr{P} := (P_n)_{n \in \mathbb{N}}$ is a uniform approximate identity. Further, notice that every two approximate projections which are due to this definition are always equivalent, that is the classes of $\mathscr{P}$-compact or $\mathscr{P}$-Fredholm operators as well as the notion of $\mathscr{P}$-strong convergence do not depend on the concrete choice of Ω. Finally, we introduce limit operators as in Section 1.3, based on the family $\mathscr{V} = (V_\alpha)_{\alpha \in \mathbb{Z}^K}$ of shift operators V_α given by the rule

$$(V_\alpha f)(x) = f(x - \alpha), \quad x \in \mathbb{Z}^K.$$

PROPOSITION 1.27. *All spaces l^p have the $\mathscr{P}$-dichotomy.*

Proof. With the identifications $(l^p(\mathbb{Z}^K, X))^* = l^q(\mathbb{Z}^K, X^*)$ for $1 < p < \infty$ and $1/p + 1/q = 1$, $(l^0(\mathbb{Z}^K, X))^* = l^1(\mathbb{Z}^K, X^*)$, as well as $(l^1(\mathbb{Z}^K, X))^* = l^\infty(\mathbb{Z}^K, X^*)$ we easily deduce that $\mathscr{P}$ is symmetric in all cases $p \neq \infty$, hence Theorem 1.20 applies. Furthermore, the spaces l^∞ fulfill the last condition in Theorem 1.20 (that is they are sequentially complete in the strict topology). $\square$

THEOREM 1.28. *Let* $A \in \mathcal{L}(l^p, \mathcal{P})$.

1. *A is $\mathcal{P}$-Fredholm if and only if it is invertible at infinity. In this case all operators $B \in \sigma_{\text{op}}(A)$ are invertible and their inverses are uniformly bounded.*

2. *If A is Fredholm then A is properly $\mathcal{P}$-Fredholm, hence $\mathcal{P}$-Fredholm. In case $\dim X < \infty$ Fredholmness and $\mathcal{P}$-Fredholmness are equivalent.*

3. *For every $B \in \sigma_{\text{op}}(A)$ it holds that $\sigma(B) \subset \sigma_{\text{ess}}(A) \subset \sigma(A)$.*

4. *Let $p = \infty$. An operator $A \in \mathcal{L}(l^\infty, \mathcal{P})$ is Fredholm if and only if its compression $A|_{l^0}$ to the space l^0 is Fredholm. In this case*

$$\dim\ker A = \dim\ker A|_{l^0},\ \dim\operatorname{coker} A = \dim\operatorname{coker} A|_{l^0},\ \textit{and}\ \operatorname{ind} A = \operatorname{ind} A|_{l^0}.$$

This is an immediate consequence of Theorem 1.16, the Corollaries 1.19 and 1.24, and the fact that in case $\dim X < \infty$ all $\mathcal{P}$-compact operators are compact (in the usual sense). Notice that these results, or at least parts of them, are already known, but only under certain additional conditions, like $1 < p < \infty$, $\dim X < \infty$, or A being band-dominated (see [25] or [15]). A recent discussion on the latter case can be found in [3], Theorem 6.28. Particularly, the last assertion clarifies the open problem no. 4 which was formulated in [3]. A very prominent example for an operator in $\mathcal{L}(l^p, \mathcal{P})$ not being band-dominated is the so-called flip operator J given by the rule $(Ju)_\alpha = u_{-\alpha}$.

DEFINITION 1.29. A band operator A is a finite sum of the form $A = \sum a_\alpha V_\alpha$, where $\alpha \in \mathbb{Z}^K$ and $a_\alpha \in l^\infty(\mathbb{Z}^K, \mathcal{L}(X))$. An operator is called band-dominated if it is the uniform limit of a sequence of band operators. We denote the class of all band-dominated operators by $\mathcal{A}_{l^p}$.

Here is a collection of important properties of band-dominated operators:

THEOREM 1.30. (see [25], Propositions 2.1.7 et seq.)

1. *$\mathcal{A}_{l^p} \subset \mathcal{L}(l^p, \mathcal{P}) \subset \mathcal{L}(l^p)$ are closed algebras as well as inverse closed.*

2. *The set $\mathcal{K} := \mathcal{K}(l^p, \mathcal{P})$ of all $\mathcal{P}$-compact operators is a closed ideal in $\mathcal{A}_{l^p}$.*

3. *$\mathcal{A}_{l^p}/\mathcal{K}$ is inverse closed in the quotient algebra $\mathcal{L}(l^p, \mathcal{P})/\mathcal{K}$.*

THEOREM 1.31. ([25], Theorem 2.1.6 and [15], Theorem 1.42)
An operator $A \in \mathcal{L}(l^p)$ is band-dominated if and only if the following holds: For every $\varepsilon > 0$, there exists an $M > 0$, such that whenever F, G are subsets of $\mathbb{Z}^K$ with $\operatorname{dist}(F,G) := \inf\{\|x-y\| : x \in F, y \in G\} > M$, *then* $\|\chi_F A \chi_G I\|_{\mathcal{L}(l^p)} < \varepsilon$.

There is another equivalent characterization of the $\mathcal{P}$-Fredholm property of rich band-dominated operators which improves Theorem 1.28.

THEOREM 1.32. *Let $A \in \mathcal{A}_{l^p}$ be rich. Then A is $\mathcal{P}$-Fredholm if and only if all limit operators of A are invertible and their inverses are uniformly bounded.*

The "only if" part obviously follows from Theorem 1.28. It was discussed in [24] and [25], Theorem 2.2.1 (for $1 < p < \infty$), in [15] (all p and with an additional assumption on the existence of a predual setting in case $p = \infty$), and in [3], Theorem 6.28 (all p). The proof of the "if" part is based on a construction of a $\mathscr{P}$-regularizer, which goes back to Simonenko [35] and can be found in [24] and [15]. Particularly notice also the pioneering paper [13] of Lange and Rabinovich.

2. Sequence algebras, Fredholm sequences and approximation numbers

We now turn our attention to sequences of operators having a certain asymptotic structure. This structure finds expression in the existence of $\mathscr{P}$-strong limits which we will call snapshots. Such sequences naturally emerge from various approximation methods. We establish a Fredholm theory for such sequences and, besides a stability criterion, we obtain a deeper understanding of the connections between the operators of a sequence and its snapshots. The results in this section are generalizations of those in [34], Section 1.2. Here we drop the assumption on the operators of the sequences to act on *finite* dimensional spaces.

2.1. Sequence algebras

Let $(\mathbf{E}_n)$ be a sequence of Banach spaces and let L_n stand for the identity operator on $\mathbf{E}_n$, respectively. We denote by $\mathscr{F}$ the set of all bounded sequences $\{A_n\}$[1] of bounded linear operators $A_n \in \mathscr{L}(\mathbf{E}_n)$. Provided with the operations

$$\alpha\{A_n\} + \beta\{B_n\} := \{\alpha A_n + \beta B_n\}, \quad \{A_n\}\{B_n\} := \{A_n B_n\},$$

and the supremum norm $\|\{A_n\}\|_{\mathscr{F}} := \sup_n \|A_n\|_{\mathscr{L}(\mathbf{E}_n)} < \infty$, $\mathscr{F}$ becomes a Banach algebra with identity $\mathbb{I} := \{L_n\}$. The set

$$\mathscr{G} := \{\{G_n\} \in \mathscr{F} : \|G_n\|_{\mathscr{L}(\mathbf{E}_n)} \to 0 \text{ as } n \to \infty\}$$

forms a closed ideal in $\mathscr{F}$.

Further, let T be a (possibly infinite) index set and suppose that, for every $t \in T$, there is a Banach space $\mathbf{E}^t$ with an approximate identity $\mathscr{P}^t$ and a bounded sequence (L_n^t) of projections $L_n^t \in \mathscr{L}(\mathbf{E}^t, \mathscr{P}^t)$ tending $\mathscr{P}^t$-strongly to the identity I^t on $\mathbf{E}^t$. Set

$$c^t := \sup\{\|L_n^t\|_{\mathscr{L}(\mathbf{E}^t)} : n \in \mathbb{N}\} < \infty \text{ for every } t \in T.$$

Further suppose that, for every $t \in T$, there is a sequence (E_n^t) of isomorphisms

$$E_n^t : \mathscr{L}(\operatorname{im} L_n^t) \to \mathscr{L}(\mathbf{E}_n),$$

such that (for brevity, we write E_n^{-t} instead of $(E_n^t)^{-1}$)

$$M^t := \sup\{\|E_n^t\|, \|E_n^{-t}\| : n \in \mathbb{N}\} < \infty. \tag{I}$$

[1] We continue to use $(\cdot)$ for sequences of elements in one common space, whereas the sequences in $\mathscr{F}$, which consist of elements coming from different spaces $\mathbf{E}_n$ in general, are denoted by $\{\cdot\}$.

We denote by $\mathscr{F}^T$ the collection of all sequences $\mathbb{A} = \{A_n\} \in \mathscr{F}$, for which there exist operators $W^t(\mathbb{A}) \in \mathscr{L}(\mathbf{E}^t, \mathscr{P}^t)$ for all $t \in T$ such that

$$A_n^{(t)} := E_n^{-t}(A_n)L_n^t \to W^t(\mathbb{A}) \quad \mathscr{P}^t\text{-strongly.}$$

These limits are uniquely determined and with the help of Theorem 1.13 it is easy to show that $\mathscr{F}^T$ is a closed subalgebra of $\mathscr{F}$ which contains the identity and the ideal $\mathscr{G}$. Both, the mappings E_n^t and $W^t : \mathscr{F}^T \to \mathscr{L}(\mathbf{E}^t, \mathscr{P}^t), \mathbb{A} \mapsto W^t(\mathbb{A})$ are unital homomorphisms.

Roughly speaking, the maps E_n^t allow us to transform a given sequence $\mathbb{A} \in \mathscr{F}^T$ and to generate snapshots $W^t(\mathbb{A})$ from different angles which outline several aspects of the asymptotic behavior of $\mathbb{A}$. In what follows, we will examine the connections between the properties of $\mathbb{A}$ and the properties of its "snapshots at infinity".

REMARK 2.1. The results of the subsequent sections remain true, if for some or all $t \in T$ the sequence (L_n^t) of projections converges $*$-strongly to the identity, and we replace $\mathscr{P}^t$-Fredholmness by Fredholmness, $\mathscr{P}^t$-compactness by compactness, and $\mathscr{P}^t$-strong convergence by $*$-strong convergence. An approximate projection $\mathscr{P}^t$ is not needed in this case.

Thus, the classical results for C^*-algebras (see [8]) and for Banach algebras of matrix sequences in [32] are completely covered by the present considerations.

2.2. $\mathscr{J}^T$-Fredholm sequences

In all what follows, we suppose that the separation condition

$$W^\tau\{E_n^t(L_n^t K^t)\} = \begin{cases} K^t & \text{if } t = \tau \\ 0 & \text{if } t \neq \tau \end{cases} \tag{II}$$

holds for all $\tau, t \in T$ and every $K^t \in \mathscr{K}^t := \mathscr{K}(\mathbf{E}^t, \mathscr{P}^t)$. This condition is very natural, since it guarantees that the angles from which we look at a sequence are sufficiently different in a sense and, by this, it prevents redundant snapshots. We put

$$\mathscr{J}^t := \{\{E_n^t(L_n^t K^t)\} + \{G_n\} : K^t \in \mathscr{K}^t, \|G_n\| \to 0\} \quad (\forall\, t \in T),$$

$$\mathscr{J}^T := \operatorname{clos}_{\mathscr{F}^T} \left\{ \sum_{i=1}^m \{J_n^{t_i}\} : m \in \mathbb{N},\, t_i \in T,\, \{J_n^{t_i}\} \in \mathscr{J}^{t_i} \right\}$$

and as in [34], Proposition 14 we check that the sets $\mathscr{J}^t$ and $\mathscr{J}^T$ are closed ideals in $\mathscr{F}^T$.

DEFINITION 2.2. A sequence $\mathbb{A} \in \mathscr{F}^T$ is said to be $\mathscr{J}^T$-Fredholm, or regularizable with respect to $\mathscr{J}^T$, if the coset $\mathbb{A} + \mathscr{J}^T$ is invertible in the quotient algebra $\mathscr{F}^T / \mathscr{J}^T$.

Notice that this property depends on the underlying algebra $\mathscr{F}^T$ and the ideal $\mathscr{J}^T$. It is obvious that the set of $\mathscr{J}^T$-Fredholm sequences is open in $\mathscr{F}^T$, the sum of a $\mathscr{J}^T$-Fredholm sequence and a sequence from the ideal $\mathscr{J}^T$ is $\mathscr{J}^T$-Fredholm and that the product of two $\mathscr{J}^T$-Fredholm sequences is $\mathscr{J}^T$-Fredholm again.

The proof for the next result on the regularization of $\mathscr{J}^T$-Fredholm sequences can again be taken from [34], Proposition 16.

PROPOSITION 2.3. *Let $\mathbb{A} \in \mathscr{F}^T$ be $\mathscr{J}^T$-Fredholm. Then there exist finite subsets $\{t_1,...,t_m\}$ and $\{\tau_1,...,\tau_l\}$ of T and a $\delta > 0$ such that the following holds: For each $\tilde{\mathbb{A}} \in \mathscr{F}^T$ with $\|\mathbb{A} - \tilde{\mathbb{A}}\| < \delta$ there are sequences $\mathbb{B}, \mathbb{C} \in \mathscr{F}^T$ and $\mathbb{G}, \mathbb{H} \in \mathscr{G}$ as well as operators $K^{t_i} \in \mathscr{K}^{t_i}$ and $K^{\tau_i} \in \mathscr{K}^{\tau_i}$ such that*

$$\mathbb{B}\tilde{\mathbb{A}} = \mathbb{I} + \sum_{i=1}^{m} \{E_n^{t_i}(L_n^{t_i} K^{t_i})\} + \mathbb{G}, \tag{2.1}$$

$$\tilde{\mathbb{A}}\mathbb{C} = \mathbb{I} + \sum_{i=1}^{l} \{E_n^{\tau_i}(L_n^{\tau_i} K^{\tau_i})\} + \mathbb{H}. \tag{2.2}$$

Applying the homomorphisms $W^t, t \in T$ to the Equations (2.1) and (2.2), and utilizing the separation condition, we see that the following theorem is in force.

THEOREM 2.4. *If a sequence $\mathbb{A} \in \mathscr{F}^T$ is $\mathscr{J}^T$-Fredholm, then all corresponding operators $W^t(\mathbb{A})$ are $\mathscr{P}^t$-Fredholm on $\mathbf{E}^t$ and the number of the non-invertible operators among them is finite.*

REMARK 2.5. Let $\mathbb{A}$ be a $\mathscr{J}^T$-Fredholm sequence. Then every snapshot $W^t(\mathbb{A})$ of $\mathbb{A}$ is $\mathscr{P}^t$-Fredholm by the previous theorem, and therefore has the $\mathscr{P}^t$-dichotomy, by Corollary 1.9. This simplifies the results of [34] in the sense, that there a $\mathscr{J}^T$-Fredholm sequence automatically has the $\mathscr{P}$-dichotomy.

2.2.1. The lower approximation numbers of A_n and the operators $W^t\{A_n\}$

For Banach spaces $\mathbf{X}, \mathbf{Y}$ and an operator $A \in \mathscr{L}(\mathbf{X}, \mathbf{Y})$ the k-th approximation number from the right $s_k^r(A)$ and the k-th approximation number from the left $s_k^l(A)$ of A are defined as

$$s_k^r(A) := \inf\{\|A - F\|_{\mathscr{L}(\mathbf{X},\mathbf{Y})} : F \in \mathscr{L}(\mathbf{X}, \mathbf{Y}), \dim\ker F \geqslant k\},$$
$$s_k^l(A) := \inf\{\|A - F\|_{\mathscr{L}(\mathbf{X},\mathbf{Y})} : F \in \mathscr{L}(\mathbf{X}, \mathbf{Y}), \dim\operatorname{coker} F \geqslant k\},$$

$(k = 0, 1, 2, ...)$, respectively. It is clear that $0 = s_0^r(A) \leqslant s_1^r(A) \leqslant s_2^r(A) \leqslant ...$ and that the same holds true for the sequence $(s_k^l(A))$. Here is a further connection between the operators A_n of a sequence $\{A_n\} \in \mathscr{F}^T$ and its snapshots $W^t\{A_n\}$.

THEOREM 2.6. *Let $\mathbb{A} = \{A_n\} \in \mathscr{F}^T$, $m \in \mathbb{N}$ and $t_1, \ldots, t_m \in T$ be such that all $W^{t_i}(\mathbb{A})$ are invertible at infinity. Then $s_k^r(A_n) \to 0$ for all $k \leqslant \sum_{i=1}^m \dim\ker W^{t_i}(\mathbb{A})$ and $s_k^l(A_n) \to 0$ for all $k \leqslant \sum_{i=1}^m \dim\operatorname{coker} W^{t_i}(\mathbb{A})$ as $n \to \infty$.*

If, for one $t \in T$, the operator $W^t(\mathbb{A})$ is properly $\mathscr{P}^t$-deficient from the right (left) then $s_k^r(A_n) \to 0$ (or $s_k^l(A_n) \to 0$, respectively) for every $k \in \mathbb{N}$ as $n \to \infty$.

To prepare the proof we recall a useful observation from [34], Section 1.2.4. For $t \in T$ let P^t be a $\mathscr{P}^t$-compact projection. Then there is an $n_t \in \mathbb{N}$ and, for every $n \geqslant n_t$, a $\mathscr{P}^t$-compact projection P_n^t such that $\operatorname{im} P_n^t \subset \operatorname{im} L_n^t$, $\|P_n^t - P^t\| < 1$, and $\|P_n^t - P^t\| \to 0$ as $n \to \infty$. Furthermore, by

$$R_n^t := \begin{cases} E_n^t(P_n^t) & : n \geqslant n_t \\ 0 & : n < n_t \end{cases}$$

we get a sequence $\{R_n^t\} \in \mathscr{J}^t$ of projections R_n^t with $\operatorname{rank} R_n^t = \operatorname{rank} P_n^t = \operatorname{rank} P^t$ for $n \geqslant n^t$. Proposition 23 in [34] states

PROPOSITION 2.7. *Let $\mathbb{A} = \{A_n\} \in \mathscr{F}^T$ and $P^t \in \mathscr{K}^t$ be a $\mathscr{P}^t$-compact projection. Then*

$$\limsup_{n\to\infty} \|A_n R_n^t\| \leqslant M^t \|W^t(\mathbb{A})P^t\|, \quad \textit{and} \quad \limsup_{n\to\infty} \|R_n^t A_n\| \leqslant M^t c^t \|P^t W^t(\mathbb{A})\|.$$

Notice that the discussion in [34] was done for $\operatorname{rank} P^t < \infty$, but this condition is redundant. Now we prove Theorem 2.6.

Proof. Proposition 1.7 reveals that, if $W^{t_i}(\mathbb{A})$ are normally solvable then, for each $i = 1, \ldots, m$ and every respective non-negative integer $k_i \leqslant \dim \ker W^{t_i}(\mathbb{A})$, we can fix a $\mathscr{P}^{t_i}$-compact projection P^{t_i} onto a k_i-dimensional subspace of $\ker W^{t_i}(\mathbb{A})$ and choose $n_{t_i}, P_n^{t_i}, R_n^{t_i}$ as above. Moreover, for each i and every $n \geqslant \max_j n_{t_j}$ we choose a normed basis $\{x_{i,l}^n\}_{l=1}^{k_i}$ of $\operatorname{im} R_n^{t_i}$, such that for arbitrary scalars $\alpha_{i,j}^n$ the following hold:

$$|\alpha_{i,p}^n| \leqslant \left\| \sum_{j=1}^{k_i} \alpha_{i,j}^n x_{i,j}^n \right\| \quad \text{for all} \quad p = 1, \ldots, k_i. \tag{2.3}$$

It is a simple consequence of Auerbach's Lemma (see [19], B.4.8) that such a basis always exists. In the same way as in Proposition 26 in [34] we prove that $\|R_n^{t_i} R_n^{t_j}\| \to 0$ as $n \to \infty$ whenever $i \neq j$, and that there is a number $N \in \mathbb{N}$, such that

$$|\alpha_{j,k}| \leqslant \gamma \left\| \sum_{i=1}^{m} \sum_{l=1}^{k_i} \alpha_{i,l} x_{i,l}^n + y \right\|, \quad \text{where} \quad \gamma = 2 \max_{i=1,\ldots,m} M^{t_i} \|P^{t_i}\|,$$

for all $j = 1, \ldots, m$, $k = 1, \ldots, k_j$, $n \geqslant N$, all scalars $\alpha_{i,l}$, and all $y \in \mathbf{Y}_n := \bigcap_{i=1}^m \ker R_n^{t_i}$. Obviously, $\mathbf{E}_n$ decomposes into the direct sum

$$\mathbf{E}_n = \operatorname{span}\{x_{1,1}^n\} \oplus \ldots \oplus \operatorname{span}\{x_{m,k_m}^n\} \oplus \mathbf{Y}_n$$

for $n \geqslant N$, and we can introduce functionals $f_{i,j}^n \in \mathbf{E}_n^*$ by the rule

$$f_{i,j}^n \left(\sum_{k=1}^{m} \sum_{l=1}^{k_k} \alpha_{k,l}^n x_{k,l}^n + y \right) := \alpha_{i,j}^n \quad 1 \leqslant i \leqslant m, \ 1 \leqslant j \leqslant k_i.$$

Then we always have $\|f_{i,j}^n\| \leqslant \gamma$ and $f_{i,j}^n(\mathbf{Y}_n) = \{0\}$. As a next step, we denote by $R_n \in \mathscr{L}(\mathbf{E}_n)$ the linear operators

$$R_n x := \sum_{i=1}^{m} \sum_{j=1}^{k_i} f_{i,j}^n(x) x_{i,j}^n.$$

The operators R_n are projections of the rank $\dim \operatorname{im} R_n = k := \sum_{i=1}^m k_i$ and they are uniformly bounded with respect to n. Moreover, for every point $x \in \mathbf{E}_n$ we have (since $x_{i,j}^n = R_n^{t_i} x_{i,j}^n$)

$$\begin{aligned} \|A_n R_n x\| &= \left\| A_n \sum_{i=1}^{m} \sum_{j=1}^{k_i} f_{i,j}^n(x) x_{i,j}^n \right\| \leqslant \sum_{i=1}^{m} \sum_{j=1}^{k_i} |f_{i,j}^n(x)| \|A_n R_n^{t_i} x_{i,j}^n\| \\ &\leqslant \sum_{i=1}^{m} \sum_{j=1}^{k_i} \gamma \|x\| \|A_n R_n^{t_i}\| \|x_{i,j}^n\| = \gamma \|x\| \sum_{i=1}^{m} k_i \|A_n R_n^{t_i}\|. \end{aligned}$$

Since $\|A_n R_n^{t_i}\| \to 0$ for each i (see Proposition 2.7) it follows that

$$\begin{aligned} s_k^r(A_n) &= \inf\{\|A_n + F\| : F \in \mathscr{L}(\mathbf{E}_n), \dim \ker F \geqslant k\} \\ &\leqslant \|A_n - A_n(L_n - R_n)\| = \|A_n R_n\| \leqslant \gamma k \max_i \|A_n R_n^{t_i}\| \to 0. \end{aligned}$$

Due to Proposition 1.7 we can also choose $\mathscr{P}^{t_i}$-compact projections $\tilde{P}^{t_i}$ with $\tilde{P}^{t_i} W^{t_i}(\mathbb{A}) = 0$ and proceed in the same way to construct the "left-hand side analogues" $\tilde{k}_i \leqslant \dim \operatorname{coker} W^{t_i}(\mathbb{A})$, $\tilde{P}_n^{t_i}$, $\tilde{R}_n^{t_i}$, $\tilde{x}_{i,j}^n$, $\tilde{f}_{i,j}^n$, $\tilde{\mathbf{Y}}_n$ and, finally, a bounded sequence $(\tilde{R}_n)$ of projections $\tilde{R}_n$, being of the rank $\tilde{k} = \sum_{i=1}^m k_i$. Then

$$\begin{aligned} \|\tilde{R}_n A_n x\| = \left\| \sum_{i=1}^{m} \sum_{j=1}^{\tilde{k}_i} \tilde{f}_{i,j}^n(A_n x) \tilde{x}_{i,j}^n \right\| &\leqslant \sum_{i=1}^{m} \sum_{j=1}^{\tilde{k}_i} |\tilde{f}_{i,j}^n(\tilde{R}_n^{t_i} A_n x) + \tilde{f}_{i,j}^n((I - \tilde{R}_n^{t_i}) A_n x)| \|\tilde{x}_{i,j}^n\| \\ &\leqslant \sum_{i=1}^{m} \sum_{j=1}^{\tilde{k}_i} \left(\|\tilde{f}_{i,j}^n\| \|\tilde{R}_n^{t_i} A_n\| + \|\tilde{f}_{i,j}^n (I - \tilde{R}t_{in})\| \|A_n\| \right) \|x\|, \end{aligned}$$

where $\|\tilde{f}_{i,j}^n(I - \tilde{R}_n^{t_i})\| \to 0$ as $n \to \infty$ due to the following observations: for each $y \in \tilde{\mathbf{Y}}_n$ we have $\tilde{f}_{i,j}^n((I - \tilde{R}_n^{t_i})y) = \tilde{f}_{i,j}^n(y) = 0$, that is $\tilde{f}_{i,j}^n(I - \tilde{R}_n^{t_i})(I - \tilde{R}_n) = 0$, and further

$$\|\tilde{f}_{i,j}^n(I - \tilde{R}_n^{t_i}) \tilde{R}_n\| \leqslant \sum_{k=1}^{m} \sum_{l=1}^{\tilde{k}_k} |\tilde{f}_{k,l}^n| \|\tilde{f}_{i,j}^n (I - \tilde{R}_n^{t_i}) \tilde{x}_{k,l}^n\| \to 0$$

since $(I - \tilde{R}_n^{t_i}) \tilde{x}_{i,l}^n = 0$ and $\|\tilde{f}_{i,j}^n (I - \tilde{R}_n^{t_i}) \tilde{x}_{k,l}^n\| = \|\tilde{f}_{i,j}^n \tilde{R}_n^{t_i} \tilde{x}_{k,l}^n\| \leqslant \|\tilde{f}_{i,j}^n\| \|\tilde{R}_n^{t_i} \tilde{R}_n^{t_k}\| \|\tilde{x}_{k,l}^n\| \to 0$ for $k \neq i$ as $n \to \infty$. Thus, we again obtain

$$s_k^l(A_n) \leqslant \|\tilde{R}_n A_n\| \to 0 \quad \text{as } n \to \infty.$$

Now, suppose that $W^t(\mathbb{A})$ is properly $\mathscr{P}^t$-deficient from the right (left). Then for each $k \in \mathbb{N}$ and each $\varepsilon > 0$ there is a projection $Q \in \mathscr{K}^t$, $\operatorname{rank} Q = k$ such that

$\|W^t(\mathbb{A})Q\| < \varepsilon$ (or $\|QW^t(\mathbb{A})\| < \varepsilon$, respectively). Choosing again suitable sequences of projections w.r.t. Q as above, we get from Proposition 2.7 that $\limsup_n s_k^r(A_n) = 0$ (or $\limsup_n s_k^l(A_n) = 0$) since ε can be chosen arbitrarily small.

Finally notice that $W^{t_i}(\mathbb{A})$ being invertible at inifinity but not normally solvable implies the $\mathscr{P}^{t_i}$-deficiency from both sides by Proposition 1.7. □

Lower Bernstein and Mityagin numbers Besides the approximation numbers there are similar geometric characteristics for bounded linear operators $A \in \mathscr{L}(\mathbf{X},\mathbf{Y})$ on Banach spaces $\mathbf{X},\mathbf{Y}$. Denote by $U_\mathbf{X}$ the closed unit ball in $\mathbf{X}$ and by

$$j(A) := \sup\{\tau \geqslant 0 : \|Ax\| \geqslant \tau\|x\| \text{ for all } x \in \mathbf{X}\},$$
$$q(A) := \sup\{\tau \geqslant 0 : A(U_\mathbf{X}) \supset \tau U_\mathbf{Y}\}$$

the injection modulus and the surjection modulus, respectively. Obviously, the relation $j(A) = \inf\{\|Ax\| : x \in \mathbf{X}, \|x\| = 1\}$ holds, and due to [19], B.3.8 we have

$$j(A^*) = q(A) \quad \text{and} \quad q(A^*) = j(A). \tag{2.4}$$

Furthermore, for given closed subspaces $V \subset \mathbf{X}$ and $W \subset \mathbf{Y}$ we let J_V denote the embedding map of V into $\mathbf{X}$ and by Q_W the canonical map of $\mathbf{Y}$ onto the quotient $\mathbf{Y}/W$. We define the lower Bernstein and Mityagin numbers by

$$B_m(A) := \sup\{j(AJ_V) : \dim \mathbf{X}/V < m\},$$
$$M_m(A) := \sup\{q(Q_W A) : \dim W < m\}.$$

These characteristics have been discussed in [27], for instance. We will show that there are estimates that connect them to the approximation numbers defined above. Furthermore, these estimates will allow to relate the approximation numbers to the Fredholm property and the norm of the inverse of an operator as well as to the lower singular values in case of a Hilbert space $\mathbf{X}$. Note also that the sequences $(B_m(A))$, $(M_m(A))$ are monotonically non-decreasing.

Recall the following observation which is well known, and can be proved as in [34], Theorem 3.

PROPOSITION 2.8. *Let $\mathbf{X},\mathbf{Y}$ be Banach spaces and $A \in \mathscr{L}(\mathbf{X},\mathbf{Y})$.*

1. *If A is normally solvable and $k \leqslant \dim\ker A$ ($k \leqslant \dim\operatorname{coker} A$) then there is a projection $P \in \mathscr{L}(\mathbf{X})$ ($P' \in \mathscr{L}(\mathbf{Y})$) of rank k and of the norm less than $k+2$ such that $AP = 0$ ($P'A = 0$, respectively).*
2. *If A is not normally solvable then, for every $k \in \mathbb{N}$ and every $\varepsilon > 0$, there are projections $P \in \mathscr{L}(\mathbf{X})$ and $P' \in \mathscr{L}(\mathbf{Y})$ of rank k and of the norm less than $k+2$ such that $\|AP\| < \varepsilon$ and $\|P'A\| < \varepsilon$.*

PROPOSITION 2.9. *Let $\mathbf{X},\mathbf{Y}$ be Banach spaces and $A \in \mathscr{L}(\mathbf{X},\mathbf{Y})$. Then, for all $m \in \mathbb{N}$,*

$$\frac{s_m^r(A)}{2^m - 1} \leqslant B_m(A) \leqslant s_m^r(A), \quad \textit{as well as} \quad \frac{s_m^l(A)}{2^m - 1} \leqslant M_m(A) \leqslant s_m^l(A).$$

Proof. Let $F \in \mathscr{L}(\mathbf{X},\mathbf{Y})$ with $\dim\ker F \geqslant m$ and let $V \subset \mathbf{X}$ be a subspace of the codimension $\operatorname{codim} V = \dim \mathbf{X}/V < m$. Then $\ker F \cap V$ is at least 1-dimensional. Thus

$$\begin{aligned}\|A-F\| &= \sup\{\|(A-F)x\| : x \in \mathbf{X},\, \|x\|=1\} \geqslant \sup\{\|Ax\| : x \in \ker F,\, \|x\|=1\}\\ &\geqslant \inf\{\|Ax\| : x \in (\ker F \cap V),\, \|x\|=1\} \geqslant \inf\{\|Ax\| : x \in V,\, \|x\|=1\}\\ &= j(AJ_V).\end{aligned}$$

Since V was chosen arbitrarily, it follows that $\|A-F\| \geqslant B_m(A)$, and since F is arbitrary, too, we find that $s_m^r(A) \geqslant B_m(A)$. Now, let $\varepsilon > 0$. We inductively construct projections S_k $(k=1,\dots,m)$ and respective elements $x_k \in \ker S_{k-1}$ with $\|x_k\|=1$ such that

$$c_{k-1} \leqslant c_k := \|Ax_k\| < j(AJ_{\ker S_{k-1}}) + \varepsilon,$$

as well as functionals $f_k \in \mathbf{X}^*$ as follows: Set $S_0 := 0$ and $c_0 := 0$. If the projections $S_0,\dots,S_{k-1}$ together with their corresponding elements and functionals are already given, we choose $x_k \in \ker S_{k-1}$ and a functional $\tilde{f}_k$ on $\ker S_{k-1}$ with $\|\tilde{f}_k\|=1$ and $\tilde{f}_k(x_k)=1$. Furthermore, we define the functional $f_k := \tilde{f}_k \circ (I-S_{k-1})$ and the projection $S_k x := \sum_{i=1}^k f_i(x)x_i$ on $\mathbf{X}$. Notice that we obviously have $f_i(x_j) = \delta_{ij}$ for all $i,j \leqslant k$ and from

$$\|f_k\| \leqslant \|I-S_{k-1}\| \leqslant 1 + \sum_{i=1}^{k-1} \|f_i\|$$

it follows that $\|f_k\| \leqslant 2^{k-1}$, hence

$$\|AS_k x\| = \left\|\sum_{i=1}^k f_i(x)Ax_i\right\| \leqslant \sum_{i=1}^k c_i\|f_i\|\|x\| \leqslant c_k \sum_{i=1}^k 2^{i-1}\|x\| = c_k(2^k-1)\|x\|.$$

Defining $V := \ker S_{m-1}$, we find

$$\begin{aligned}s_m^r(A) &= \inf\{\|A-F\| : \dim\ker F \geqslant m\}\\ &\leqslant \|A - A(I-S_m)\| = \|AS_m\| \leqslant c_m(2^m-1)\\ &< (2^m-1)(j(AJ_V)+\varepsilon) \leqslant (2^m-1)(B_m(A)+\varepsilon),\end{aligned} \tag{2.5}$$

which completes the proof of the first estimate, since ε is arbitrary. Next, we prove that

$$M_m(A) = B_m(A^*) \leqslant s_m^r(A^*) \leqslant s_m^l(A). \tag{2.6}$$

For this let $W \subset \mathbf{Y}$ be a subspace with $\dim W < m$. Define $U := \{f \in \mathbf{Y}^* : f(W) = \{0\}\}$ and an operator $T : U \to (\mathbf{Y}/W)^*$ by $Tf = \tilde{f}$, $\tilde{f}(y+W) := f(y)$. T is well defined, surjective and even an isometry, since

$$\begin{aligned}\|Tf\| &= \sup_{y\in\mathbf{Y}\setminus W} \frac{|(Tf)(Q_W(y))|}{\|Q_W(y)\|} = \sup_{y\in\mathbf{Y}\setminus W} \frac{|f(y)|}{\inf_{z\in W}\|y+z\|}\\ &= \sup_{y\in\mathbf{Y}\setminus W}\sup_{z\in W} \frac{|f(y)|}{\|y+z\|} = \sup_{y\in\mathbf{Y}\setminus W}\sup_{z\in W} \frac{|f(y+z)|}{\|y+z\|} = \|f\|.\end{aligned}$$

Moreover, we check that $(Q_W^*(Tf))(y) = (Tf)(Q_W y) = f(y)$ for all $f \in U$ and all $y \in \mathbf{Y}$, hence the first equality in (2.6) follows with (2.4) from

$$\begin{aligned} q(Q_W A) = j(A^* Q_W^*) &= \inf\{\|A^* Q_W^* g\| : g \in (\mathbf{Y}/W)^*, \|g\| = 1\} \\ &= \inf\{\|A^* Q_W^*(Tf)\| : f \in U, \|f\| = 1\} \\ &= \inf\{\|A^* f\| : f \in U, \|f\| = 1\} = j(A^* J_U), \end{aligned}$$

since every finite dimensional subspace $W \subset \mathbf{Y}$ yields a subspace $U \subset \mathbf{Y}^*$ of the same codimension (as above) and, conversely, every subspace $U \subset \mathbf{Y}^*$ of finite codimension induces a corresponding subspace $W \subset \mathbf{Y}$ via $W := \{x \in \mathbf{Y} : f(x) = 0 \forall f \in U\}$. We now fix $\varepsilon > 0$ and an operator $F \in \mathscr{L}(\mathbf{X}, \mathbf{Y})$ of the codimension $\dim \operatorname{coker} F \geqslant m$ such that $\|A - F\| \leqslant s_m^l(A) + \varepsilon$. Choose a rank m projection Q such that $\|QF\| < \varepsilon$ (see Proposition 2.8). Then the kernel of $F^*(I-Q)^*$ is at least m-dimensional and

$$s_m^r(A^*) \leqslant \|A^* - F^*(I-Q)^*\| \leqslant \|A^* - F^*\| + \|F^* Q^*\| = \|A - F\| + \|QF\| \leqslant s_m^l(A) + 2\varepsilon$$

which completes the proof of (2.6) since ε was chosen arbitrarily.

For the remaining part of the second estimate in the assertion we use a construction similar to the above one. Fix $\varepsilon > 0$ and set $R_0 := 0$, $d_0 := 0$. For $k = 1, \ldots, m$ we gradually choose functionals $g_k \in \ker R_{k-1}^*$ with $\|g_k\| = 1$ such that

$$d_{k-1} \leqslant d_k := \|A^* g_k\| < j(A^* J_{\ker R_{k-1}^*}) + \varepsilon,$$

as well as elements $\tilde{y}_k \in \mathbf{Y}$ with $\|\tilde{y}_k\| = 1$ and $|g_k(\tilde{y}_k)| \geqslant 1 - \varepsilon$, respectively. Furthermore, we always define $y_k := \frac{1}{g_k(\tilde{y}_k)}(I - R_{k-1})\tilde{y}_k$ and an operator $R_k y := \sum_{i=1}^k g_i(y) y_i$ on $\mathbf{Y}$. We easily check that $g_i(y_j) = \delta_{ij}$ for all $i, j \leqslant k$ hence R_k is a projection of rank k, and from

$$\|y_k\| \leqslant \frac{1}{1-\varepsilon}\|I - R_{k-1}\| \leqslant \frac{1}{1-\varepsilon}\left(1 + \sum_{i=1}^{k-1}\|y_i\|\right)$$

we conclude $\|y_k\| \leqslant \frac{2^{k-1}}{(1-\varepsilon)^k}$. Thus, for all $g \in \mathbf{Y}^*$ and all $x \in \mathbf{X}$,

$$\begin{aligned} \|(A^* R_k^* g)(x)\| = \|g(R_k A x)\| &= \left\|\sum_{i=1}^k g_i(Ax) g(y_i)\right\| = \left\|\left(\sum_{i=1}^k g(y_i) A^* g_i\right)(x)\right\| \\ &\leqslant \sum_{i=1}^k \|g\|\|y_i\| d_i \|x\| \leqslant d_k \sum_{i=1}^k \frac{2^{i-1}}{(1-\varepsilon)^i}\|g\|\|x\| \leqslant d_k \frac{2^k - 1}{(1-\varepsilon)^k}\|g\|\|x\|. \end{aligned}$$

The assertion then follows by

$$\begin{aligned} s_m^l(A) = \inf\{\|A - F\| : \dim \operatorname{coker} F \geqslant m\} &\leqslant \|R_m A\| = \|A^* R_m^*\| \leqslant d_m \frac{2^m - 1}{(1-\varepsilon)^m} \\ &\leqslant \frac{2^m - 1}{(1-\varepsilon)^m}(B_m(A^*) + \varepsilon) = \frac{2^m - 1}{(1-\varepsilon)^m}(M_m(A) + \varepsilon) \end{aligned}$$

where $\varepsilon > 0$ is arbitrary. □

Now we have $s_1^r(A) = B_1(A) = j(A)$ and $s_1^l(A) = M_1(A) = q(A) = j(A^*)$, hence we deduce

COROLLARY 2.10. *Let $A \in \mathscr{L}(\mathbf{X},\mathbf{Y})$. Then*

$$s_1^{r/l}(A) = \begin{cases} \|A^{-1}\|^{-1} & \text{if } A \text{ is invertible} \\ 0 & \text{if } A \text{ is not invertible from the left/right.} \end{cases}$$

With the help of Proposition 2.8 we even find more.

COROLLARY 2.11. *Let $A \in \mathscr{L}(\mathbf{X},\mathbf{Y})$.*

1. *If A is normally solvable and k is greater than the dimension of the kernel (cokernel) of A then $s_k^r(A)$ ($s_k^l(A)$, resp.) is non-zero. Otherwise $s_k^r(A)$ ($s_k^l(A)$) is equal to zero.*
2. *If A is not normally solvable then all approximation numbers are equal to zero.*

In particular, A is Fredholm if and only if the number of vanishing approximation numbers of A is finite.

Hilbert spaces and singular values Suppose now that $\mathbf{X},\mathbf{Y}$ are Hilbert spaces and $A \in \mathscr{L}(\mathbf{X},\mathbf{Y})$. Let $d := \sqrt{\inf \sigma_{\mathrm{ess}}(A^*A)}$ denote the square root of the infimum of the essential spectrum of A^*A and

$$\sigma_1(A) \leqslant \sigma_2(A) \leqslant \dots$$

be the sequence of the non-negative square roots of the eigenvalues of A^*A less than d, counted according to their algebraic multiplicities. If there are only N $(= 0,1,2,\dots)$ such eigenvalues, we put $\sigma_{N+1}(A) = \sigma_{N+2}(A) = \dots = d$. These numbers may be called the lower singular values of A.

COROLLARY 2.12. *Let $\mathbf{X},\mathbf{Y}$ be Hilbert spaces and $A \in \mathscr{L}(\mathbf{X},\mathbf{Y})$. Then, for all m,*

$$s_m^r(A) = B_m(A) = \sigma_m(A), \quad \text{as well as} \quad s_m^l(A) = M_m(A) = \sigma_m(A^*).$$

Proof. Set $B := A^*A$ and for a subspace $U \subset \mathbf{X}$ let $U^\perp$ be its orthogonal complement. Then

$$\begin{aligned}(B_m(A))^2 &= \sup\{\inf\{\|Ax\|^2 : x \in U^\perp, \|x\| = 1\} : \dim U < m\} \\ &= \sup\{\inf\{\langle x, Bx\rangle : x \in U^\perp, \|x\| = 1\} : \dim U < m\}\end{aligned}$$

which equals $(\sigma_m(A))^2$ by the Min-Max Principle (see [28], Theorem XIII.1).

Moreover, for a subspace U of the dimension m and P_U the orthogonal projection onto U,

$$\begin{aligned}(s_m^r(A))^2 \leqslant \|AP_U\|^2 &= \sup\{\langle Ax, Ax\rangle : x \in U, \|x\| \leqslant 1\} \\ &= \sup\{\langle x, Bx\rangle : x \in U,\ \|x\| \leqslant 1\} \\ &\leqslant \sup\{\|x\|\|Bx\| : x \in U,\ \|x\| \leqslant 1\} \leqslant \|BJ_U\|.\end{aligned}$$

If $m \leqslant N$, then there is an orthonormal system $\{x_i\}_{i=1}^m$ of eigenvectors, that means $Bx_i = (\sigma_i(A))^2 x_i$. Set $U := \operatorname{span}\{x_i\}_{i=1}^m$ and find that

$$(s_m^r(A))^2 \leqslant \sup\left\{ \sum_{i=1}^m \alpha_i^2 (\sigma_i(A))^2 : x = \sum_{k=1}^m \alpha_k x_k,\ \|x\| \leqslant 1 \right\} \leqslant (\sigma_m(A))^2.$$

For the case $m > N$ we note that $C := B - d^2 I$ is not Fredholm, since $d^2 \in \sigma_{\mathrm{ess}}(B)$. That is, C is not normally solvable or $\dim\ker C = \dim\operatorname{coker} C = \infty$. Thus, by Proposition 2.8, for every $\varepsilon > 0$ there is a subspace U of the dimension m such that $\|CJ_U\| < \varepsilon$, hence $(s_m^r(A))^2 \leqslant \|BJ_U\| < d^2 + \varepsilon$. Together with Proposition 2.9 This finishes the proof of the first assertion and the second one follows by duality and Equation (2.6), since $s_m^l(A) = s_m^r(A^*)$ in case of Hilbert spaces. □

In this sense the approximation numbers and the results of the subsequent sections can be regarded as generalizations for the singular values in the theory of Fredholm sequences in so-called Standard algebras in [8], Chapters 6.1 and 6.2.

REMARK 2.13. Besides the lower approximation, Bernstein and Mityagin numbers there are also their much more famous “upper relatives”. For example, the upper approximation numbers $s_k(A)$ of an operator A are given by

$$s_k(A) := \inf\{\|A - F\|_{\mathcal{L}(\mathbf{X},\mathbf{Y})} : F \in \mathcal{L}(\mathbf{X},\mathbf{Y}), \operatorname{rank} F < k\}.$$

These numbers form a decreasing sequence $\|A\| = s_0(A) \geqslant s_1(A) \geqslant \ldots \geqslant 0$ and, roughly speaking, they sound out the spectrum of an operator from above, and not from below as the lower approximation numbers do. For the other characteristics the definitions are similar. Notice that all of these “big brothers” are so-called s-numbers, for which a well developed theory is available that provides many results on the relations between them. We think that the modern books [18] and [5] provide a good overview on that business. Unfortunately, we do not know such results in the literature for the lower versions in case of (infinite dimensional) Banach spaces, except those of [27], from where we borrowed the equalities $B_m(A) = \sigma_m(A)$ and $M_m(A) = \sigma_m(A^*)$.

2.2.2. Regular $\mathcal{J}^T$-Fredholm sequences

DEFINITION 2.14. We introduce the nullity $\alpha(\mathbb{A})$ and deficiency $\beta(\mathbb{A})$ of a sequence $\mathbb{A} \in \mathcal{F}^T$ by

$$\alpha(\mathbb{A}) := \sum_{t\in T} \dim\ker W^t(\mathbb{A}) \quad \text{and} \quad \beta(\mathbb{A}) := \sum_{t\in T} \dim\operatorname{coker} W^t(\mathbb{A}).$$

A $\mathcal{J}^T$-Fredholm sequence $\mathbb{A} \in \mathcal{F}^T$ is said to be regular, if all operators $W^t(\mathbb{A})$ are Fredholm (in the usual sense). Of course, due to Theorem 2.4, $\mathbb{A}$ is regular if and only if $\alpha(\mathbb{A})$ and $\beta(\mathbb{A})$ are finite, hence we are in a position to introduce the index of a regular sequence $\mathbb{A}$ by

$$\operatorname{ind}\mathbb{A} := \alpha(\mathbb{A}) - \beta(\mathbb{A}).$$

Applying the well known properties of Fredholm operators (see [2], [20], or [7]) and Proposition 2.3 it is not hard to prove the following proposition.

PROPOSITION 2.15. *Let $\mathbb{A} \in \mathscr{F}^T$ be regular and $\mathbb{B} \in \mathscr{F}^T$.*

- *If $\|\mathbb{B}\|$ is sufficiently small then $\alpha(\mathbb{A}+\mathbb{B}) \leqslant \alpha(\mathbb{A})$, $\beta(\mathbb{A}+\mathbb{B}) \leqslant \beta(\mathbb{A})$ and $\operatorname{ind}(\mathbb{A}+\mathbb{B}) = \operatorname{ind}(\mathbb{A})$.*
- *If $\mathbb{B} \in \mathscr{J}^T$ has only compact snapshots then $\mathbb{A}+\mathbb{B}$ is a regular $\mathscr{J}^T$-Fredholm sequence and $\operatorname{ind}(\mathbb{A}+\mathbb{B}) = \operatorname{ind}\mathbb{A}$.*
- *If $\mathbb{B} \in \mathscr{G}$ then $\alpha(\mathbb{A}+\mathbb{B}) = \alpha(\mathbb{A})$ and $\beta(\mathbb{A}+\mathbb{B}) = \beta(\mathbb{A})$.*
- *If $\mathbb{B} \in \mathscr{F}^T$ is a regular $\mathscr{J}^T$-Fredholm sequence then $\operatorname{ind}(\mathbb{A}\mathbb{B}) = \operatorname{ind}\mathbb{A} + \operatorname{ind}\mathbb{B}$.*

2.2.3. The splitting property and the index formula

In all what follows, let $\mathscr{P}^t$ be uniform approximate identities for all $t \in T$.

PROPOSITION 2.16. *Let $\mathbb{A} = \{A_n\} \in \mathscr{F}^T$ be a $\mathscr{J}^T$-Fredholm sequence and suppose that all of its snapshots $W^t(\mathbb{A})$ are properly $\mathscr{P}^t$-Fredholm operators, respectively. If $\alpha(\mathbb{A})$ is a finite number then $\liminf_n s^r_{\alpha(\mathbb{A})+1}(A_n) > 0$, and if $\beta(\mathbb{A})$ is finite then $\liminf_n s^l_{\beta(\mathbb{A})+1}(A_n) > 0$.*

Proof. Assume that $\alpha(\mathbb{A})$ is finite. Due to Equation (2.1)

$$\mathbb{B}\mathbb{A} = \mathbb{I} + \sum_{i=1}^{m} \{E_n^{t_i}(L_n^{t_i} K^{t_i})\} + \mathbb{G}$$

from Proposition 2.3, all operators $W^t(\mathbb{A})$ with $t \in T \setminus \{t_1, ..., t_m\}$ have trivial kernels. Moreover, for every $i = 1, ..., m$, Corollary 1.17 provides an operator $B^i \in \mathscr{L}(\mathbf{E}^{t_i}, \mathscr{P}^{t_i})$ such that the operator $P^i := I^{t_i} - B^i W^{t_i}(\mathbb{A}) \in \mathscr{K}^{t_i}$ is a projection onto the kernel of $W^{t_i}(\mathbb{A})$. Now we prove that $\liminf_n s^r_{\alpha(\mathbb{A})+1}(A_n) > 0$ in the same manner as in [34], Proposition 28, where we make use of Corollary 2.10.

If we start with Equation (2.2) we can proceed analogously to obtain the estimate $s^l_{\beta(\mathbb{A})+1}(A_n) \geqslant \text{const} > 0$ for all sufficiently large n. □

Since for a regular $\mathscr{J}^T$-Fredholm sequence $\mathbb{A}$ the operators $W^t(\mathbb{A})$ are always properly $\mathscr{P}^t$-Fredholm, due to Corollary 1.9, and in view of Theorem 2.6 this yields the following theorem.

THEOREM 2.17. *Let $\mathbb{A} = \{A_n\} \in \mathscr{F}^T$ be a regular $\mathscr{J}^T$-Fredholm sequence. Then the approximation numbers from the right have the $\alpha(\mathbb{A})$-splitting property, i.e.*

$$\lim_{n\to\infty} s^r_{\alpha(\mathbb{A})}(A_n) = 0 \quad \textit{and} \quad \liminf_{n\to\infty} s^r_{\alpha(\mathbb{A})+1}(A_n) > 0.$$

Furthermore, the approximation numbers from the left have the $\beta(\mathbb{A})$-splitting property.

THEOREM 2.18. *Let* $\mathbb{A} = \{A_n\} \in \mathscr{F}^T$ *be a regular* $\mathscr{J}^T$*-Fredholm sequence. Then, for sufficiently large* n*, the operators* A_n *are Fredholm and their indices coincide with* $\operatorname{ind}\mathbb{A}$*, in other words*

$$\lim_{n\to\infty} \operatorname{ind} A_n = \sum_{t\in T} \operatorname{ind} W^t(\mathbb{A}).$$

Proof. Theorem 2.17 shows that $s^r_{\alpha(\mathbb{A})+1}(A_n) > 0$ and $s^l_{\beta(\mathbb{A})+1}(A_n) > 0$ for sufficiently large n, that is A_n is Fredholm, by Corollary 2.11. Moreover, for large n there is always an operator F_n with $\dim\operatorname{coker} F_n \geqslant \beta(\mathbb{A})$ such that $\|A_n - F_n\| \leqslant s^l_{\beta(\mathbb{A})}(A_n) + \frac{1}{n}$. We can choose a projection S_n of rank $\beta(\mathbb{A})$ and of the norm less than $\beta(\mathbb{A})+2$ such that $\|S_nF_n\| < \frac{1}{n}$ (see Proposition 2.8) and we deduce from Theorem 2.17 that

$$\frac{\|S_nA_n\|}{\beta(\mathbb{A})+2} < \frac{\|S_nA_n\|}{\|S_n\|} \leqslant \frac{\|S_n(A_n-F_n)\| + \|S_nF_n\|}{\|S_n\|} \leqslant s^l_{\beta(\mathbb{A})}(A_n) + \frac{2}{n} \to_{n\to\infty} 0.$$

We set $S_n := 0$ for the remaining smaller n and conclude that $\{S_nA_n\} \in \mathscr{G}$, that is the sequence $\{\tilde{A}_n\} := \{(L_n - S_n)A_n\} \in \mathscr{F}^T$ is $\mathscr{J}^T$-Fredholm with $\alpha(\{\tilde{A}_n\}) = \alpha(\mathbb{A})$, $\beta(\{\tilde{A}_n\}) = \beta(\mathbb{A})$ and $\operatorname{ind}\{\tilde{A}_n\} = \operatorname{ind}\mathbb{A}$ (cf. Proposition 2.15). Analogously, we choose a sequence $\{R_n\} \in \mathscr{F}$ of projections R_n of the rank $\alpha(\mathbb{A})$ and $\|R_n\| \leqslant \alpha(\mathbb{A})+2$ for large n, such that $\{\tilde{A}_nR_n\} \in \mathscr{G}$.

We now consider the sequence $\mathbb{C} = \{C_n\} := \{(L_n - S_n)A_n(L_n - R_n)\}$ and we find that it is a regular $\mathscr{J}^T$-Fredholm sequence with $\alpha(\mathbb{C}) = \alpha(\mathbb{A})$, $\beta(\mathbb{C}) = \beta(\mathbb{A})$ and $\operatorname{ind}\mathbb{C} = \operatorname{ind}\mathbb{A}$. More precisely, there is an $N \in \mathbb{N}$ and a constant $C > 0$ such that for all $n \geqslant N$

$$s^r_{\alpha(\mathbb{A})}(C_n) = s^l_{\beta(\mathbb{A})}(C_n) = 0 \text{ and } s^r_{\alpha(\mathbb{A})+1}(C_n), s^l_{\beta(\mathbb{A})+1}(C_n) > C,$$

which shows that the C_n are Fredholm of the index $\alpha(\mathbb{A}) - \beta(\mathbb{A}) = \operatorname{ind}\mathbb{A}$. Since $L_n - R_n$ and $L_n - S_n$ are Fredholm of index zero, we find that A_n is Fredholm of index $\operatorname{ind}\mathbb{A}$. □

REMARK 2.19. Note that there is a bounded sequence $\{D_n\}$ such that for sufficiently large n the operator D_n is a generalized inverse for C_n. Moreover, $\mathbb{A} - \mathbb{C}$ belongs to $\mathscr{G}$, hence we find that $\mathbb{A} + \mathscr{G}$ is generalized invertible in $\mathscr{F}/\mathscr{G}$, whenever $\mathbb{A}$ is a regular $\mathscr{J}^T$-Fredholm sequence.

Furthermore, this shows that the nullity, deficiency and the index of a structured operator sequence $\mathbb{A}$ being $\mathscr{J}^T$-Fredholm as introduced in Definition 2.14 are universal characteristics of this sequence, in the sense that if these numbers exist for the $\mathscr{J}^T$-Fredholm sequence $\mathbb{A}$ in one setting $\mathscr{F}^T$ then they are the same in every setting $\mathscr{F}^{\tilde{T}}$ where $\mathbb{A}$ is $\mathscr{J}^{\tilde{T}}$-Fredholm. In Section 2.4 we will recover this observation from a more general point of view.

2.3. Stability of sequences $\mathbb{A} \in \mathscr{F}^T$

DEFINITION 2.20. A sequence $\mathbb{A} = \{A_n\} \in \mathscr{F}$ is called stable, if there is an index n_0 such that all operators A_n, $n \geqslant n_0$, are invertible and

$$\sup_{n\geqslant n_0} \|A_n^{-1}\| < \infty.$$

It is well known, that a sequence $\mathbb{A} \in \mathscr{F}$ is stable if and only if the coset $\mathbb{A}+\mathscr{G}$ is invertible in $\mathscr{F}/\mathscr{G}$. Utilizing the higher structure of the given setting, namely the existence of $\mathscr{P}^t$-strong limits $W^t(\mathbb{A})$, we can prove a stronger result for the case that all $\mathscr{P}^t$ are uniform.

THEOREM 2.21. *A sequence $\mathbb{A} \in \mathscr{F}^T$ is stable and all $W^t(\mathbb{A})$, $t \in T$, have the $\mathscr{P}^t$-dichotomy if and only if $\mathbb{A}$ is $\mathscr{J}^T$-Fredholm and all $W^t(\mathbb{A})$, $t \in T$, are invertible.*

Proof. From Corollary 2.10 we deduce that $\mathbb{A} = \{A_n\} \in \mathscr{F}^T$ is stable if and only if

$$\text{there is a constant } C > 0 \text{ such that } s_1^r(A_n) \geqslant C \text{ and } s_1^l(A_n) \geqslant C \text{ for large } n. \tag{2.7}$$

Let $\mathbb{A} = \{A_n\}$ be $\mathscr{J}^T$-Fredholm and all $W^t(\mathbb{A})$ be invertible. Then $\mathbb{A}$ is regular and Theorem 2.17 tells us that (2.7) is in force.

Conversely, let $\mathbb{A} = \{A_n\} \in \mathscr{F}^T$ be stable and let all snapshots $W^t(\mathbb{A})$ have the $\mathscr{P}^t$-dichotomy. Then for large n and every $K \in \mathscr{K}^t$

$$\|E_n^{-t}(A_n)L_n^t K\| = \frac{\|E_n^{-t}(A_n^{-1})L_n^t\|}{\|E_n^{-t}(A_n^{-1})L_n^t\|}\|E_n^{-t}(A_n)L_n^t K\| \geqslant \frac{1}{\|E_n^{-t}(A_n^{-1})L_n^t\|}\|L_n^t K\|.$$

For $n \to \infty$, we obtain

$$\|W^t(\mathbb{A})K\| \geqslant C^t\|K\| \text{ and analogously } \|KW^t(\mathbb{A})\| \geqslant C^t\|K\|$$

for every $K \in \mathscr{K}^t$, where $C^t \geqslant 1/(M^t c^t \sup\|A_n^{-1}\|) > 0$ is constant. Thus, $W^t(\mathbb{A})$ is not $\mathscr{P}^t$-deficient, hence properly $\mathscr{P}^t$-Fredholm. Suppose that the kernel of $W^t(\mathbb{A})$ is not trivial. Then there is a projection $P \in \mathscr{K}^t$, $P \neq 0$ s.t. $0 = \|W^t(\mathbb{A})P\| \geqslant C^t\|P\| \geqslant C^t$, a contradiction. Thus $W^t(\mathbb{A})$ is injective. Analogously one shows that $W^t(\mathbb{A})$ is surjective and hence invertible, due to the Banach inverse mapping theorem. Define a sequence $\{B_n\}$ by $B_n := A_n^{-1}$ if A_n is invertible and $B_n := L_n$ otherwise. Then one easily checks that $E_n^{-t}(B_n)L_n^t$ tends $\mathscr{P}^t$-strongly to $(W^t(\mathbb{A}))^{-1}$ for every $t \in T$, that is $\{B_n\} \in \mathscr{F}^T$ and $\{B_n\} + \mathscr{J}^T$ is the inverse of $\mathbb{A} + \mathscr{J}^T$. Thus, $\mathbb{A}$ is a $\mathscr{J}^T$-Fredholm sequence. For more details see the proof of [34], Theorem 21. □

2.4. A general Fredholm property

The notion of $\mathscr{J}^T$-Fredholmness for structured operator sequences $\mathbb{A} \in \mathscr{F}^T$ as it was introduced in the preceding sections depends on the underlying setting $\mathscr{F}^T$ and $\mathscr{J}^T$. On the one hand, it is convenient to work in this context because there we can describe the Fredholm properties of a sequence $\mathbb{A}$ quite well in terms of Fredholm properties of its snapshots $W^t(\mathbb{A})$. On the other hand, the main disadvantage lies in the fact that a sequence which is Fredholm in one setting does not need to be Fredholm in another setting. For an example see Section 2.4.5 in [34]. Thus, in what follows, we introduce a "universal" Fredholm property for the larger framework of all bounded operator sequences in $\mathscr{F}$. This approach has been extensively studied in the case of C^*-algebras of operator sequences on finite dimensional spaces by S. Roch in [8], Chapter

6.3. In particular, we will recover the mentioned universal characteristics in this larger framework again.

To avoid degenerated cases we assume that $\limsup_n \dim \mathbf{E}_n = \infty$.

DEFINITION 2.22. A sequence $\{K_n\} \in \mathscr{F}$ is said to be of almost uniformly bounded rank if

$$\limsup_{n\to\infty} \operatorname{rank} K_n < \infty.$$

Let $\mathscr{I}$ denote the closure of the set containing all sequences of almost uniformly bounded rank. The elements of $\mathscr{I}$ are referred to as compact sequences.

One can easily check that $\mathscr{I}$ forms a proper closed two-sided ideal in $\mathscr{F}$ which contains $\mathscr{G}$.

DEFINITION 2.23. Now we are in a position to introduce a class of Fredholm sequences in $\mathscr{F}$ by calling $\mathbb{A} = \{A_n\} \in \mathscr{F}$ Fredholm if $\mathbb{A} + \mathscr{I}$ is invertible in $\mathscr{F}/\mathscr{I}$.

Evidently, we have

- Stable sequences are Fredholm and never compact.
- Products of Fredholm sequences are Fredholm.
- The sum of a Fredholm sequence and a compact sequence is Fredholm.
- The set of all Fredholm sequences is open in $\mathscr{F}$.
- If $\{A_n\}$ is Fredholm, then $\{A_n^*\}$ is of Fredholm type.

For an equivalent characterization of Fredholm sequences we need the following definition.

DEFINITION 2.24. Let $\mathbb{A} = \{A_n\} \in \mathscr{F}$. If there is a finite number $\alpha \in \mathbb{Z}_+$ with

$$\liminf_{n\to\infty} s^r_\alpha(A_n) = 0 \text{ and } \liminf_{n\to\infty} s^r_{\alpha+1}(A_n) > 0,$$

then this number is called the α-number of $\mathbb{A}$ and it is denoted by $\alpha(\mathbb{A})$. Analogously, we introduce $\beta(\mathbb{A})$, the β-number of $\mathbb{A}$, as the $\beta \in \mathbb{Z}_+$ with

$$\liminf_{n\to\infty} s^l_\beta(A_n) = 0 \text{ and } \liminf_{n\to\infty} s^l_{\beta+1}(A_n) > 0.$$

Besides the well known result of Kozak [12], by applying Corollary 2.10, we immediately get the following characterization of stability in the large algebra $\mathscr{F}$.

THEOREM 2.25. *For a sequence $\mathbb{A} \in \mathscr{F}$ the following are equivalent.*

- $\mathbb{A}$ *is stable.*
- $\mathbb{A}+\mathscr{G}$ *is invertible in* $\mathscr{F}/\mathscr{G}$.
- $\alpha(\mathbb{A}) = \beta(\mathbb{A}) = 0$.

THEOREM 2.26. *For a sequence $\mathbb{A} \in \mathscr{F}$ the following are equivalent.*

- $\mathbb{A}$ *is Fredholm.*
- *There are sequences $\mathbb{B}_1, \mathbb{B}_2 \in \mathscr{F}$ such that $\mathbb{B}_1\mathbb{A} - \mathbb{I}$ and $\mathbb{A}\mathbb{B}_2 - \mathbb{I}$ are of almost uniformly bounded rank.*
- $\mathbb{A}$ *has an α-number and a β-number.*

Proof. Let $\mathbb{A} = \{A_n\}$ be Fredholm. Then there are sequences $\mathbb{D}, \mathbb{H}_i, \mathbb{G}_i \in \mathscr{F}$ (with $i = 1,2$) such that $\mathbb{D}\mathbb{A} = \mathbb{I} + \mathbb{H}_1 + \mathbb{G}_1$ and $\mathbb{A}\mathbb{D} = \mathbb{I} + \mathbb{H}_2 + \mathbb{G}_2$, where $\mathbb{H}_i$ are of almost uniformly bounded rank and $\|\mathbb{G}_i\| < 1/2$. Since $\mathbb{I} + \mathbb{G}_i$ are invertible, we can define $\mathbb{B}_1 := (\mathbb{I} + \mathbb{G}_1)^{-1}\mathbb{D}$, $\mathbb{K}_1 := (\mathbb{I} + \mathbb{G}_1)^{-1}\mathbb{H}_1$ and $\mathbb{B}_2 := \mathbb{D}(\mathbb{I} + \mathbb{G}_2)^{-1}$, $\mathbb{K}_2 := \mathbb{H}_2(\mathbb{I} + \mathbb{G}_2)^{-1}$. This implies the second assertion.

Now let $\{K_n\} = \{B_n\}\{A_n\} - \mathbb{I}$ and $n_0 \in \mathbb{N}$ with $k := \sup_{n \geqslant n_0} \operatorname{rank} K_n < \infty$. For each $n \geqslant n_0$ we can introduce a projection R_n with kernel of the dimension k and norm $\|R_n\| \leqslant k+1$ such that $R_n K_n = 0$ (see [19], B4.9). Then $R_n B_n A_n = R_n$. Moreover, for each $n \geqslant n_0$ we observe that $s^r_{k+1}(R_n) \geqslant 1$, because otherwise there would exist an operator F with $\dim\ker F \geqslant k+1$ such that $\|R_n - F\| < 1$, hence $L_n - R_n + F$ would be invertible, but since $\operatorname{rank}(L_n - R_n) = k$ this yields a contradiction. Thus

$$\begin{aligned} 1 \leqslant s^r_{k+1}(R_n) &= \inf\{\|R_n - F\| : \dim\ker F \geqslant k+1\} \\ &= \inf\{\|R_n B_n A_n - F\| : \dim\ker F \geqslant k+1\} \\ &\leqslant \inf\{\|R_n B_n A_n - R_n B_n F\| : \dim\ker F \geqslant k+1\} \leqslant \|R_n\|\|B_n\| s^r_{k+1}(A_n), \end{aligned}$$

hence $\alpha(\mathbb{A}) \leqslant k+1$ exists. Analogously we find a β-number for $\mathbb{A}$, that is, the third assertion holds.

Finally, let $\mathbb{A}$ have an α-number and a β-number and let $N \in \mathbb{N}$ be such that

$$\inf\{s^r_{\alpha(\mathbb{A})+1}(A_n), s^l_{\beta(\mathbb{A})+1}(A_n) : n \geqslant N\} > 0.$$

Then A_n, $n \geqslant N$, are Fredholm operators by Corollary 2.11. In view of Equation (2.5) there are a constant $C > 0$ and a sequence $\{R_n\} \in \mathscr{F}$ of projections R_n with kernels of the dimension $\alpha(\mathbb{A})$ such that $\inf\{\|A_n x\| : x \in \operatorname{im} R_n, \|x\| = 1\} \geqslant C$ for all $n \geqslant N$. We consider the restrictions $A_n|_{\operatorname{im} R_n}$ of A_n to $\operatorname{im} R_n$ which are injective. The spaces $\operatorname{im}(A_n|_{\operatorname{im} R_n})$ are of the codimension not greater than $\alpha(\mathbb{A}) + \beta(\mathbb{A})$, hence they are closed and we can choose projections S_n onto $\operatorname{im}(A_n|_{\operatorname{im} R_n})$, which are uniformly bounded with respect to $n \geqslant N$. The operators $A_n|_{\operatorname{im} R_n} : \operatorname{im} R_n \to \operatorname{im} S_n$ are invertible and their inverses $A_n^{(-1)}$ are (uniformly) bounded by C. For the operators $B_n := R_n A_n^{(-1)} S_n$ we conclude that

$$\begin{aligned} A_n B_n &= A_n R_n A_n^{(-1)} S_n = A_n A_n^{(-1)} S_n = S_n, \\ B_n A_n &= B_n A_n R_n + B_n A_n (L_n - R_n) = R_n A_n^{(-1)} S_n A_n R_n + B_n A_n (L_n - R_n) \\ &= R_n + B_n A_n (L_n - R_n). \end{aligned} \tag{2.8}$$

Since the latter term is of uniformly bounded rank, this proves the Fredholmness of $\mathbb{A}$. □

COROLLARY 2.27. *Let $\mathbb{A} = \{A_n\} \in \mathscr{F}$ be a Fredholm sequence and let T, as well as $\mathbf{E}^t$, $\mathscr{P}^t$, (L_n^t) and (E_n^t) be given as in Section 2.1 such that the Conditions* (I), (II) *are in force. Suppose that, for one $t \in T$, a subsequence of $(E_n^t(A_n)L_n^t)$ converges $\mathscr{P}^t$-strongly to an operator $A^t \in \mathscr{L}(\mathbf{E}^t, \mathscr{P}^t)$ which has the $\mathscr{P}^t$-dichotomy. Then A^t is Fredholm. If for all t in a certain subset $\tilde{T} \subset T$ and with respect to one common subsequence of $\mathbb{A}$ such operators A^t exist then*

$$\sum_{t \in \tilde{T}} \dim \ker A^t \leqslant \alpha(\mathbb{A}), \qquad \sum_{t \in \tilde{T}} \dim \operatorname{coker} A^t \leqslant \beta(\mathbb{A}).$$

Proof. The assertion immediately results from Theorem 2.6. □

For a setting $\mathscr{F}^T$ where all $\mathscr{P}^t$ are uniform we get from Theorem 2.17

COROLLARY 2.28. *If $\mathbb{A} \in \mathscr{F}^T$ is a regular $\mathscr{J}^T$-Fredholm sequence then $\mathbb{A}$ is a Fredholm sequence in $\mathscr{F}$, that is $\mathbb{A}$ is invertible modulo $\mathscr{I}$. Moreover, the numbers $\alpha(\mathbb{A})$ and $\beta(\mathbb{A})$ in the Definitions 2.14 and 2.24 are consistent.*

3. Applications

3.1. Standard finite sections of band-dominated operators on $l^p(\mathbb{Z}, X)$

Recall the definitions and notations from Section 1.4 and let $\mathscr{H}_+$ ($\mathscr{H}_-$) denote the set of all sequences $h : \mathbb{N} \to \mathbb{N}$ ($h : \mathbb{N} \to \mathbb{Z} \setminus \mathbb{N}$) tending to $+\infty$ ($-\infty$, respectively). Moreover, set $\mathscr{H} := \mathscr{H}_+ \cup \mathscr{H}_-$. For a rich band-dominated operator $A \in \mathscr{L}(l^p(\mathbb{Z}, X))$ and a given sequence $h \in \mathscr{H}$ there is a subsequence j of h such that A_j exists. By the same argument, j itself contains a subsequence g such that both limit operators A_g and A_{-g} exist.

Therefore, it seems to be feasible and valuable to consider appropriate subsequences for which the limit operators exist and hence the ideas of the general theory of Section 2 apply:

For a given sequence $h = (h_n)_{n \in \mathbb{N}} \in \mathscr{H}_+$ we define operators $L_{h_n} := \chi_{\{-h_n, \ldots, h_n\}} I$ and obtain a sequence (L_{h_n}) of projections converging $\mathscr{P}$-strongly to the identity. Further, set $\mathbf{E}_{h_n} := \operatorname{im} L_{h_n}$, $T := \{-1, 0, +1\}$, $I^0 := I$, $I^{\pm 1} := \chi_{\mathbb{Z}_\mp} I$ and

$$\begin{aligned}
\mathbf{E}^0 &:= l^p(\mathbb{Z}, X) & L_{h_n}^0 &:= L_{h_n} \\
& & E_{h_n}^0 &: \mathscr{L}(\operatorname{im} L_{h_n}^0) \to \mathscr{L}(\mathbf{E}_{h_n}), B \mapsto B \\
\mathbf{E}^{\pm 1} &:= \operatorname{im} I^{\pm 1} & L_{h_n}^{\pm 1} &:= V_{\mp h_n} L_{h_n} V_{\pm h_n} \\
& & E_{h_n}^{\pm 1} &: \mathscr{L}(\operatorname{im} L_{h_n}^{\pm 1}) \to \mathscr{L}(\mathbf{E}_{h_n}), B \mapsto V_{\pm h_n} B V_{\mp h_n}
\end{aligned}$$

for every n. By $\mathscr{P}^x := (L_n^x)$ uniform approximate identities on $\mathbf{E}^x$ are given and the sequences $(L_{h_n}^x)$ converge $\mathscr{P}^x$-strongly to the identities I^x on $\mathbf{E}^x$. We let $\mathscr{F}_h^T$ denote the Banach algebra of all bounded sequences $\{A_{h_n}\}$ of bounded linear operators $A_{h_n} \in \mathscr{L}(\mathbf{E}_{h_n})$ for which there exist operators $W^x\{A_{h_n}\} \in \mathscr{L}(\mathbf{E}^x, \mathscr{P}^x)$ for each $x \in T$ such that for $n \to \infty$

$$E_{h_n}^{-x}(A_{h_n}) L_{h_n}^x \to W^x\{A_{h_n}\} \quad \mathscr{P}^x\text{-strongly.}$$

Further, we introduce closed ideals $\mathscr{G}_h$ and $\mathscr{J}_h^T$ in $\mathscr{F}_h^T$ by

$$\begin{aligned}
\mathscr{G}_h &:= \{\{G_{h_n}\} : \|G_{h_n}\| \to 0\}, \\
\mathscr{J}_h^T &:= \operatorname{span}\{\{E_{h_n}^x(L_{h_n}^x K L_{h_n}^x)\}, \{G_{h_n}\} : x \in T, K \in \mathscr{K}^x, \{G_{h_n}\} \in \mathscr{G}_h\}.
\end{aligned}$$

A sequence $\{A_{h_n}\} \in \mathscr{F}_h^T$ is said to be $\mathscr{J}_h^T$-Fredholm, if $\{A_{h_n}\} + \mathscr{J}_h^T$ is invertible in $\mathscr{F}_h^T / \mathscr{J}_h^T$.

The finite section algebra $\mathscr{F}_{\mathscr{A}_{l^p}}$ Let $\mathscr{F}$ be the algebra of all bounded sequences $\{A_n\}$ of bounded linear operators $A_n \in \mathscr{L}(\mathbf{E}_n)$ and let $\mathscr{F}_{\mathscr{A}_{l^p}}$ denote the smallest closed subalgebra of $\mathscr{F}$ containing all sequences $\{L_n A L_n\}$ with rich $A \in \mathscr{A}_{l^p}$. For $\mathbb{A} = \{A_n\} \in \mathscr{F}_{\mathscr{A}_{l^p}}$ and a sequence $h = (h_n)_{n \in \mathbb{N}} \in \mathscr{H}_+$ let $\mathbb{A}_h$ denote the subsequence $\{A_{h_n}\}$. It is obvious, that for each $\mathbb{A} = \{A_n\} \in \mathscr{F}_{\mathscr{A}_{l^p}}$ and every $h \in \mathscr{H}_+$ the following operator exists independently of the choice of h:

$$W(\mathbb{A}) := \mathscr{P}\text{-}\lim_{n \to \infty} A_n L_n = W^0(\mathbb{A}_h) = \mathscr{P}\text{-}\lim_{n \to \infty} A_{h_n} L_{h_n}.$$

Appropriate subsequences Let $\mathbb{A} \in \mathscr{F}_{\mathscr{A}_{l^p}}$ and $h \in \mathscr{H}_+$. By $\mathscr{H}_{\mathbb{A}_h}$ we denote the collection of all subsequences g of h such that the following hold

- $\mathbb{A}_g \in \mathscr{F}_g^T$ (which means that the operators $W(\mathbb{A})$ and $W^{\pm 1}(\mathbb{A}_g)$ exist).
- The limit operators $(W(\mathbb{A}))_{\pm g}$ exist.
- $\mathbb{A}_g - \{L_{g_n} W(\mathbb{A}) L_{g_n}\} \in \mathscr{J}_g^T$.

We note that for the finite section sequence $\mathbb{A} := \{L_n A L_n\}$ of a single rich band-dominated operator A the existence of such an appropriate subsequence is easy to prove: Pass to a subsequence j of h, such that A_j exists. Then pass to a subsequence g of j such that A_g exists, too. Now, the last condition is automatically fulfilled. Fortunately, there is also an analogon for the more general $\mathscr{F}_{\mathscr{A}_{l^p}}$-sequences.

PROPOSITION 3.1. *Let $\mathbb{A} \in \mathscr{F}_{\mathscr{A}_{lp}}$ and let $h \in \mathscr{H}_+$. Then $\mathscr{H}_{\mathbb{A}_h}$ is not empty. If $W(\mathbb{A})$ is $\mathscr{P}$-Fredholm and $g \in \mathscr{H}_{\mathbb{A}_h}$ then $\mathbb{A}_g$ is a $\mathscr{J}_g^T$-Fredholm sequence in $\mathscr{F}_g^T$ and if B is a $\mathscr{P}$-regularizer for $W(\mathbb{A})$ then $\{L_{g_n} B L_{g_n}\} \in \mathscr{F}_g^T$ is a $\mathscr{J}_g^T$-regularizer for $\mathbb{A}_g$, too.*

Proof. With the help of Theorem 1.31 and Proposition 1.23 this can be proven in the same way as [34], Proposition 63. □

Now Theorems 2.4, 2.17, 2.18, 2.6 and 2.21 provide the following result.

THEOREM 3.2. *Let $\mathbb{A} = \{A_n\} \in \mathscr{F}_{\mathscr{A}_{lp}}$ and $g \in \mathscr{H}_{\mathbb{A}}$.*

- *If $W(\mathbb{A})$ is $\mathscr{P}$-Fredholm, then $W^{\pm 1}(\mathbb{A}_g)$ are $\mathscr{P}$-Fredholm.*
- *If $W(\mathbb{A}), W^{\pm 1}(\mathbb{A}_g)$ are Fredholm, then*

$$\lim_{n\to\infty} \operatorname{ind} A_{g_n} = \operatorname{ind} W(\mathbb{A}) + \operatorname{ind} W^{+1}(\mathbb{A}_g) + \operatorname{ind} W^{-1}(\mathbb{A}_g)$$

  and the approximation numbers from the right/left of the entries of $\mathbb{A}_g$ have the α-/β-splitting-property with

$$\alpha = \dim\ker W(\mathbb{A}) + \dim\ker W^{+1}(\mathbb{A}_g) + \dim\ker W^{-1}(\mathbb{A}_g),$$
$$\beta = \dim\operatorname{coker} W(\mathbb{A}) + \dim\operatorname{coker} W^{+1}(\mathbb{A}_g) + \dim\operatorname{coker} W^{-1}(\mathbb{A}_g).$$

- *If one of the operators $W(\mathbb{A})$, $W^{\pm 1}(\mathbb{A}_g)$ is not Fredholm then $\lim_{n\to\infty} s_k^r(A_{g_n}) = 0$ or $\lim_{n\to\infty} s_k^l(A_{g_n}) = 0$ for each $k \in \mathbb{N}$.*
- *$\mathbb{A}_g$ is stable if and only if $W(\mathbb{A})$ and $W^{\pm 1}(\mathbb{A}_g)$ are invertible.*

The complete sequences Theorem 3.2 provides some information on the behavior of subsequences of a sequence $\mathbb{A} \in \mathscr{F}_{\mathscr{A}_{lp}}$. We now want to state similar results for $\mathbb{A}$ itself, its Fredholm property in the sense of Definition 2.23 as well as its stability.

THEOREM 3.3. *Let $\mathbb{A} = \{A_n\} \in \mathscr{F}_{\mathscr{A}_{lp}}$ and let $\tilde{\mathscr{H}}_{\mathbb{A}} \subset \mathscr{H}_+$ denote the set of all sequences h for which $W^{\pm 1}(\mathbb{A}_h)$ exist. Then, $\mathbb{A}$ is a Fredholm sequence if and only if $W(\mathbb{A})$ and all operators $W^{\pm 1}(\mathbb{A}_h)$ with $h \in \tilde{\mathscr{H}}_{\mathbb{A}}$ are Fredholm. In this case the α- and β-number of $\mathbb{A}$ equal*

$$\begin{aligned}
\alpha(\mathbb{A}) &= \dim\ker W(\mathbb{A}) + \max_{h\in\tilde{\mathscr{H}}_{\mathbb{A}}} \left[\dim\ker W^{+1}(\mathbb{A}_h) + \dim\ker W^{-1}(\mathbb{A}_h)\right], \\
\beta(\mathbb{A}) &= \dim\operatorname{coker} W(\mathbb{A}) + \max_{h\in\tilde{\mathscr{H}}_{\mathbb{A}}} \left[\dim\operatorname{coker} W^{+1}(\mathbb{A}_h) + \dim\operatorname{coker} W^{-1}(\mathbb{A}_h)\right].
\end{aligned} \tag{3.1}$$

Proof. Suppose all operators $W(\mathbb{A})$ and $W^{\pm 1}(\mathbb{A}_h)$ with $h \in \tilde{\mathscr{H}}_{\mathbb{A}}$ are Fredholm, but for each $n \in \mathbb{N}$ there is a number h_n such that $h_n > h_{n-1}$ and $s_n^r(A_{h_n}) < \frac{1}{n}$. Choose a subsequence $g \in \mathscr{H}_{\mathbb{A}_h}$ and find a splitting number for $(s_k^r(A_{g_n}))$ by Theorem 3.2. A contradiction. Thus, $\mathbb{A}$ has an α-number and, analogously, a β-number, and Theorem 2.26 yields the Fredholm property of $\mathbb{A}$.

On the other hand, let $\alpha(\mathbb{A}) < \infty$ and $\beta(\mathbb{A}) < \infty$ be given. Then we deduce from Corollary 2.27 that all operators $W^t(\mathbb{A}_h)$, $h \in \tilde{\mathscr{H}}_{\mathbb{A}}$ are Fredholm as well as the relations "$\geqslant$" in (3.1). Furthermore, there is a sequence $h \in \mathscr{H}_+$ such that $\lim_n s^l_{\alpha(\mathbb{A})}(A_{h_n}) = 0$. Choose a subsequence $g \in \mathscr{H}_{\mathbb{A}_h}$ of h. Then Theorem 3.2 applies to $\mathbb{A}_g$ and yields the equalities in (3.1). $\square$

Notice that every $\mathscr{P}$-Fredholm operator is Fredholm if $\dim X < \infty$. Thus, Theorem 3.2 reveals that the Fredholm property of $W(\mathbb{A})$ is already sufficient for the Fredholm property of $\mathbb{A}$ in this case.

COROLLARY 3.4. *A sequence $\mathbb{A} \in \mathscr{F}_{\mathscr{A}_{lp}}$ is stable if and only if all operators $W(\mathbb{A})$ and $W^{\pm 1}(\mathbb{A}_h)$ with $h \in \tilde{\mathscr{H}}_{\mathbb{A}}$ are invertible.*

This easily follows by combination of Theorems 3.3 and 2.25 and we reformulate it as follows. For $\mathbb{A} = \{A_n\} \in \mathscr{F}_{\mathscr{A}_{lp}}$ we denote the set of all operators B_h which are $\mathscr{P}$-strong limits of one of the sequences

$$(V_{-h_n}[(I - L_{h_n}) + L_{h_n} A_{h_n} L_{h_n}] V_{h_n}) \text{ with } h \in \mathscr{H}_+ \ (\mathscr{H}_-)$$

by $\sigma^+_{\text{stab}}(\mathbb{A})$ (or $\sigma^-_{\text{stab}}(\mathbb{A})$, respectively). Of course, if B_h is in $\sigma^+_{\text{stab}}(\mathbb{A})$ or $\sigma^-_{\text{stab}}(\mathbb{A})$ then we can pass to a subsequence $g \in \tilde{\mathscr{H}}_{\mathbb{A}}$ of h and identify B_h with the respective operator $W^{\pm 1}(\mathbb{A}_g)$.

THEOREM 3.5. *A sequence $\mathbb{A} \in \mathscr{F}_{\mathscr{A}_{lp}}$ is stable if and only if all operators in*

$$\sigma_{\text{stab}}(\mathbb{A}) := \{W(\mathbb{A})\} \cup \sigma^+_{\text{stab}}(\mathbb{A}) \cup \sigma^-_{\text{stab}}(\mathbb{A})$$

are invertible.

REMARK 3.6. The latter result was recently proved in [14] for the finite sections sequence $(L_n A L_n)$ of a single band-dominated operator A, also by studying its subsequences. It has a lot of predecessors, which required additional restrictions like $p = 2$, $1 < p < \infty$, or the existence of a predual setting (see [23], [26], [16], [15], [3], for example).

Now having this result for sequences in the whole algebra $\mathscr{F}_{\mathscr{A}_{lp}}$, we get much more flexibility in constructing efficient algorithms for specific operators. Assume, for example, that the operator A admits a decomposition

$$A = \sum_{i=1}^{m} \prod_{j=1}^{k} A_{ij}$$

into band-dominated operators of simple structure (e.g. banded, triangular, or Toeplitz). This structure, which could permit the application of fast algorithms, gets lost if one applies the usual finite section method $L_n A L_n$, but it can be preserved, if one uses the composition of the finite sections $L_n A_{ij} L_n$ instead. The results above provide the desired information on the stability and convergence also for such compositions.

3.2. Adapted finite sections

In the preceding section the aim was to check if for arbitrarily given band-dominated operators A the "standard" finite section method, based on the projections L_n which arise from inflating the set $[-1,1]$ in a sense, applies. Providing that A is rich, the answer is that one has to check the Fredholm properties and the invertibility of a possibly infinite set $\sigma_{\mathrm{stab}}(\mathbb{A})$ of operators. Of course, we have seen that we can pass to subsequences to reduce the number of limit operators, but since the projections L_n are always chosen symmetric w.r.t. $\mathbb{Z}$, the limiting processes towards ∞ and $-\infty$ are somehow coupled, which seems to be artificial.

Now we let $A \in \mathscr{A}_{lp}$ be fixed and we ask if there is a more adapted sequence of projections which provides a specific "finite section like" method for this single operator, such that we only have to check one limit operator at ∞ and one limit operator at $-\infty$ and such that both can be chosen independently from each other.

The idea is very natural and simple: Suppose that for $A \in \mathscr{A}_{lp}$ (not necessarily rich) there is a sequence $l \in \mathscr{H}_-$, such that A_l exists and, independently of l, let $u \in \mathscr{H}_+$ be another sequence such that A_u exists. We show, that the properties of the finite sections $\{L_n^{l,u}AL_n^{l,u}\}$, where

$$L_n^{l,u} = \chi_{\{l(n),\dots,u(n)\}}I,$$

are determined by the three operators A, A_l and A_u.

THEOREM 3.7. *Let $A \in \mathscr{A}_{lp}$, and let $l \in \mathscr{H}_-, u \in \mathscr{H}_+$ be strictly decreasing or increasing, respectively, such that $A_l, A_u \in \sigma_{\mathrm{op}}(A)$ exist. Further let $\mathbb{A} := \{A_n^{l,u}\}$ denote the sequence of the operators $A_n^{l,u} := L_n^{l,u}AL_n^{l,u} \in \mathscr{L}(\operatorname{im} L_n^{l,u})$. Then*

- *The operators $A^+ := \chi_{\mathbb{Z}_+}A_l\chi_{\mathbb{Z}_+}I + (1-\chi_{\mathbb{Z}_+})I$, $A^- := \chi_{\mathbb{Z}_-}A_u\chi_{\mathbb{Z}_-}I + (1-\chi_{\mathbb{Z}_-})I$ are $\mathscr{P}$-Fredholm, whenever A is $\mathscr{P}$-Fredholm.*
- *If A and $A^\pm$ are Fredholm then $\lim_n \operatorname{ind} A_n^{l,u} = \operatorname{ind} A + \operatorname{ind} A^+ + \operatorname{ind} A^-$ and the approximation numbers from the right/left of $A_n^{l,u}$ have the α-/β-splitting-property with*
$$\alpha = \dim\ker A + \dim\ker A^+ + \dim\ker A^-,$$
and β w.r.t. the cokernels instead of the kernels.
- *If one of the operators A, $A^\pm$ is not Fredholm then, for all $k \in \mathbb{N}$, it holds $\lim_n s_k^r(A_n^{l,u}) = 0$ or $\lim_n s_k^l(A_n^{l,u}) = 0$.*
- *$\mathbb{A}$ is stable if and only if A and $A^\pm$ are invertible.*

Proof. Notice that $(L_n^{l,u})$ converges $\mathscr{P}$-strongly to the identity and further introduce $T := \{-1,0,+1\}$, homomorphisms $E_n^t : \mathscr{L}(\operatorname{im} L_n^{l,u,t}) \to \mathscr{L}(\mathbf{E}_n^{l,u})$ and sequence algebras $\mathscr{F}^{l,u,T}$, $\mathscr{J}^{l,u,T}$ in the same way as in Section 3.1. Then the finite section sequence of each $\mathscr{P}$-regularizer of A is again a $\mathscr{J}^{l,u,T}$-regularizer for $\mathbb{A}$ and Theorems 2.4, 2.17, 2.18, and 2.21 give the claim. $\square$

Of course, these two slightly different approaches can be combined to get more appropriate methods for classes of band-dominated operators having a certain common structure: First, one chooses the sequence $(L_n^{l,u})$ of projections which align with the common structure in a sense and then one proceeds as in Section 3.1 to prove Theorems 3.2 and 3.3 also in this setting.

Furthermore, [33] and [34] present an idea how one can pass to modified finite sections which need not to be stable, but which are still generalized invertible (e.g. Moore-Penrose invertible).

Also note that the equations $Ax = y$ and $V_\kappa Ax = V_\kappa y$ are equivalent for every $\kappa \in \mathbb{Z}$ since V_κ is invertible, whereas at most one of them leads to a stable finite section sequence and therefore can be solved by the finite section method. The simplest example of an operator for which such a preconditioning procedure is indicated to get a stable finite section sequence is the operator $A = V_1$ itself. This method is known as *index cancellation* and was already studied in [6]. A comprehensive discussion can also be found in [10].

REMARK 3.8. Also notice, that all considerations and results of Section 3.1 remain the same, if we replace the finite section projections L_{h_n} by $L_{h_n} := \chi_{\{-h_n,\ldots,h_n-1\}}I$, I^{+1} by $I^{+1} := \chi_{\{\ldots,-2,-1\}}I$ and $\mathbf{E}^{+1}$ by $\mathbf{E}^{+1} := \operatorname{im} I^{+1}$. To see this, use that the arising approximate identities $\{L_{h_n}\}$ are equivalent and that $A_h \in \sigma_{\mathrm{op}}(A)$ if and only if $V_{-1}A_hV_1 \in \sigma_{\mathrm{op}}(A)$.

This observation will help us to simplify some notations within the next section.

3.3. Band-dominated operators on $L^p(\mathbb{R})$

3.3.1. Discretization and the main results

Let P_n stand for the operator of multiplication by the characteristic function of the interval $[-n,n)$ acting on $\mathbf{X} := L^p = L^p(\mathbb{R})$ $(1 \leqslant p \leqslant \infty)$. Then $\hat{\mathscr{P}} := (P_n)$ forms a uniform approximate identity. For $\alpha \in \mathbb{R}$ we consider the operator

$$U_\alpha : L^p(\mathbb{R}) \to L^p(\mathbb{R}),\ (U_\alpha f)(t) := f(t-\alpha)$$

of shift by α. Let $A \in \mathscr{L}(L^p)$ and $h = (h_n)_{n\in\mathbb{N}} \subset \mathbb{Z}$ be a sequence tending to infinity, i.e. $|h_n| \to \infty$ as $n \to \infty$. The operator A_h is called limit operator of A with respect to h if

$$U_{-h_n}AU_{h_n} \to A_h \ \hat{\mathscr{P}}\text{-strongly.}$$

The set $\sigma(A)$ of all limit operators of A is again called the operator spectrum of A.

Discretization Now, let χ_0 denote the characteristic function of the interval $I_0 := [0,1)$ and set $X := L^p(I_0)$. The mapping G which sends the function $f \in L^p$ to the sequence

$$Gf = ((Gf)_k)_{k\in\mathbb{Z}}, \ \text{ where } (Gf)_k := \chi_0 U_{-k} f$$

is an isometric isomorphism from L^p onto $l^p(\mathbb{Z},X)$. Thus, the mapping

$$\Gamma : \mathscr{L}(L^p) \to \mathscr{L}(l^p(\mathbb{Z},X)),\ A \mapsto GAG^{-1}$$

is an isometric algebra isomorphism. Further, having the definition $L_n := \chi_{\{-n,\dots,n-1\}}I$ for the finite section projections on l^p and Remark 3.8 in mind, we have $\Gamma(P_m) = L_m$ and the sets of compact ($\hat{\mathscr{P}}/\mathscr{P}$-compact) operators translate under the discretization operator Γ. Hence, $A \in \mathscr{L}(L^p)$ is Fredholm ($\hat{\mathscr{P}}$-Fredholm, properly $\hat{\mathscr{P}}$-Fredholm, or properly $\hat{\mathscr{P}}$-deficient) if and only if $\Gamma(A)$ is so, w.r.t $\mathscr{P}$ instead of $\hat{\mathscr{P}}$. Particularly, L^p has the $\hat{\mathscr{P}}$-dichotomy. As in [25], Proposition 3.1.4 we find

PROPOSITION 3.9. *Let $A \in \mathscr{L}(L^p)$ and $h = (h_n)_{n\in\mathbb{N}} \subset \mathbb{Z}$ be a sequence tending to infinity. The limit operator A_h exists if and only if the limit operator $(\Gamma(A))_h$ of $\Gamma(A)$ w.r.t. h exists. Then $(\Gamma(A))_h = \Gamma(A_h)$. In particular, A is rich iff $\Gamma(A)$ is rich.*

DEFINITION 3.10. An operator $A \in \mathscr{L}(L^p)$ is called a band-dominated operator, if its discretization $\Gamma(A)$ is so. Let $\mathscr{A}_{L^p}$ denote the set of all such operators.

REMARK 3.11. For each function $\varphi \in \mathrm{BUC}$, the algebra of all bounded and uniformly continuous functions on the real line, and for each $t > 0$, set $\varphi_t(x) := \varphi(tx)$. By [15], Theorem 1.42 the algebra $\mathscr{A}_{L^p}$ coincides with the algebra of all operators $A \in \mathscr{L}(L^p)$ with $\|[A, \varphi_t I]\| \to 0$ for all $\varphi \in \mathrm{BUC}$ as $t \to 0$ (For $1 < p < \infty$ this can also be found in [25], Propositions 3.1.6f). Furthermore every multiplication operator aI with a function $a \in L^\infty(\mathbb{R})$ and every shift operator U_α with $\alpha \in \mathbb{R}$ belong to $\mathscr{A}_{L^p}$. The considerations in [15], Example 1.45 guarantee that $\mathscr{A}_{L^p}$ also contains all operators of convolution with a function in L^1. We will address such operators in Section 3.3.3.

Let $\hat{\mathbf{E}}_n = L^p(\operatorname{im} P_n)$ and let $\mathscr{F}_{\mathscr{A}_{L^p}}$ denote the Banach algebra which is generated by all sequences $\{P_nAP_n\}$ with rich $A \in \mathscr{A}_{L^p}$. There is a natural identification of sequences $\mathbb{A} = \{A_n\} \in \mathscr{F}_{\mathscr{A}_{L^p}}$ with sequences in $\mathscr{F}_{\mathscr{A}_{l^p}}$ via Γ and we shortly write $\Gamma(\mathbb{A})$ for the discrete version $\{L_n\Gamma(A_nP_n)L_n\}$. Then, for $\mathbb{A} \in \mathscr{A}_{L^p}$ also the notions of subsequences $\mathbb{A}_h$, of appropriate subsequences $\mathbb{A}_h$ with $h \in \mathscr{H}_{\mathbb{A}}$ as well as the limit $W(\mathbb{A})$ translate to the L^p-case. Furthermore, let χ_+ and χ_- stand for the characteristic functions of the sets $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively and introduce, for $h \in \tilde{\mathscr{H}}_{\mathbb{A}} := \tilde{\mathscr{H}}_{\Gamma(\mathbb{A})}$,

$$W^{+1}(\mathbb{A}_h) := \chi_-\Gamma^{-1}(W^{+1}(\Gamma(\mathbb{A}_h))I^{+1}) \text{ on } L^p(\mathbb{R}_-),$$
$$W^{-1}(\mathbb{A}_h) := \chi_+\Gamma^{-1}(W^{-1}(\Gamma(\mathbb{A}_h))I^{-1}) \text{ on } L^p(\mathbb{R}_+).$$

Of course, for rich $A \in \mathscr{A}_{L^p}$, $\mathbb{A} := \{P_nAP_n\}$ and a sequence $h \in \mathscr{H}_+$ there is a subsequence $g \in \mathscr{H}_{\mathbb{A}} := \mathscr{H}_{\Gamma(\mathbb{A})}$ of h such that $A_{\pm g}$ exist and then

$$W^{+1}(\mathbb{A}_g) = \chi_-A_g, \quad W^{-1}(\mathbb{A}_g) = \chi_+A_{-g}. \tag{3.2}$$

Now we are in a position to reformulate Theorems 3.2 - 3.5 in terms of operators in $\mathscr{L}(L^p(\mathbb{R}))$. For $\mathbb{A} = \{A_n\} \in \mathscr{F}_{\mathscr{A}_{L^p}}$ denote the set of all operators B_h which are $\mathscr{P}$-strong limits of one of the sequences

$$(U_{-h_n}[(I - P_{h_n}) + P_{h_n}A_{h_n}P_{h_n}]U_{h_n}) \text{ with } h \in \mathscr{H}_+ \ (\mathscr{H}_-)$$

by $\sigma^+_{\mathrm{stab}}(\mathbb{A})$ (or $\sigma^-_{\mathrm{stab}}(\mathbb{A})$, resp.) and set $\sigma_{\mathrm{stab}}(\mathbb{A}) := \{W(\mathbb{A})\} \cup \sigma^+_{\mathrm{stab}}(\mathbb{A}) \cup \sigma^-_{\mathrm{stab}}(\mathbb{A})$.

THEOREM 3.12. *A sequence $\mathbb{A} \in \mathscr{F}_{\mathscr{A}_{L^p}}$ is stable iff all operators in $\sigma_{\mathrm{stab}}(\mathbb{A})$ are invertible.*

Also Theorems 3.2 and 3.3 for $\mathbb{A} = \{A_n\} \in \mathscr{F}_{\mathscr{A}_{L^p}}$ are literally the same. Let us only discuss the index formula in more detail.

REMARK 3.13. If $\mathbb{A} = \{P_nAP_n\}$ is the finite section sequence of a single rich band-dominated operator A and $h \in \mathscr{H}_+$ such that A_h, A_{-h} exist (that is $h \in \tilde{\mathscr{H}}_{\mathbb{A}}$), then we particularly get the formula

$$\lim_{n\to\infty} \operatorname{ind} P_{h_n}AP_{h_n} = \operatorname{ind} A + \operatorname{ind} \chi_- A_h + \operatorname{ind} \chi_+ A_{-h} \tag{3.3}$$

where the latter operators are considered as operators acting on $L^p(\mathbb{R}_-)$ or $L^p(\mathbb{R}_+)$, respectively.

If A is a (not necessarily rich) band-dominated operator and $l \in \mathscr{H}_-, u \in \mathscr{H}_+$ are strictly decreasing or increasing sequences, respectively, such that $A_l, A_u \in \sigma_{\mathrm{op}}(A)$ exist, then one can also consider the adapted finite section sequence $\{P_n^{l,u}AP_n^{l,u}\}$ with $P_n^{l,u} = \chi_{[l(n),u(n))}I$ and finds as in Theorem 3.7

$$\lim_{n\to\infty} \operatorname{ind} P_n^{l,u}AP_n^{l,u} = \operatorname{ind} A + \operatorname{ind} \chi_- A_u + \operatorname{ind} \chi_+ A_l.$$

3.3.2. Locally compact operators

In [21] band-dominated operators of the form $A = I + K$ where K is locally compact were considered. At this, a band-dominated operator K is said to be locally compact if φA and $A\varphi I$ are compact operators for each function $\varphi \in \mathrm{BUC}$ with bounded support.

The operators $\chi_+ K \chi_- I$ and $\chi_- K \chi_+ I$ are compact for each locally compact K (see [21]). Thus, the operators $\chi_+ A \in \mathscr{L}(L^p(\mathbb{R}_+))$ and $\chi_- A \in \mathscr{L}(L^p(\mathbb{R}_-))$ are Fredholm operators, whenever $A = I + K$ is Fredholm. We call

$$\operatorname{ind}_+ A := \operatorname{ind}(\chi_+ A) \quad \text{and} \quad \operatorname{ind}_- A := \operatorname{ind}(\chi_- A)$$

the plus- and the minus-index of A and find that $\operatorname{ind} A = \operatorname{ind}_+ A + \operatorname{ind}_- A$. Further notice that the limit operators of a locally compact operator are locally compact again.

Then the main result of [21] for operators on the spaces L^p, $1 < p < \infty$ reads as follows:

THEOREM 3.14. *Let $A = I + K$ with K being a rich locally compact operator.*

1. *The operator A is Fredholm iff all limit operators of A are invertible and their inverses are uniformly bounded.*

2. *If A is Fredholm then, for arbitrary limit operators $B, C \in \sigma(A)$ with respect to sequences $u \in \mathscr{H}_+$, $l \in \mathscr{H}_-$, respectively,*

$$\operatorname{ind}_+ A = \operatorname{ind}_+ B, \quad \operatorname{ind}_- A = \operatorname{ind}_- C, \quad \textit{hence} \quad \operatorname{ind} A = \operatorname{ind}_+ B + \operatorname{ind}_- C.$$

Applying the results of the present paper this can be generalized.

It is easy to check that for all band-dominated operators $A = C + K$ with C invertible and K locally compact, and for all $p \in [1,\infty]$ the invertibility at infinity of A already implies its Fredholmness (see e.g. [15], Proposition 2.15). Furthermore, the $\hat{\mathscr{P}}$-dichotomy of A also yields the reverse implication. Consequently, for every $p \in [1,\infty]$ and every rich band-dominated operator of the form $A = C + K$ we have that A is Fredholm iff its limit operators are invertible and their inverses are uniformly bounded.

If $A = I + K$ is Fredholm then $\mathrm{ind}_+ B$ and $\mathrm{ind}_- B$ are finite numbers for every limit operator B of A and their sum equals zero since B is invertible. Moreover, the finite sections of locally compact operators are compact, hence the finite sections of $A = I + K$ are Fredholm of index 0. Thus, Remark 3.13 covers the second part of Theorem 3.14 and extends it to spaces L^p with $p \in [1,\infty]$ and to operators $A = I + K$ which need not to be rich but only possess at least one limit operator at $+\infty$ and at $-\infty$. Moreover, Formula (3.3) gives an extension to all band-dominated operators.

3.3.3. Convolution type operators

We turn to a further more concrete subclass of operators which were already studied by several authors. Thus, we omit repeating some details and refer the reader to e.g. [15], Section 4.2.

With every function $k \in L^1$, we associate the operator that maps a function $f \in L^p$ to the so-called convolution $k * f$ which is given by

$$(k * f)(x) = \int_{\mathbb{R}} k(x-y) f(y) dy, \quad x \in \mathbb{R}.$$

This operator is band-dominated and bounded by $\|k\|_1$, and it is usually denoted by C_a where a, the symbol of C_a, is the Fourier transform of k. Notice further that C_a is always shift invariant, hence rich.

Let $\mathscr{B}_p$ denote the Banach subalgebra which is generated by all such convolution operators and all rich operators bI of multiplication by a function $b \in L^\infty$. Moreover, introduce $\mathscr{B}_p^0$, the Banach algebra generated by all operators of the form $b_1 C_a b_2 I$ where, again, a is the Fourier transform of an L^1-function and $b_1 I, b_2 I$ are rich multiplication operators with functions $b_1, b_2 \in L^\infty$.

PROPOSITION 3.15. (cf. [15], Lemma 4.10, Proposition 4.11)

- *All operators in $\mathscr{B}_p^0$ are locally compact.*
- *The decomposition $\mathscr{B}_p = \{bI : b \in L^\infty, bI \text{ rich}\} \oplus \mathscr{B}_p^0$ holds.*

Hence every operator $A \in \mathscr{B}_p$ is of the form $A = bI + B$ with $B \in \mathscr{B}_p^0$ and if bI is invertible then the assumptions of Theorem 3.14 and its generalizations which were mentioned above hold. In particular, the Fredholm property and the Fredholm index of A are determined by its limit operators. For this notice that A can be written in the form $A = bI(I + \tilde{B})$ with $\tilde{B} \in \mathscr{B}_p^0$.

Further, Theorem 3.12 applies to the finite section sequence $\{P_nAP_n\}$ and states that the invertibility of A and all snapshots $W^{\pm 1}\{P_{g_n}AP_{g_n}\}$ (see (3.2)) is necessary and sufficient for its stability. This is even true for sequences in the finite section algebra. Also Theorems 3.2 and 3.3 on the Fredholm property of a sequence and the asymptotic behavior of the approximation numbers translate to this setting.

The paper [4] deals with several applications of these results for boundary integral equations.

3.4. On harmonic approximation of Fredholm Toeplitz operators

If $a \in L^\infty(\mathbb{T})$ and $T(a) \in \mathscr{L}(l^2(\mathbb{Z}_+))$ is the familiar Toeplitz operator with generating function a, then it is often convenient to study the family $\{T(h_ra)\}_{0<r<1}$ in order to study the Fredholm properties of $T(a)$, where h_ra is, by definition,

$$(h_ra)(e^{\mathrm{i}x}) = \sum_{l\in\mathbb{Z}} r^{|l|}a_le^{\mathrm{i}lx}, \quad x \in [0,2\pi).$$

Notice that h_ra can also be written as

$$(h_ra)(e^{\mathrm{i}x}) = \int_0^{2\pi} k_r(x-t)a(e^{\mathrm{i}t})dt, \quad x \in [0,2\pi),$$

where

$$k_r(x) = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos x+r^2}, \quad x \in \mathbb{R}.$$

The family $(k_r)_{0<r<1}$ forms a so-called approximate identity (in the classical sense, which differs from the notion we employed in Section 1). The study of harmonic extensions as well as the study of further approximate identities played an important role in Toeplitz operator theory (see [2], Chapters 3 and 4, which is likely the most complete source). Without going into great detail, we like to mention that the results of Section 2 give a slightly different understanding of this matter. The point is that $\{T(h_ra)\}_{0<r<1}$ can be embedded into an algebra $\mathscr{F}^T$, $T = \{1\}$, of generalized sequences $\{A_r\}_{0<r<1}$, with $A_r \in \mathbf{E}_r = l^2(\mathbb{Z}_+)$ for every $r \in (0,1)$. It is well known that $T(h_ra)$ converges $*$-strongly to $T(a)$ as $r \to 1$. The ideal $\mathscr{J}^T$ we have to deal with is the family

$$\mathscr{J}^T = \{\{K+G_r\}_{0<r<1} : K \in \mathscr{K}(l^2(\mathbb{Z}_+)), \{G_r\} \subset \mathscr{L}(l^2(\mathbb{Z}_+)), \|G_r\| \to 0 \text{ as } r \to 1\}.$$

Thus, if $\{T(h_ra)\}$ is a Fredholm sequence then $T(a)$ as well as all $T(h_ra)$, with r sufficiently close to 1, are Fredholm and

$$\operatorname{ind} T(a) = \lim_{r\to 1} \operatorname{ind} T(h_ra) = -\lim_{r\to 1} \operatorname{wind} h_ra$$

by Theorem 2.18. Notice that the results and proofs in Section 2 translate to the case of $*$-strongly converging sequences (see Remark 2.1) as well as to such algebras of generalized sequences. Alternatively, one might also pass to sequences $(r_n)_{n\in\mathbb{N}} \subset (0,1)$ which converge (increasingly) to 1, and apply Theorem 2.18 in the present form.

There are many instances where it can be proved that $\{T(h_ra)\}$ is a Fredholm sequence, for instance if a is locally sectorial, or a is invertible in the algebra $C+H^\infty$.

REFERENCES

[1] A. BÖTTCHER, *On the approximation numbers of large Toeplitz operators*, Documenta Mathematica **2** (1997), 1–29.

[2] A. BÖTTCHER, B. SILBERMANN, *Analysis of Toeplitz Operators*, Second edition, Springer-Verlag, Berlin, 2006.

[3] S. N. CHANDLER-WILDE, M. LINDNER, *Limit Operators, Collective Compactness and the Spectral Theory of Infinite Matrices*, Mem. Amer. Math. Soc. **210** (2011), No. 989.

[4] S. N. CHANDLER-WILDE, M. LINDNER, *Boundary integral equations on unbounded rough surfaces: Fredholmness and the Finite Section Method*, J. Integral Equations Appl. **20** (2008), 13–48.

[5] D. E. EDMUNDS, W. D. EVANS, *Spectral Theory and Differential Operators*, Oxford University Press, New York, 1987.

[6] I. C. GOHBERG, I. A. FELDMAN, *Convolution equations and projection methods for their solution*, Nauka, Moscow 1971; Engl. transl.: Amer. Math. Soc. Transl. of Math. Monographs 41, Providence, R. I., 1974; German transl.: Akademie-Verlag, Berlin 1974.

[7] I. C. GOHBERG, N. KRUPNIK, *One-dimensional Linear Singular Integral Equations*, Birkhäuser Verlag, Basel, Boston, Berlin, 1992.

[8] R. HAGEN, S. ROCH, B. SILBERMANN, *C^*-Algebras and Numerical Analysis*, Marcel Dekker, Inc., New York, Basel, 2001.

[9] R. HAGEN, S. ROCH, B. SILBERMANN, *Spectral Theory of Approximation Methods for Convolution Equations*, Operator Theory: Advances and Applications, Vol. 74, Birkhäuser Verlag, Basel, 1995.

[10] G. HEINIG, F. HELLINGER, *The finite section method for Moore-Penrose inversion of Toeplitz operators*, Integral Equations Oper. Theory **19**, 4 (1994), 419–446.

[11] P. JUNGHANNS, S. ROCH, B. SILBERMANN, *Collocation methods for systems of Cauchy singular integral equations on an interval*, Comp. Techn. **6** (2001), 88–124.

[12] A. V. KOZAK, *A local principle in the theory of projection methods*, Dokl. Akad. Nauk SSSR **212**, 6 (1973), 1287–1289; Engl. transl.: Sov. Math. Dokl. **14**, 5 (1974), 1580–1583.

[13] B. V. LANGE, V. S. RABINOVICH, *On the Noether property of multidimensional discrete convolutions*, Mat. Zametki **37**, 3 (1985), 407–421.

[14] M. LINDNER, *The finite section method and stable subsequences*, Appl. Num. Math. **60**, 4 (2010), 501–512.

[15] M. LINDNER, *Infinite Matrices and their Finite Sections*, Birkhäuser Verlag, Basel, Boston, Berlin, 2006.

[16] M. LINDNER, *Limit Operators and Applications on the Space of Essentially Bounded Functions*, Dissertationsschrift, Chemnitz, 2003.

[17] H. MASCARENHAS, B. SILBERMANN, *Spectral approximation and index for convolution type operators on cones on $L^p(\mathbb{R}^2)$*, Integral Equations Oper. Theory **65**, 3 (2009), 415–448.

[18] A. PIETSCH, *History of Banach spaces and linear operators*, Birkhäuser Boston, Inc., Boston, MA, 2007.

[19] A. PIETSCH, *Operator Ideals*, Verlag der Wissenschaften, Berlin, 1978.

[20] S. PRÖSSDORF, B. SILBERMANN, *Numerical Analysis for Integral and Related Operator Equations*, Birkhäuser Verlag, Basel, Boston, Berlin, 1991.

[21] V. S. RABINOVICH, S. ROCH, *The Fredholm index of locally compact band-dominated operators on $L^p(\mathbb{R})$*, Integral Equations Oper. Theory **57** (2007), 263–281.

[22] V. S. RABINOVICH, S. ROCH, J. ROE, *Fredholm indices of band-dominated operators*, Integral Equations Oper. Theory **49**, 2 (2004), 221–238.

[23] V. S. RABINOVICH, S. ROCH, B. SILBERMANN, *Band-dominated operators with operator-valued coefficients, their Fredholm properties and finite sections*, Integral Equations Oper. Theory **40**, 3 (2001), 342–381.

[24] V. S. RABINOVICH, S. ROCH, B. SILBERMANN, *Fredholm theory and finite section method for band-dominated operators*, Integral Equations Oper. Theory **30**, 4 (1998), 452–495.

[25] V. S. RABINOVICH, S. ROCH, B. SILBERMANN, *Limit Operators and Their Applications in Operator Theory*, Birkhäuser Verlag, Basel, Boston, Berlin, 2004.

[26] V. S. RABINOVICH, S. ROCH, B. SILBERMANN, *On finite sections of band-dominated operators*, Operator Theory: Advances and Applications **181** (2008), 385–391.

[27] V. RAKOČEVIĆ, J. ZEMÁNEK, *Lower s-numbers and their asymptotic behaviour*, Studia Math. **91**, 3 (1988), 231–239.

[28] M. REED, B. SIMON, *Methods of Modern Mathematical Physics IV: Analysis of Operators*, Academic Press, New York, 1978.

[29] S. ROCH, *Finite Sections of Band-Dominated Operators*, Memoirs of the AMS **191** (2008), No. 895.

[30] S. ROCH, B. SILBERMANN, *Non-strongly converging approximation methods*, Demonstr. Math. **22**, 3 (1989), 651–676.

[31] A. ROGOZHIN, B. SILBERMANN, *Approximation numbers for the finite sections of Toeplitz operators with piecewise continuous symbols*, J. Funct. Anal. **237**, 1 (2006), 135–149.

[32] A. ROGOZHIN, B. SILBERMANN, *Banach algebras of operator sequences: Approximation numbers*, J. Oper. Theory **57**, 2 (2007), 325–346.

[33] M. SEIDEL, B. SILBERMANN, *Finite sections of band-dominated operators: l^p-theory*, Complex Analysis and Operator Theory **2** (2008), 683–699.

[34] M. SEIDEL, B. SILBERMANN, *Banach algebras of structured matrix sequences*, Lin. Alg. Appl. **430** (2009), 1243–1281.

[35] I. B. SIMONENKO, *On multidimensional discrete convolutions*, Mat. Issled. **3**, 1 (1968), 108–127 (Russian).

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Markus Seidel
Technische Universität Chemnitz
Fakultät für Mathematik
09107 Chemnitz
Germany
e-mail: markus.seidel@mathematik.tu-chemnitz.de

Bernd Silbermann
Technische Universität Chemnitz
Fakultät für Mathematik
09107 Chemnitz
Germany
e-mail: bernd.silbermann@mathematik.tu-chemnitz.de