Operators
and
Matrices
Volume 6, Number 3 (2012), 433–464
doi:10.7153/oam-06-29

PRESERVERS OF MATRIX PAIRS WITH A FIXED INNER PRODUCT VALUE

CHI-KWONG LI, LUCIJAN PLEVNIK AND PETER ŠEMRL

(*Communicated by J. Pečarić*)

Abstract. Let $\mathscr{V}$ be the set of $n\times n$ hermitian matrices, the set of $n\times n$ symmetric matrices, the set of all effects, or the set of all projections of rank one. Let c be a real number. We characterize bijective maps $\phi:\mathscr{V}\to\mathscr{V}$ satisfying $\operatorname{tr}(AB)=c \iff \operatorname{tr}(\phi(A)\phi(B))=c$ with some additional restrictions on c, depending on the underlying set of matrices.

1. Introduction

We denote by $M_n(\mathbb{F})$ the set of all $n\times n$ matrices with coefficients from $\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}$ and by $\mathscr{H}_n\subset M_n(\mathbb{C})$ and $\mathscr{S}_n\subset M_n(\mathbb{R})$ the sets of hermitian and symmetric matrices, respectively. We further denote by $\mathscr{E}_n$ the set of all effects, by $\mathscr{P}_n$ the set of all projections and $\mathscr{P}_n^1$ the set of all projections of rank one, that is

$$\mathscr{E}_n=\{A\in M_n(\mathbb{F}) : 0\leqslant A\leqslant I\},$$

$$\mathscr{P}_n=\{P\in\mathscr{E}_n : P^2=P\},$$

$$\mathscr{P}_n^1=\{P\in\mathscr{P}_n : \operatorname{rank}P=1\}.$$

Let us remark that $\mathscr{E}_n$ may denote $\mathscr{E}_n(\mathbb{R})$ or $\mathscr{E}_n(\mathbb{C})$, that is the set of all real effects or the set of all complex effects. In our discussion, we sometimes consider both cases simultaneously. The precise meaning of $\mathscr{E}_n$ will be clear from the context.

Let c be a real number. In this paper we study bijective maps ϕ acting on any of the sets $\mathscr{H}_n,\mathscr{S}_n,\mathscr{E}_n$, and $\mathscr{P}_n^1$ satisfying

$$\operatorname{tr}(AB)=c \iff \operatorname{tr}(\phi(A)\phi(B))=c \tag{1}$$

for a given real number c. Our motivation is twofold. Let us recall that Wigner's unitary-antiunitary theorem can be formulated in the following way. If ϕ is a bijective

Mathematics subject classification (2010): 15A86, 15B57.
Keywords and phrases: hermitian matrix; real symmetric matrix; projection; effect; trace; preserver.
Research of Li was supported by a USA NSF grant, a HK RCG grant, and the Key Disciplines of Shanghai Municipality Grant S30104. He is an honorary professor of the University of Hong Kong, the Taiyuan University of Technology, and the Shanghai University. The third author was partially supported by a grant from the Ministry of Science of Slovenia.

Paper OaM-06-29

map defined on the set of all rank one projections acting on a Hilbert space H with the property that

$$\operatorname{tr}(PQ) = \operatorname{tr}(\phi(P)\phi(Q)), \tag{2}$$

then there is a unitary or an antiunitary operator $U : H \to H$ such that $\phi(P) = UPU^*$ for all rank one projections P. Identifying projections of rank one with one-dimensional subspaces of H we can reformulate Wigner's theorem by saying that every bijective map on one-dimensional subspaces (rays) of H which preserves the angles between rays is induced by a unitary or an anti-unitary operator on H. Uhlhorn's generalization [13] of Wigner's theorem states that the same conclusion holds under the weaker assumption that only the orthogonality of rays is preserved. More precisely, we get the same conclusion if we replace (2) by the weaker condition

$$\operatorname{tr}(PQ) = 0 \iff \operatorname{tr}(\phi(P)\phi(Q)) = 0. \tag{3}$$

Let us mention that in the mathematical foundations of quantum mechanics projections of rank one are called pure states and $\operatorname{tr}(PQ)$ corresponds to the transition probability between P and Q. It is natural to ask what happens if we replace the transition probability 0 in (3) by some other fixed value c, $0 < c < 1$. For more information on mathematical and physical background of this problem we refer to Molnár's book [12].

Another motivation comes from the study of 2-local automorphisms of operator algebras. Once again we refer to [12] for the details. Let us just mention that a main step in the characterization of 2-local automorphisms of certain standard operator algebras is the description of the general form of maps ϕ on matrix algebras satisfying the condition

$$\operatorname{tr}(\phi(A)\phi(B)) = \operatorname{tr}(AB) \tag{4}$$

for all matrices A and B (see Section 3.4 of [12], and in particular, (3.4.2) on page 189). Molnár's approach to the study of 2-local automorphisms based on maps satisfying (4) initiated a series of papers studying spectral conditions similar to (4), see for example [2, 3, 5, 7, 8]. We consider here a new direction by studying maps that preserve not the trace of all products, but just those having a given fixed value.

Our study can also be viewed as a special case of the study of non-linear preserver (also referred to as general preserver) problems, which concern the study of maps on matrices or operators with special properties. For example, for any given function f on matrices or operators, one seeks characterization of maps ϕ such that $f(\phi(A)\phi(B)) = f(AB)$ for all matrices in the domain of ϕ; for example, see [1] and the references therein. In many cases, the maps will simply be a multiplicative map composed by a simple operation such as multiplying by scalars. In our case, we consider the trace function of matrices, i.e., $f(A) = \operatorname{tr} A$, and impose a weaker condition $\operatorname{tr}(\phi(A)\phi(B)) = c$ whenever $\operatorname{tr}(AB) = c$ rather than assuming that $\operatorname{tr}(\phi(A)\phi(B)) = \operatorname{tr}(AB)$ for all (A,B) pairs. It is interesting to note that our results show that the maps also demonstrate a strong link with multiplicative maps. Of course, $\operatorname{tr}(A)$ is just the sum of eigenvalues of the matrix A. Thus, our study can also be viewed as a refinement of the study of maps preserving the eigenvalues or spectra of product of matrices; see [2] and the references therein. Also, one may consider special subset $\mathscr{S}$ of matrices or operators, and consider maps ϕ such that $\phi(A)\phi(B) \in \mathscr{S}$ whenever (if and only if) $AB \in \mathscr{S}$; for example, see [11]. Our problem is the special case when $\mathscr{S}$ is the set of matrices with trace equal to c.

More generally, for a binary operation $A * B$ on matrices or operators such as $A * B = A + B, A - B, AB, ABA, AB + BA, AB - BA$, or the Schur (entrywise) product $A \circ B$, there is interest in characterizing maps ϕ such that

1) $f(\phi(A) * \phi(B)) = f(A * B)$ for all (A,B) pairs,

2) $f(\phi(A) * \phi(B)) = c$ whenever (if and only if) $f(A * B) = c$, or

3) $\phi(A) * \phi(B) \in \mathscr{S}$ whenever (if and only if) $A * B \in \mathscr{S}$;

see [3]–[10] and the references therein.

Let us briefly explain our main results. We study bijective maps ϕ acting on any of the sets $\mathscr{H}_n, \mathscr{S}_n, \mathscr{E}_n$, and $\mathscr{P}_n^1$ satisfying the property (1). The cases when ϕ acts on $\mathscr{H}_n$ and $\mathscr{S}_n$ are relatively easy. We first observe that on real-linear spaces $\mathscr{H}_n$ and $\mathscr{S}_n$ we have the usual inner product defined by $\langle A, B \rangle = \operatorname{tr}(AB)$. Clearly, orthogonal transformations acting on $\mathscr{H}_n$ ($\mathscr{S}_n$) satisfy (1). Using the fundamental theorem of affine geometry we can prove that there are no other maps satisfying (1) provided that $c \neq 0$. In the case when $c = 0$ every orthogonal transformation multiplied by a scalar-valued everywhere nonzero function satisfy (1). And once again we are able to prove that these obvious examples are the only maps satisfying our assumptions. In this case the proof depends on the fundamental theorem of projective geometry.

The problem becomes much more intricate when we treat maps on the set of effects. First of all, it is easy to see that $\operatorname{tr}(AB) \leqslant \operatorname{tr} A$ for all $A, B \in \mathscr{E}_n$. So, the assumption (1) tells nothing about the behavior of ϕ on the subset of all effects whose trace is less than c. Let $c \in (0,1]$ and assume that a bijective map $\phi : \mathscr{E}_n \to \mathscr{E}_n$ satisfies (1). We will show that the set of all effects whose trace is not larger than c is invariant under ϕ. The behavior of ϕ on the set of all effects whose trace is $< c$ is arbitrary. But on the set of all effects whose trace is larger than c the map ϕ has the nice expected form. In the case of effects not only the result is more interesting, but also the proof is much more involved than that in the case of maps on $\mathscr{H}_n$ and $\mathscr{S}_n$.

In the context of quantum physics, it is interesting to study the case when the underlying set of matrices is $\mathscr{P}_n^1$. The study of this case turns out to be very challenging. We were able to get some results only in the real case. Of course, in the case of projections of rank one the condition (1) makes sense only for constants c satisfying $0 \leqslant c < 1$. Even in the real case we were not able to solve the problem completely. With our proof techniques we were able to cover only the cases when $c \geqslant 1/2$.

So, there are still a lot of open questions. Let us mention the most important ones: the complex case when dealing with maps on $\mathscr{P}_n^1$, the remaining values of c when treating the maps on effects and rank one projections, and the infinite-dimensional case.

2. Maps on $\mathscr{H}_n$ and $\mathscr{S}_n$

It is much easier to describe the general form of bijective maps satisfying (1) on $\mathscr{H}_n$ and $\mathscr{S}_n$ than on the subsets $\mathscr{E}_n$ and $\mathscr{P}_n^1$. The reason is that $\mathscr{H}_n$ and $\mathscr{S}_n$ are vector spaces. We will prove a more general result using a geometrical approach. We will start with the case when $c \neq 0$, and formulate the problem in terms of linear functionals.

LEMMA 2.1. *Let V be a finite-dimensional real vector space of dimension at least 2, V' its dual space, c a nonzero real number and $\tau : V \to V$ and $\sigma : V' \to V'$ maps. Then the following two statements are equivalent:*

- *τ and σ are bijective and for every pair $x \in V$ and $f \in V'$ we have*

$$f(x) = c \iff \sigma(f)(\tau(x)) = c, \tag{5}$$

- *τ and σ are linear and*

$$\sigma(f)(\tau(x)) = f(x), \quad x \in V, \quad f \in V'.$$

Proof. One direction is clear. So, assume that the first condition is fulfilled. Then we have $\tau(0) = 0$. Indeed, assume that this was not true. Then one could find a functional $g \in V'$ such that $g(\tau(0)) = c$. Applying the bijectivity of σ and (5) we get a contradiction. Similarly we show that $\sigma(0) = 0$.

We next show that if $x_1, \ldots, x_r \in V$ are linearly independent then $\tau(x_1), \ldots, \tau(x_r)$ are linearly independent as well. Every linearly independent subset of V can be extended to a basis of V. Thus, we may, and we will assume that $r = k = \dim V$. Then there is a unique $g \in V'$ such that $g(x_p) = c$ for every p, $1 \leqslant p \leqslant k$. Using (5) we now see that $f = \sigma(g)$ is the unique linear functional with the property that $f(\tau(x_1)) = \ldots = f(\tau(x_k)) = c$, and consequently, $\tau(x_1), \ldots, \tau(x_k)$ are linearly independent as well. In the same way we see that if the functionals $f_1, \ldots, f_p \in V'$ are linearly independent then $\sigma(f_1), \ldots, \sigma(f_p)$ are linearly independent as well. Clearly, the equation (5) holds with τ^{-1} and σ^{-1} instead of τ and σ. Thus, $x_1, \ldots, x_r \in V$ are linearly independent if and only if $\tau(x_1), \ldots, \tau(x_r)$ are linearly independent and an analogue holds for the map σ.

Let $x, u \in V$ be linearly independent. We will show that for the line $L = \{x + tu : t \in \mathbb{R}\}$ in V there exist $k-1$ linearly independent functionals $f_1, \ldots, f_{k-1} \in V'$ such that for $z \in V$ we have

$$z \in L \iff f_p(z) = c, \quad p = 1, \ldots, k-1.$$

Indeed, as x and u are linearly independent we can find $f_1 \in V'$ such that $f_1(x) = c$ and $f_1(u) = 0$. We can further find $k-2$ linearly independent functionals $g_2, \ldots, g_{k-1}$ such that $g_j(x) = g_j(u) = 0$. The functionals $f_1, f_2 = f_1 + g_2, \ldots, f_{k-1} = f_1 + g_{k-1}$ are linearly independent. Clearly, if $z \in L$, then $f_p(z) = c$, $p = 1, \ldots, k-1$. Assume next that $f_p(z) = c$, $p = 1, \ldots, k-1$. Then $g_2(z) = \ldots = g_{k-1}(z) = 0$. As the g_j's are linearly independent, the intersection of their kernels is two-dimensional. Thus, this intersection is the linear span of x and u. It follows that $z = sx + tu$ for some real numbers s and t. From $f_1(z) = c$ we conclude that $s = 1$. Hence, $z \in L$, as desired.

On the other hand, if $f_1, \dots, f_{k-1} \in V'$ are linearly independent functionals then we can find $x \in V$ such that $f_1(x) = \dots = f_{k-1}(x) = c$ and a nonzero vector u which spans the one-dimensional intersection of the kernels of these functionals. Of course, x and u are linearly independent and the set of all vectors $z \in V$ satisfying $f_1(z) = \dots = f_{k-1}(z) = c$ is exactly the line $\{z = x + tu : t \in \mathbb{R}\}$.

Let $L \subset V$ be a line that does not contain the origin of V. It follows from the previous three paragraphs that $\tau(L)$ is a line in V such that $0 \notin \tau(L)$. We already know that vectors $x, y \in V$ are linearly dependent if and only if $\tau(x)$ and $\tau(y)$ are linearly dependent. Thus, τ maps lines through the origin (one-dimensional subspaces) to lines of the same type.

Using the fundamental theorem of affine geometry [14] together with the fact that the identity is the only automorphism of the field of real numbers we conclude that τ is a bijective linear map. Similarly, σ is a bijective linear map.

The linearity together with (5) yields that

$$\sigma(f)(\tau(x)) = f(x)$$

for every $x \in V$ and every $f \in V'$. In other words, σ is the adjoint of the inverse of τ. $\square$

The desired descriptions of bijective preservers of matrix pairs with a fixed nonzero inner product value, acting on $\mathscr{H}_n$ and $\mathscr{S}_n$, are now straightforward consequences. All we need is to recall that the dual of the inner product space $\mathscr{H}_n$ can be identified with itself because of Riesz's representation of linear functionals with the inner product.

THEOREM 2.2. *Let n be a positive integer larger than 1, c a nonzero real number, and $\phi : \mathscr{H}_n \to \mathscr{H}_n$ a map. Then the following two statements are equivalent:*

- *ϕ is bijective and for every pair $A, B \in \mathscr{H}_n$ we have* $\operatorname{tr}(AB) = c \iff \operatorname{tr}(\phi(A)\phi(B)) = c$,
- *ϕ is an orthogonal transformation on $\mathscr{H}_n$ with respect to the usual inner product.*

THEOREM 2.3. *Let n be a positive integer larger than 1, c a nonzero real number, and $\phi : \mathscr{S}_n \to \mathscr{S}_n$ a map. Then the following two statements are equivalent:*

- *ϕ is bijective and for every pair $A, B \in \mathscr{S}_n$ we have* $\operatorname{tr}(AB) = c \iff \operatorname{tr}(\phi(A)\phi(B)) = c$,
- *ϕ is an orthogonal transformation on $\mathscr{S}_n$ with respect to the usual inner product.*

To solve completely our problem for $\mathscr{H}_n$ and $\mathscr{S}_n$ we have to treat the remaining case when $c = 0$. Once again we will prove a more general result considering pairs of maps acting on a general real vector space and its dual.

LEMMA 2.4. *Let V be a finite-dimensional real vector space of dimension at least 3, V' its dual space and $\tau : V \to V$ and $\sigma : V' \to V'$ maps. Then the following two statements are equivalent:*

- *τ and σ are bijective and for every pair $x \in V$ and $f \in V'$ we have*

$$f(x) = 0 \iff \sigma(f)(\tau(x)) = 0, \tag{6}$$

- *there exist bijective linear maps $\varphi : V \to V$ and $\eta : V' \to V'$, a nonzero real number a and functions $\xi : V \to \mathbb{R}^* = \mathbb{R} \setminus \{0\}$, $\zeta : V' \to \mathbb{R}^*$ such that*

$$\eta(f)(\varphi(x)) = af(x), \quad x \in V, \quad f \in V',$$

$$\tau(x) = \xi(x)\varphi(x),\ x \in V,$$

$$\sigma(f) = \zeta(f)\eta(f),\ f \in V',$$

and functions $t \mapsto t\xi(tx)$ and $t \mapsto t\zeta(tf)$ are bijections of $\mathbb{R}$ onto $\mathbb{R}$ for every nonzero $x \in V$ and every nonzero $f \in V'$, respectively.

Proof. Clearly, $\tau(0) = 0$ and $\sigma(0) = 0$. Set $k = \dim V$. Functionals $f_1, \ldots, f_k \in V'$ are linearly independent if and only if for every $x \in V$ we have $f_1(x) = \ldots = f_k(x) = 0 \iff x = 0$. It follows that functionals $g_1, \ldots, g_p \in V'$ are linearly independent if and only if $\sigma(g_1), \ldots, \sigma(g_p)$ are linearly independent. Similarly, a certain subset of vectors in V is linearly independent if and only if its τ-image is linearly independent.

Let $x \in V$ be a nonzero vector. We denote by $[x]$ the one-dimensional subspace of V spanned by x and by $\mathbb{P}V$ the projective space over V, $\mathbb{P}V = \{[x] : x \in V \setminus \{0\}\}$. Let $x \in V$ be any nonzero vector. Then there exist linearly independent functionals $f_1, \ldots, f_{k-1}$ such that $f_1(x) = \ldots = f_{k-1}(x) = 0$. We have $[x] = \{z \in V : f_1(z) = \ldots = f_{k-1}(z) = 0\}$. Therefore, $\tau([x]) = \{z \in V : \sigma(f_1)(z) = \ldots = \sigma(f_{k-1})(z) = 0\} = [\tau(x)]$. Hence, τ induces a bijective map $\mathbb{P}\tau : \mathbb{P}V \to \mathbb{P}V$ by the formula $\mathbb{P}\tau([x]) = [\tau(x)]$, $x \in V \setminus \{0\}$. Similarly, σ induces in a natural way the map $\mathbb{P}\sigma$ on the projective space $\mathbb{P}V'$.

We will now show that for every $x, y, z \in V \setminus \{0\}$ we have

$$[x] \subset [y] + [z] \iff \tau([x]) \subset \tau([y]) + \tau([z]).$$

We will prove only one direction, $[x] \subset [y] + [z] \Rightarrow \tau([x]) \subset \tau([y]) + \tau([z])$. There is nothing to prove if y and z are linearly dependent. So, assume that y and z are linearly independent. Then we can find linearly independent functionals $f_1, \ldots, f_{k-2}$ such that $f_1(y) = f_1(z) = \ldots = f_{k-2}(y) = f_{k-2}(z) = 0$. It follows that $f_1(x) = \ldots = f_{k-2}(x) = 0$. Clearly, $\tau([y])$ and $\tau([z])$ are two linearly independent one-dimensional subspaces of V that span the two-dimensional subspace $W = \{z \in V : \sigma(f_1)(z) = \ldots = \sigma(f_{k-2})(z) = 0\}$. Since $\tau([x]) \subset W$ we have $\tau([x]) \subset \tau([y]) + \tau([z])$, as desired.

By the fundamental theorem of projective geometry [14] there exists a bijective linear map $\varphi : V \to V$ such that $\mathbb{P}\tau([x]) = [\varphi(x)]$, $x \in V \setminus \{0\}$. Similarly, there exists a bijective linear map $\eta : V' \to V'$ such that $\mathbb{P}\sigma([f]) = [\eta(f)]$, $f \in V' \setminus \{0\}$. Clearly, for every $x \in V$ and $f \in V'$ we have

$$f(x) = 0 \iff \eta(f)(\varphi(x)) = 0.$$

By linearity, there exists a nonzero real number a such that

$$\eta(f)(\varphi(x)) = af(x), \quad x \in V, \quad f \in V',$$

and therefore, $\eta = a(\varphi^{-1})'$.

Moreover, there exists a function $\xi : V \to \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ such that $\tau(x) = \xi(x)\varphi(x)$, $x \in V$. Bijectivity of τ implies that for every nonzero $x \in V$ the function $t \mapsto t\xi(tx)$, $t \in \mathbb{R}$, is a bijection on $\mathbb{R}$. Similarly, there exists a function $\zeta : V' \to \mathbb{R}^*$ such that $\sigma(f) = \zeta(f)\eta(f)$, $f \in V'$. The function $t \mapsto t\zeta(tf)$, $t \in \mathbb{R}$, is a bijection of $\mathbb{R}$ onto $\mathbb{R}$ for every nonzero $f \in V'$. $\square$

We are now ready to treat our special cases $\mathscr{H}_n$ and $\mathscr{S}_n$.

THEOREM 2.5. *Let n be an integer larger than 1, and $\phi : \mathscr{H}_n \to \mathscr{H}_n$ a map. Then the following two statements are equivalent:*

- *ϕ is bijective and for every pair $A, B \in \mathscr{H}_n$ we have*

$$\operatorname{tr}(AB) = 0 \iff \operatorname{tr}(\phi(A)\phi(B)) = 0,$$

- *there exist an orthogonal (with respect to the usual inner product) transformation $\varphi : \mathscr{H}_n \to \mathscr{H}_n$ and a function $\xi : \mathscr{H}_n \to \mathbb{R}^*$ such that*

$$\phi(A) = \xi(A)\varphi(A), \quad A \in \mathscr{H}_n,$$

  and the function

$$t \mapsto t\xi(tA), \quad t \in \mathbb{R},$$

  is a bijection of $\mathbb{R}$ onto $\mathbb{R}$ for every nonzero $A \in \mathscr{H}_n$.

Proof. We will prove the theorem using Lemma 2.4 for n^2-dimensional real vector space $\mathscr{H}_n$. We consider bijective maps $\tau = \phi : \mathscr{H}_n \to \mathscr{H}_n$ and $\sigma : \mathscr{H}_n' \to \mathscr{H}_n'$, defined by $\sigma(B \mapsto \operatorname{tr}(AB)) = (B \mapsto \operatorname{tr}(\phi(A)B))$. By Lemma 2.4 we have

$$\phi(A) = \xi(A)\varphi(A), \quad A \in \mathscr{H}_n,$$

and

$$\phi(A) = \zeta(A)\left(\varphi^{-1}\right)^*(A), \quad A \in \mathscr{H}_n.$$

It follows that for every $A \in \mathscr{H}_n$ the matrices $\varphi(A)$ and $\left(\varphi^{-1}\right)^*(A)$ are linearly dependent. It is well known (and easy to check) that this yields

$$\left(\varphi^{-1}\right)^* = d\varphi$$

for some nonzero real constant d. In other words, we have $d\varphi\varphi^* = I$. Consequently, $d > 0$, and φ is an orthogonal transformation up to a multiplicative constant, which can be absorbed in the function ξ. $\square$

With almost the same proof we get the result in the real case.

THEOREM 2.6. *Let n be an integer larger than 1, and $\phi : \mathscr{S}_n \to \mathscr{S}_n$ a map. Then the following two statements are equivalent:*

- *ϕ is bijective and for every pair $A, B \in \mathscr{S}_n$ we have*

$$\operatorname{tr}(AB) = 0 \iff \operatorname{tr}(\phi(A)\phi(B)) = 0,$$

- *there exist an orthogonal (with respect to the usual inner product) transformation $\varphi : \mathscr{S}_n \to \mathscr{S}_n$ and a function $\xi : \mathscr{S}_n \to \mathbb{R}^*$ such that*

$$\phi(A) = \xi(A)\varphi(A), \quad A \in \mathscr{S}_n,$$

and the function

$$t \mapsto t\xi(tA), \quad t \in \mathbb{R},$$

is a bijection of $\mathbb{R}$ onto $\mathbb{R}$ for every nonzero $A \in \mathscr{S}_n$.

3. Maps on $\mathscr{E}_n$

In this section we will identify $n \times n$ matrices with linear operators acting on $\mathbb{F}^n$, the space of all $n \times 1$ matrices. For a given matrix A we denote by $\operatorname{Im} A$ the image of the corresponding operator. We will deal with the real and the complex case simultaneously. Whenever doing so, we will simply use the term unitary matrix (operator) U to denote a unitary matrix (operator) in the complex case and an orthogonal matrix (operator) in the real case. Further, A^* will denote the adjoint of operator A. Hence, in the matrix language A^* stands for the conjugate transpose of A in the complex case and for the transpose in the real case. And of course, when treating scalars, $\overline{\mu}$ will denote the conjugate of μ in the complex case, while $\overline{\mu} = \mu$ in the real case.

We will start with the special case $c = 0$. Thus, we are interested in bijective maps $\phi : \mathscr{E}_n \to \mathscr{E}_n$ with the property that

$$\operatorname{tr}(AB) = 0 \iff \operatorname{tr}(\phi(A)\phi(B)) = 0. \tag{7}$$

We first observe that for $A, B \in \mathscr{E}_n$ the condition $\operatorname{tr}(AB) = 0$ is equivalent to $AB = 0$. Indeed, let us assume that $\operatorname{tr}(AB) = 0$. After applying the unitary similarity we may assume that $A = \operatorname{diag}(t_1, \ldots, t_r, 0, \ldots, 0)$, where all the t_j's are positive and $r = \operatorname{rank} A$. The diagonal entries of B are nonnegative. Hence, $\operatorname{tr}(AB) = 0$ yields that the first r diagonal entries of B are zero. All principal 2×2 minors of B are nonnegative, and consequently, the first r rows and the first r columns of B must be zero. In fact, we have shown that for every pair $A, B \in \mathscr{E}_n$ we have $\operatorname{tr}(AB) = 0 \iff AB = 0 \iff \operatorname{Im} A \perp \operatorname{Im} B$.

Clearly, if U is any $n \times n$ unitary matrix, then the map $A \mapsto UAU^*$ is a bijection of $\mathscr{E}_n$ onto itself with the property (7). The same is true for the map $A \mapsto A^t$. Let now $\varphi : \mathscr{E}_n \to \mathscr{E}_n$ be any bijective image preserving map, that is, for every $A \in \mathscr{E}_n$ we have $\operatorname{Im} \varphi(A) = \operatorname{Im} A$. Clearly, such a map also satisfies the condition (7).

THEOREM 3.1. *Let n be an integer larger than 2, $\mathbb{F} = \mathbb{C}$, and $\phi : \mathscr{E}_n \to \mathscr{E}_n$ a map. Then the following conditions are equivalent:*

- *ϕ is bijective and for every pair $A, B \in \mathscr{E}_n$ we have*
$$\operatorname{tr}(AB) = 0 \iff \operatorname{tr}(\phi(A)\phi(B)) = 0,$$
- *there exist a unitary $n \times n$ matrix U and a bijective image preserving map $\varphi : \mathscr{E}_n \to \mathscr{E}_n$ such that either*
  - *$\phi(A) = U\varphi(A)U^*$, $A \in \mathscr{E}_n$, or*
  - *$\phi(A) = U\varphi(A)^t U^*$, $A \in \mathscr{E}_n$.*

THEOREM 3.2. *Let n be an integer larger than 2, $\mathbb{F} = \mathbb{R}$, and $\phi : \mathscr{E}_n \to \mathscr{E}_n$ a map. Then the following conditions are equivalent:*

- *ϕ is bijective and for every pair $A, B \in \mathscr{E}_n$ we have*
$$\operatorname{tr}(AB) = 0 \iff \operatorname{tr}(\phi(A)\phi(B)) = 0,$$
- *there exist an orthogonal $n \times n$ matrix O and a bijective image preserving map $\varphi : \mathscr{E}_n \to \mathscr{E}_n$ such that $\phi(A) = O\varphi(A)O^t$, $A \in \mathscr{E}_n$.*

Note that in the complex case the map $A \mapsto A^t, A \in \mathscr{E}_n$, is the entrywise complex conjugation. It is therefore not surprising that when describing the general form of bijective maps preserving trace zero products on effects, we have two possibilities in the complex case and only one in the real case.

Observe also that it is trivial to describe the general form of bijective image preserving maps. Namely, we introduce an equivalence relation on $\mathscr{E}_n$ by $A \sim B$ if and only if $\operatorname{Im} A = \operatorname{Im} B$. So, every such map has to act like bijection on each $\sim$ equivalence class.

Proof of Theorems 3.1 *and* 3.2. We will prove both theorems simultaneously.

All we need to do is to prove that the first condition implies the second one. Obviously, $A \sim B$ if and only if $A^\perp = \{C \in \mathscr{E}_n : AC = 0\} = \{C \in \mathscr{E}_n : BC = 0\} = B^\perp$. By our assumption, $\phi(A^\perp) = \phi(A)^\perp$. It follows that we have $A \sim B$ if and only if $\phi(A) \sim \phi(B)$.

In each equivalence class with respect to the relation $\sim$ there is a unique projection P. It follows that ϕ induces a bijective map $\psi : \mathscr{P}_n \to \mathscr{P}_n$ with the property that for every $P, Q \in \mathscr{P}_n$ we have $PQ = 0$ if and only if $\psi(P)\psi(Q) = 0$. Similarly as above we define $P^\perp = \{Q \in \mathscr{P}_n : PQ = 0\}$, $P \in \mathscr{P}_n$. Clearly, $\psi(P^\perp) = \psi(P)^\perp$. As $P^\perp = \mathscr{P}_n$ if and only if $P = 0$, we have $\psi(0) = 0$. For a nonzero $P \in \mathscr{P}_n$ the set $P^\perp$ is maximal among orthogonal complements of nonzero projections if and only if P is a rank one projection. Hence, ψ maps the set of projections of rank one onto itself. This restriction preserves orthogonality. By Uhlhorn's theorem [12] there exists a unitary matrix U (let us remind here that U is an orthogonal matrix in the real case) such that either $\psi(P) = UPU^*$ for all projections of rank one, or $\psi(P) = UP^tU^*$ for all projections of rank one (note that in the real case these two possibilities coincide). After composing the map ϕ by $A \mapsto U^*AU$ and by the transposition, if necessary, we may assume with no loss of generality that ψ maps every projection of rank one into itself. Using the fact that the map ψ preserves orthogonality on $\mathscr{P}_n$ we conclude that $\psi(P) = P$ for every projection $P \in \mathscr{P}_n$. It follows from $A \sim B \iff \mathrm{Im}A = \mathrm{Im}B$ and $A \sim B \iff \phi(A) \sim \phi(B)$ that ϕ is an image preserving map. $\square$

In order to formulate the main result of this section we need some more notation. Let c be a real number, $0 < c \leqslant 1$. We set $\mathscr{E}_n(c^-) = \{A \in \mathscr{E}_n : \mathrm{tr}A < c\}$, $\mathscr{E}_n(c^+) = \{A \in \mathscr{E}_n : \mathrm{tr}A > c\}$, and $\mathscr{E}_n(c) = \{A \in \mathscr{E}_n : \mathrm{tr}A = c\}$. If $A \in \mathscr{E}_n$, set $A(c) = \{B \in \mathscr{E}_n : \mathrm{tr}(AB) = c\}$. For an arbitrary set $\mathscr{P} \subset \mathscr{E}_n$ and a matrix $A \in \mathscr{E}_n$ denote further $\mathscr{P}(c,A) = A(c) \cap \mathscr{P} = \{P \in \mathscr{P} : \mathrm{tr}(AP) = c\}$.

We start with two technical lemmas. We will denote by $\{E_{11}, E_{12}, \dots, E_{nn}\}$ the standard basis of the space of $n \times n$ matrices.

LEMMA 3.3. *Let n be an integer larger than 2. Suppose that $\mathscr{P} \subset \mathscr{E}_n$ is a set such that $U\mathscr{P}U^* = \mathscr{P}$ for any unitary U. Let $D \in \mathscr{E}_n$ be a diagonal matrix and $0 < c \leqslant 1$. Assume that $Q = [q_{ij}] \in \mathscr{P}(c,D)$ and $q_{ij} \neq 0$ for some $i \neq j$. Then $\mu E_{ij} + \overline{\mu} E_{ji} \in \mathrm{span}\,\mathscr{P}(c,D)$ for any $\mu \in \mathbb{F}$.*

Proof. First observe that the diagonality of D yields that $U\mathscr{P}(c,D)U^* = \mathscr{P}(c,D)$ for any diagonal unitary matrix U. Consequently, $U\mathrm{span}\,\mathscr{P}(c,D)U^* = \mathrm{span}\,\mathscr{P}(c,D)$.

Without loss of generality, assume that $(i,j) = (1,2)$. Consider the diagonal matrices $U_1 = 2E_{11} - I$ and $U_2 = 2E_{11} + 2E_{22} - I$. We already know that $U_1QU_1 \in \mathscr{P}(c,D)$, so $X = Q - U_1QU_1 \in \mathrm{span}\,\mathscr{P}(c,D)$ and $Y = X + U_2XU_2 \in \mathrm{span}\,\mathscr{P}(c,D)$. Since X has nonzero entries only in the first column and in the first row, but not in the $(1,1)$ position, $Y = \gamma E_{12} + \overline{\gamma} E_{21}$ for some $\gamma \in \mathbb{F}$. Note that $\gamma \neq 0$ because $q_{12} \neq 0$. So, for $U_3 = (\nu - 1)E_{11} + I$, where $|\nu| = 1$, we have $U_3YU_3^* = \nu\gamma E_{12} + \overline{\nu\gamma} E_{21} \in \mathrm{span}\,\mathscr{P}(c,D)$. $\square$

LEMMA 3.4. *Let n be an integer larger than 2, $0 < c \leqslant 1$ and suppose that $\phi : \mathscr{E}_n \to \mathscr{E}_n$ is a bijective map such that*

- $\mathrm{tr}(AB) = c \iff \mathrm{tr}(\phi(A)\phi(B)) = c, \quad A, B \in \mathscr{E}_n$,
- $\phi(P) = P$ *for any* $P \in \mathscr{P}_n \setminus \{0\}$.

Then $\phi(A) = A$ for every $A \in \mathscr{E}_n(c^+)$.

Proof. Introduce the set

$$\Lambda = \left\{ \lambda P : P \in \mathscr{P}_n, \mathrm{rank}P = n-1, \frac{c}{n-1} < \lambda \leqslant 1, \frac{c}{\lambda} \notin \mathbb{N} \right\} \subset \mathscr{E}_n\left(c^+\right).$$

The first step in the proof is to show that $\phi(\lambda P) = \lambda P$ for every $\lambda P \in \Lambda$ with $\lambda < 1$ (we already know that the statement is true for $\lambda = 1$). When doing so, we may assume with no loss of generality that $P = \mathrm{diag}(0, 1, \dots, 1)$.

Recall that $\mathscr{P}_n(c,\lambda P)=\{R\in\mathscr{P}_n : \operatorname{tr}(\lambda PR)=c\}$. Our aim is to prove that $\operatorname{span}\mathscr{P}_n(c,\lambda P)=\mathscr{H}_n$ (or $\mathscr{S}_n$ in the real case) with the help of Lemma 3.3 (for the set $\mathscr{P}=\mathscr{P}_n$). Set $m=\lceil\frac{c}{\lambda}\rceil\leqslant n-1$ (here, $\lceil t\rceil=\min\{k\in\mathbb{Z} : k\geqslant t\}$) and note that $0<m-\frac{c}{\lambda}<1$. Let $Q=[q_{ij}]$ be any rank m projection such that $q_{11}=m-\frac{c}{\lambda}$. Then $\operatorname{tr}(\lambda PQ)=\lambda\operatorname{tr}(Q-(I-P)Q)=\lambda(m-q_{11})=c$, so $Q\in\mathscr{P}_n(c,\lambda P)$. Now Lemma 3.3 ensures that $\operatorname{span}\mathscr{P}_n(c,\lambda P)$ contains all hermitian matrices with zero diagonals. It is also clear that we can find projections $Q_1,\ldots,Q_n\in\mathscr{P}_n(c,\lambda P)$ with linearly independent vectors of diagonals.

We proved that

- $\operatorname{span}\mathscr{P}_n(c,\lambda P)=\mathscr{H}_n$, if $\mathbb{F}=\mathbb{C}$, and
- $\operatorname{span}\mathscr{P}_n(c,\lambda P)=\mathscr{S}_n$, if $\mathbb{F}=\mathbb{R}$,

which implies that $\operatorname{tr}(\lambda PH)=\operatorname{tr}(\phi(\lambda P)H)$ for every $H\in\mathscr{H}_n$ or $\mathscr{S}_n$, respectively. It is clear that $\phi(\lambda P)=\lambda P$ in both cases.

Denote the eigenvalues of $A\in\mathscr{E}_n$ by $\lambda_1(A)\geqslant\ldots\geqslant\lambda_n(A)$. Set $\mathscr{I}=\left\{\mu I : \frac{c}{n-1}\leqslant\mu\leqslant 1\right\}$ and

$$\mathscr{A}=\left\{A\in\mathscr{E}_n\left(c^+\right) : \sum_{j=2}^{n}\lambda_j(A)\geqslant c\right\}\setminus\mathscr{I}.$$

In this step we show that $\phi(A)=A$ for every $A\in\mathscr{A}$. We will again use Lemma 3.3, this time for $\mathscr{P}=\Lambda$. So, let $A\in\mathscr{A}$ be arbitrary. We may assume that $A=\operatorname{diag}(a_1,\ldots,a_n)$ with $1\geqslant a_1\geqslant\ldots\geqslant a_n\geqslant 0$, $a_1>a_n$ and $\sum_{j=2}^n a_j\geqslant c$. Consider a matrix $X=\lambda(I-xx^t)\in\Lambda$, where $x=(x_1,\ldots,x_n)^t\in\mathbb{F}^n$ is a unit vector with positive entries, $\frac{c}{n-1}<\lambda<1$ and $\frac{c}{\lambda}\notin\mathbb{N}$. Then $X\in\Lambda(c,A)$ if and only if

$$\operatorname{tr}A-\sum_{j=1}^{n}a_jx_j^2=\frac{c}{\lambda}\in(c,n-1)\setminus\mathbb{N}. \tag{8}$$

Observe that $\operatorname{tr}A-\sum_{j=1}^n a_jx_j^2\in\left(\sum_{j=2}^n a_j,\sum_{j=1}^{n-1}a_j\right)\subset(c,n-1)$. Let

$$\mathscr{S}=\left\{x=(x_1,\ldots,x_n)^t\in\mathbb{F}^n : x\in(0,1)^n,\ \|x\|=1\right\}$$

and consider the function $f:\mathscr{S}\to\left(\sum_{j=2}^n a_j,\sum_{j=1}^{n-1}a_j\right)$, given by $f(x)=\operatorname{tr}A-\sum_{j=1}^n a_jx_j^2$. Set $U=f^{-1}\left(\left(\sum_{j=2}^n a_j,\sum_{j=1}^{n-1}a_j\right)\setminus\mathbb{N}\right)$, which is clearly not empty. Since f is continuous, U is an open set in $\mathscr{S}$ and consequently open in S^{n-1}. We proved that for any $x\in U$ there exists $\lambda_x\in(\frac{c}{n-1},1)\setminus\frac{c}{\mathbb{N}}$, such that (8) holds. It is now clear that we can find matrices $X_1,\ldots,X_n\in\Lambda(c,A)$ with all entries nonzero and linearly independent vectors of diagonals. Now Lemma 3.3 yields that $\phi(A)=A$.

Next, we show that ϕ fixes each element in the set $\mathscr{B}=\left\{B\in\mathscr{E}_n\left(c^+\right) : \sum_{j=2}^n\lambda_j(B)<c\right\}$. Let $B=\operatorname{diag}(b_1,\ldots,b_n)$ be a diagonal matrix from $\mathscr{B}$ with $b_1\geqslant\ldots\geqslant b_n$. Set $A=I-axx^t\in\mathscr{A}$, where $x=(x_1,\ldots,x_n)^t\in\mathbb{F}^n$ is a unit vector with nonnegative entries

and $0 < a \leqslant 1$. Then $A \in \mathscr{A}(c,B)$ if and only if

$$a\sum_{j=1}^{n} b_j x_j^2 = \sum_{j=1}^{n} b_j - c = b_1 - \left(c - \sum_{j=2}^{n} b_j\right). \tag{9}$$

Note that $\sum_{j=1}^{n} b_j x_j^2 \leqslant b_1$, where equality holds if $x_1 = 1$. If we choose x_1 large enough, then for any choice of $x_2,\ldots,x_n$ we can find $a \in (0,1]$ satisfying (9). Hence, there exist matrices $A, A_1,\ldots,A_n \in \mathscr{A}(c,B)$ such that A has all entries nonzero and vectors of diagonals of $A_1,\ldots,A_n$ are linearly independent. So, $\phi(B) = B$.

In the last step it remains to consider the set $\mathscr{I}$. We have proved so far that ϕ acts like the identity on the set $\mathscr{D} = \mathscr{E}_n(c^+) \setminus \mathscr{I}$. But for an arbitrary $\mu \in \left[\frac{c}{n-1}, 1\right)$ we have $\mathscr{D}(c,\mu I) = \left\{A \in \mathscr{E}_n(c^+) : \operatorname{tr} A = \frac{c}{\mu}\right\} \setminus \left\{\frac{c}{\mu n} I\right\}$. We see that $\operatorname{span}\mathscr{D} = \mathscr{H}_n$ (or $\mathscr{S}_n$ respectively) and consequently, $\phi(\mu I) = \mu I$. This completes the proof. $\square$

THEOREM 3.5. *Let n be an integer larger than 2, $\mathbb{F} = \mathbb{C}$, $c \in (0,1]$, and $\phi : \mathscr{E}_n \to \mathscr{E}_n$ a map. Then the following conditions are equivalent:*

- *ϕ is bijective and for every pair $A, B \in \mathscr{E}_n$ we have*

$$\operatorname{tr}(AB) = c \iff \operatorname{tr}(\phi(A)\phi(B)) = c, \tag{10}$$

- *ϕ maps $\mathscr{E}_n(c^-)$ bijectively onto $\mathscr{E}_n(c^-)$, ϕ maps $\mathscr{E}_n(c)$ bijectively onto $\mathscr{E}_n(c)$, and there exists a unitary $n \times n$ matrix U such that either*
  - *$\phi(A) = UAU^*$ for every $A \in \mathscr{E}_n(c^+)$ and $\operatorname{Im}\phi(A) = \operatorname{Im} UAU^*$ for every $A \in \mathscr{E}_n(c)$, or*
  - *$\phi(A) = UA^tU^*$ for every $A \in \mathscr{E}_n(c^+)$ and $\operatorname{Im}\phi(A) = \operatorname{Im} UA^tU^*$ for every $A \in \mathscr{E}_n(c)$.*

THEOREM 3.6. *Let n be an integer larger than 2, $\mathbb{F} = \mathbb{R}$, $c \in (0,1]$, and $\phi : \mathscr{E}_n \to \mathscr{E}_n$ a map. Then the following conditions are equivalent:*

- *ϕ is bijective and for every pair $A, B \in \mathscr{E}_n$ we have*

$$\operatorname{tr}(AB) = c \iff \operatorname{tr}(\phi(A)\phi(B)) = c,$$

- *ϕ maps $\mathscr{E}_n(c^-)$ bijectively onto $\mathscr{E}_n(c^-)$, ϕ maps $\mathscr{E}_n(c)$ bijectively onto $\mathscr{E}_n(c)$, and there exists an orthogonal $n \times n$ matrix O such that*

$$\phi(A) = OAO^t$$

  for every $A \in \mathscr{E}_n(c^+)$ and $\operatorname{Im}\phi(A) = \operatorname{Im} OAO^t$ for every $A \in \mathscr{E}_n(c)$.

Proof of Theorems 3.5 *and* 3.6. We will prove both theorems simultaneously.

We start with the assumption that ϕ satisfies the first condition in our theorem. In the first step of the proof we establish that the set $\mathscr{E}_n(c^-)$ is invariant for ϕ. Let $A,B \in \mathscr{E}_n$. When calculating $\operatorname{tr}(AB)$ we may always assume that A is diagonal. Of course, all the diagonal entries of A are $\leqslant 1$. It follows that $\operatorname{tr}(AB) \leqslant \operatorname{tr} B$. Hence, we have $B \in \mathscr{E}_n(c^-)$ if and only if there is no $A \in \mathscr{E}_n$ such that $\operatorname{tr}(AB) = c$. It follows that a bijective map $\phi : \mathscr{E}_n \to \mathscr{E}_n$ satisfying (10) maps $\mathscr{E}_n(c^-)$ bijectively onto itself (and, of course, the behavior of such a ϕ on this subset is completely arbitrary).

In the next step we consider the set $\mathscr{E}_n(c)$. Recall that for an arbitrary $A \in \mathscr{E}_n$ the set $A(c)$ was defined by $A(c) = \{B \in \mathscr{E}_n : \operatorname{tr}(AB) = c\}$. We claim that $A(c)$ is a singleton if and only if $\operatorname{tr} A = c$ and A is strictly positive (0 is not an eigenvalue of A). Assume first that $A \in \mathscr{E}_n$ is a strictly positive matrix with $\operatorname{tr} A = c$. There is no loss of generality in assuming that A is diagonal, $A = \operatorname{diag}(t_1,\ldots,t_n)$ with $t_j > 0$, $j = 1,\ldots,n$, and $t_1 + \ldots + t_n = c$. Clearly, $I \in A(c)$. We have to prove that $B \in \mathscr{E}_n$ and $\operatorname{tr}(AB) = c$ yields $B = I$. This is easy as the diagonal entries $b_{11},\ldots,b_{nn}$ of the matrix B are all $\leqslant 1$. It follows from $t_1 b_{11} + \ldots + t_n b_{nn} = c$ that all diagonal entries of B are actually equal to 1. Since $B \leqslant I$, we necessarily have $B = I$. Assume next that $A(c)$ is a singleton. Then $\operatorname{tr} A \geqslant c$. We may, and we will assume that A is diagonal with the sum of diagonal entries $\geqslant c$. It is straightforward to see that if $\operatorname{tr} A > c$ or if one of the diagonal entries of A is zero, then there are infinitely many diagonal matrices in $A(c)$. Hence, A is an invertible matrix with trace c.

It follows that ϕ maps the set of invertible matrices with trace c onto itself. And therefore, $\phi(I) = I$, which further yields that $\phi(\mathscr{E}_n(c)) = \mathscr{E}_n(c)$.

We now start to investigate the behavior of ϕ on the set $\mathscr{P}_n$. In fact, we will first show that ϕ induces in a natural way a bijective map ψ on $\mathscr{P}_n$. When doing so, the following sets will be of the significant importance. For every nonzero projection $P \in \mathscr{P}_n$ we set $\mathscr{E}_n(P) = \{A \in \mathscr{E}_n : AP = P\}$. Thus, $\mathscr{E}_n(P)$ is the set of all effects A which act like the identity on $\operatorname{Im} P$. For every such effect A we conclude from $AP = P$ and $A \leqslant I$ that A maps the orthogonal complement of $\operatorname{Im} P$ into itself. Hence, $A \in \mathscr{E}_n$ belongs to $\mathscr{E}_n(P)$ if and only if $Ax = x$ for every $x \in \operatorname{Im} P$ and $Ax \in \operatorname{Ker} P$ for every $x \in \operatorname{Ker} P$.

Let now $A \in \mathscr{E}_n$ be any member of $\mathscr{E}_n(c)$. We denote by P_A the orthogonal projection onto the image of A. We claim that $A(c) = \mathscr{E}_n(P_A)$. Indeed, all we need to do is to show that $A(c) \subset \mathscr{E}_n(P_A)$, since the opposite inclusion trivially holds true. There is no loss of generality in assuming that A is diagonal, $A = \operatorname{diag}(t_1,\ldots,t_r,0,\ldots,0)$, $t_1 + \ldots + t_r = c$, and $t_j > 0$, $j = 1,\ldots r$. Then, as above, $B \in \mathscr{E}_n$ belongs to $A(c)$ if and only if the first r diagonal entries of B are equal to 1. Since $B \leqslant I$, we conclude that B is a 2×2 block diagonal matrix with the upper left entry I_r, the $r \times r$ identity matrix. As $P_A = \operatorname{diag}(I_r, 0)$, we clearly have $B \in \mathscr{E}_n(P_A)$, as desired.

Recall that for $A,B \in \mathscr{E}_n$ we write $A \sim B$ if and only if $\operatorname{Im} A = \operatorname{Im} B$. For $A,B \in \mathscr{E}_n(c)$ we know by the previous paragraph that $A \sim B$ if and only if $A(c) = B(c)$. As $A(c) = B(c)$ if and only if $(\phi(A))(c) = (\phi(B))(c)$ we conclude that for every pair $A,B \in \mathscr{E}_n(c)$ we have

$$A \sim B \iff \phi(A) \sim \phi(B).$$

It follows that the map ϕ induces in a natural way a map $\psi : \mathscr{P}_n \to \mathscr{P}_n$, namely, for an arbitrary nonzero $P \in \mathscr{P}_n$ choose any $A \in \mathscr{E}_n(c)$ with $P = P_A$ and define $\psi(P) = P_{\phi(A)}$ and set $\psi(0) = 0$. Obviously, ψ is bijective. Clearly, we have $P \leqslant Q$ for some $P, Q \in \mathscr{P}_n$ if and only if $\mathscr{E}_n(Q) \subset \mathscr{E}_n(P)$. As each $\mathscr{E}_n(P)$, $P \in \mathscr{P}_n \setminus \{0\}$, is equal to $A(c)$ for some $A \in \mathscr{E}_n(c)$, we have for every pair $P, Q \in \mathscr{P}_n$ that

$$P \leqslant Q \iff \psi(P) \leqslant \psi(Q).$$

By the fundamental theorem of projective geometry there exists a bijective semilinear map $L : \mathbb{F}^n \to \mathbb{F}^n$ such that

$$\psi(Q) = P_{L(\operatorname{Im} Q)}, \quad Q \in \mathscr{P}_n,$$

where P_U denotes the orthogonal projection of $\mathbb{F}^n$ onto the subspace $U \subset \mathbb{F}^n$. It follows that

$$\phi(\mathscr{E}_n(Q)) = \mathscr{E}_n(P_{L(\operatorname{Im} Q)}), \quad Q \in \mathscr{P}_n \setminus \{0\}. \tag{11}$$

Recall that the semilinearity of L means that L is additive and $L(\lambda x) = \omega(\lambda) Lx$, $\lambda \in \mathbb{F}$, $x \in \mathbb{F}^n$, for some field automorphism $\omega : \mathbb{F} \to \mathbb{F}$. Note also that $\omega = id$ is the only field automorphism of $\mathbb{R}$. It is well-known (and easy to see) that there exists a unique semilinear map $L^* : \mathbb{F}^n \to \mathbb{F}^n$ such that

$$\langle Lx, y \rangle = \omega(\langle x, L^* y \rangle)$$

for all $x, y \in \mathbb{F}^n$. Clearly, the automorphism of the field $\mathbb{F}$ that corresponds to the semilinear map L^* is $\lambda \mapsto \overline{\omega^{-1}\left(\overline{\lambda}\right)}$, $\lambda \in \mathbb{F}$.

Our next goal is to show that the set $\mathscr{P}_n \setminus \{0\}$ is invariant for ϕ. We will achieve this goal in a few steps. We have to distinguish two cases. We start with the case when $c = 1$. Let $P, Q \in \mathscr{E}_n$ be nonzero projections such that $\operatorname{rank} P + \operatorname{rank} Q = n + 1$. Let further $A \in \mathscr{E}_n(P)$ and $B \in \mathscr{E}_n(Q)$. We claim that $\operatorname{tr}(AB) = 1$ if and only if $A = P$, $B = Q$, and P and Q commute and PQ is a projection of rank one. Indeed, denote $R = I - Q$. Then $\operatorname{rank} R = \operatorname{rank} P - 1$. From $A \in \mathscr{E}_n(P)$ and $B \in \mathscr{E}_n(Q)$ we conclude that $A = P + A_1$ and $B = Q + B_1$ with $PA_1 = A_1 P = 0$ and $QB_1 = B_1 Q = 0$. Hence,

$$\operatorname{tr}(AB) = \operatorname{tr}(PQ + A_1 Q + PB_1 + A_1 B_1) \geqslant \operatorname{tr}(PQ)$$

where $\geqslant$ is an equality if and only if $\operatorname{tr}(A_1 Q + PB_1 + A_1 B_1) = 0$ which is equivalent to $A_1 Q = PB_1 = A_1 B_1 = 0$. Moreover,

$$\operatorname{tr}(PQ) = \operatorname{tr}(P(I - R)) = \operatorname{rank} P - \operatorname{tr}(PR) \geqslant \operatorname{rank} P - \operatorname{tr}(R) = \operatorname{rank} P - \operatorname{rank} R = 1,$$

where $\geqslant$ is an equality if and only if $\operatorname{tr}(PR) = \operatorname{tr} R$ if and only if $R \leqslant P$. It is now clear that $\operatorname{tr}(AB) = 1$ yields that P and Q commute and that PQ is a projection of rank one. It then follows from $PA_1 = 0$ that $A_1 = QA_1$. As we know that $QA_1 = 0$ we conclude that $A_1 = 0$, and similarly, $B_1 = 0$. Thus, $A = P$ and $B = Q$, as desired. The converse is trivial.

We next claim that ϕ maps nonzero projections into nonzero projections and for every pair of nonzero projections P,R we have $P \leqslant R$ if and only if $\phi(P) \leqslant \phi(R)$. Indeed, let P be a nonzero projection. Choose a projection Q that commutes with P such that $\operatorname{rank} P + \operatorname{rank} Q = n+1$ and PQ is a projection of rank one. It follows that $\operatorname{tr}(\phi(P)\phi(Q)) = 1$. As

$$\phi(P) \in \mathscr{E}_n\left(P_{L(\operatorname{Im} P)}\right) \quad \text{and} \quad \phi(Q) \in \mathscr{E}_n\left(P_{L(\operatorname{Im} Q)}\right)$$

the previous paragraph yields that $\phi(P) = P_{L(\operatorname{Im} P)}$ is a projection. Moreover, $\phi(P)$ and $\phi(Q)$ commute and $\operatorname{rank}(\phi(P)\phi(Q)) = 1$. It is also clear that $P \leqslant R$ if and only if $\phi(P) \leqslant \phi(R)$, $P,R \in \mathscr{P}_n$. Further, ϕ maps every projection of rank one to a projection of rank one. Moreover, if P_1 and P_2 are projections of rank one, then $P_1P_2 = 0$ if and only if $\phi(P_1)\phi(P_2) = 0$. Indeed, it is enough to check that $P_1P_2 = 0$ yields $\phi(P_1)\phi(P_2) = 0$. Let P be a rank two projection such that $P_1 \leqslant P$ and $P_2 \leqslant I - P$. We have $P_2 \leqslant Q$ for every projection Q of rank $n-1$ satisfying $PQ = QP$ and $\operatorname{rank}(PQ) = 1$. By what we have already proved, $\phi(P_2) \leqslant Q'$ for every projection Q' of rank $n-1$ satisfying $\phi(P)Q' = Q'\phi(P)$ and $\operatorname{rank}(\phi(P)Q') = 1$. It follows that $\phi(P_2) \leqslant I - \phi(P)$, and since $\phi(P_1) \leqslant \phi(P)$, we have $\phi(P_1)\phi(P_2) = 0$, as desired.

Using Uhlhorn's theorem we see that there exists a unitary operator $U : \mathbb{F}^n \to \mathbb{F}^n$ such that either $\phi(P) = UPU^*$ for every projection of rank one, or $\phi(P) = UP^tU^*$ for every projection of rank one. After composing ϕ with a unitary similarity and the transposition, if necessary, we may assume that $\phi(P) = P$ for every projection of rank one. It follows from $P \leqslant Q \iff \phi(P) \leqslant \phi(Q)$, $P,Q \in \mathscr{P}_n \setminus \{0\}$, that $\phi(P) = P$ for every nonzero projection in $\mathscr{E}_n$. For $A \in \mathscr{E}_n(1)$ and a projection $P \in \mathscr{P}_n$ we have $\operatorname{tr}(AP) = 1 \iff \operatorname{Im} A \subset \operatorname{Im} P$. It follows that $\operatorname{Im}\phi(A) = \operatorname{Im} A$ for every A of trace 1. Lemma 3.4 now tells us that $\phi(A) = A$ for every $A \in \mathscr{E}_n(1^+)$.

We now turn to the case when $0 < c < 1$. In this case we will first show that ϕ acts nice on the set of rank one projections. We start with some technicalities. Let $P \in \mathscr{E}_n$ be a projection of rank one. Then P can be written as $P = xx^*$, where $x \in \mathbb{F}^n$ is a column vector of norm one. Let further $Q \in \mathscr{E}_n$ be a projection of rank $n-1$, that is, $Q = I - yy^*$ for some $y \in \mathbb{F}^n$ of norm one. Set $a = \sqrt{1-c} \in (0,1)$. We will prove that the following two statements are equivalent:

- there exists a unique pair of matrices $A \in \mathscr{E}_n(P)$ and $B \in \mathscr{E}_n(Q)$ such that $\operatorname{tr}(AB) = c$,
- $|\langle x,y\rangle| = a$, $\operatorname{tr}(PQ) = c$, and $\operatorname{tr}(AB) > c$ for every pair of matrices $A \in \mathscr{E}_n(P)$ and $B \in \mathscr{E}_n(Q)$ such that $A \neq P$ or $B \neq Q$.

In order to prove this equivalence we start with some simple observations. We have

$$\operatorname{tr}(PQ) = \operatorname{tr}(xx^* - xx^*yy^*) = 1 - |\langle x,y\rangle|^2.$$

Let $y, y_1, \ldots y_{n-1} \in \mathbb{F}^n$ be an orthonormal basis, that is, the linear span of $y_1, \ldots, y_{n-1}$ is equal to the image of Q. Then the vectors $x, y_1, \ldots, y_{n-1}$ are linearly independent if and only if x does not belong to the linear span of $y_1, \ldots, y_{n-1}$ which is equivalent to

$x \not\perp y$. This is equivalent to $\mathrm{tr}\,(PQ) < 1$. For $A \in \mathscr{E}_n(P)$, $A = P + A_1$, and $B \in \mathscr{E}_n(Q)$, $B = Q + B_1$, we have

$$\mathrm{tr}\,(AB) = \mathrm{tr}\,(PQ) + \mathrm{tr}\,(A_1Q + PB_1 + A_1B_1).$$

If $\mathrm{tr}\,(PQ) < 1$ and $A_1 \neq 0$, then $x, y_1, \ldots, y_{n-1}$ are linearly independent and since the image of A_1 is orthogonal to x, it cannot be orthogonal to the image of Q which yields that $A_1Q \neq 0$, and consequently, $\mathrm{tr}\,(A_1Q) > 0$. Similarly, if $\mathrm{tr}\,(PQ) < 1$ and $B_1 \neq 0$, then $\mathrm{tr}\,(PB_1) > 0$. It is clear that the second condition above implies the first one with $A = P$ and $B = Q$. So, assume that the first condition is satisfied. We have three possibilities for $\mathrm{tr}\,(PQ)$, namely, $\mathrm{tr}\,(PQ) > c$, $\mathrm{tr}\,(PQ) < c$, and $\mathrm{tr}\,(PQ) = c$. In the first case we would have $\mathrm{tr}\,(AB) > c$ for every pair $A \in \mathscr{E}_n(P)$, $B \in \mathscr{E}_n(Q)$, a contradiction. In the second case we have $P + t(I-P) \in \mathscr{E}_n(P)$ for every $t \in [0,1]$ and $Q + s(I-Q) \in \mathscr{E}_n(Q)$ for every $s \in [0,1]$. The equation

$$c = \mathrm{tr}\,((P + t(I-P))(Q + s(I-Q))) = \mathrm{tr}\,(PQ) + tc_1 + sc_2 + tsc_3$$

is fulfilled for infinitely many pairs of real numbers $t, s \in [0,1]$ because $\mathrm{tr}\,(PQ) < c$, $c_1 = \mathrm{tr}\,((I-P)Q) \geqslant n-2 \geqslant 1$ (note that both Q and $I-P$ are projections of rank $n-1$), $c_2 = \mathrm{tr}\,(P(I-Q)) > 0$, and $c_3 \geqslant 0$. This is impossible as there exists only one pair of matrices $A \in \mathscr{E}_n(P)$ and $B \in \mathscr{E}_n(Q)$ such that $\mathrm{tr}\,(AB) = c$. Hence, we must have the third possibility $\mathrm{tr}\,(PQ) = c$. It is now straightforward to check that the second condition holds true.

Let $x, y \in \mathbb{F}^n$ be any vectors of norm one satisfying $|\langle x, y\rangle| = a$, and let $P = xx^*$ be a projection of rank one. We will prove that $\phi(P)$ is a projection of rank one,

$$\phi(P) = \frac{1}{\|Lx\|^2}(Lx)(Lx)^*. \tag{12}$$

Set $Q = I - yy^*$. Clearly, $P \in \mathscr{E}_n(P)$ and $Q \in \mathscr{E}_n(Q)$ is the unique pair of matrices from these two sets with the property $\mathrm{tr}\,(PQ) = c$. It follows from (11) that $\phi(P) \in \mathscr{E}_n(P_{L(\mathrm{Im}\,P)})$ and $\phi(Q) \in \mathscr{E}_n(P_{L(\mathrm{Im}\,Q)})$ is the unique pair whose product has trace c. Hence,

$$\phi(P) = P_{L(\mathrm{Im}\,P)} = \frac{1}{\|Lx\|^2}(Lx)(Lx)^*.$$

Our next aim is to show that L is either linear or conjugate linear. If Q is as before, we have

$$\phi(Q) = P_{L(\mathrm{Im}\,Q)} = P_{L\left(\{y\}^\perp\right)} = I - \frac{1}{\left\|(L^*)^{-1}y\right\|^2}\left((L^*)^{-1}y\right)\left((L^*)^{-1}y\right)^*.$$

It follows from $\mathrm{tr}\,(\phi(P)\phi(Q)) = c$ that

$$\left|\left\langle \frac{1}{\|Lx\|}Lx, \frac{1}{\left\|(L^*)^{-1}y\right\|}(L^*)^{-1}y \right\rangle\right| = a,$$

which yields

$$a = \frac{1}{\|Lx\| \left\|(L^*)^{-1} y\right\|} \left|\left\langle Lx, (L^*)^{-1} y\right\rangle\right| = \frac{1}{\|Lx\| \left\|(L^*)^{-1} y\right\|} |\omega(\langle x, y\rangle)|. \tag{13}$$

In particular, if $\langle x, y\rangle = a$, then

$$\|Lx\| \left\|(L^*)^{-1} y\right\| = \frac{|\omega(a)|}{a}. \tag{14}$$

Suppose now that $\mathbb{F} = \mathbb{C}$. We will prove that ω is either the identity or the complex conjugation. Choose $y = (1, 0, \ldots, 0)^t$ and $x(t) = \left(a, \sqrt{1-a^2}e^{it}, 0, \ldots, 0\right)^t$. It follows from (14) that the norm

$$\left\|Lx(t)\right\| = \left\|u + \omega(e^{it})v\right\|$$

has a constant value independent of t. Here,

$$u = L\left((a, 0, \ldots, 0)^t\right) \quad \text{and} \quad v = L\left((0, \sqrt{1-a^2}, 0, \ldots, 0)^t\right).$$

We conclude that the set $\{\omega\left(e^{it}\right) : t \in \mathbb{R}\}$ is bounded. Recall that ω is an automorphism of the complex field. Hence, for every real t the set $\{\omega\left(e^{int}\right) : n \in \mathbb{Z}\} = \left\{\left(\omega\left(e^{it}\right)\right)^n : n \in \mathbb{Z}\right\}$ is bounded, and consequently,

$$\left|\omega\left(e^{it}\right)\right| = 1$$

for every real t. Hence, for every real t there is a real s such that $\omega(\cos t + i \sin t) = \cos s + i \sin s$. But then

$$\omega(\cos t - i \sin t) = \omega\left(\left(\cos t + i \sin t\right)^{-1}\right) = (\cos s + i \sin s)^{-1} = \cos s - i \sin s,$$

and therefore, $\omega(\cos t) \in \mathbb{R}$, $t \in \mathbb{R}$. As $\omega(n\lambda) = n\omega(\lambda)$, $\lambda \in \mathbb{C}$, $n \in \mathbb{N}$, we conclude that $\omega(\mathbb{R}) \subset \mathbb{R}$. Hence, the restriction of ω to the subfield $\mathbb{R}$ is a nonzero endomorphism of $\mathbb{R}$. It is well-known that the identity is the only nonzero endomorphism of the real field. Thus, $\omega(t) = t$ for every real t. As $\omega(i) \in \{-i, i\}$, we conclude that ω is either the identity or the complex conjugation.

In the next step we prove that L is a scalar multiple of a unitary or anti-unitary operator. It follows from (13) that $x, y \in \mathbb{F}^n$, $\|x\| = \|y\| = 1$, and $|\langle x, y\rangle| = a$ yields

$$\|Lx\| \left\|(L^*)^{-1} y\right\| = 1.$$

Thus, if x_1, x_2 are two unit vectors such that there exists a unit vector $y \in \mathbb{F}^n$ satisfying $|\langle x_1, y\rangle| = |\langle x_2, y\rangle| = a$, then $\|Lx_1\| = \|Lx_2\|$. If $x, z \in \mathbb{F}^n$ are any two unit vectors, we can find two chains of unit vectors $x = x_1, x_2, \ldots, x_{n-1}, x_n = z$ and $y_1, \ldots, y_{n-1}$ such that

$$|\langle x_k, y_k\rangle| = |\langle x_{k+1}, y_k\rangle| = a, \quad k = 1, \ldots, n-1.$$

Hence, $\|Lx\| = \|Lz\|$ whenever $\|x\| = \|z\| = 1$. Consequently, $L = pU$ for some nonzero $p \in \mathbb{F}$ and some unitary or anti-unitary operator U, so

$$\phi(P) = UPU^*$$

for every projection of rank one. When $\mathbb{F} = \mathbb{C}$, we further observe that every anti-unitary operator U can be written as $U = VJ$, where V is a unitary operator and $J : \mathbb{C}^n \to \mathbb{C}^n$ is the entry-wise complex conjugation. Thus, if U is anti-unitary, then $UPU^* = (VJ)P(VJ)^* = V(JPJ)V^* = VP^tV^*$.

After composing ϕ with a unitary similarity and the transposition, if necessary, we may, and we will assume that

$$\phi(P) = P$$

for every projection P of rank one.

We now show that the same is true also for projections of higher rank. Let $P = I_r \oplus O_{n-r}$ be a projection with $1 < r < n$. Consider a rank one projection $Q = xx^t$, where $x = (x_1, \ldots, x_n)^t$ is a unit vector with nonnegative entries. Then $Q \in \mathscr{P}_n^1(c, P)$ if and only if $\sum_{j=1}^r x_j^2 = c$. It is now clear that we can find rank one projections $Q, Q^{(1)}, \ldots, Q^{(n)} \in \mathscr{P}_n^1(c, P)$ such that Q has no zero entries and vectors of diagonal entries of $Q^{(1)}, \ldots, Q^{(n)}$ are linearly independent. By Lemma 3.3 we have $\phi(P) = P$. Furthermore, it follows from Lemma 3.4 that $\phi(A) = A$ for every $A \in \mathscr{E}_n(c^+)$. And finally, if $A \in \mathscr{E}_n(c)$ and $P \in \mathscr{P}_n$, then $\mathrm{tr}\,(AP) = c$ if and only if $\mathrm{Im}\, A \subset \mathrm{Im}\, P$. Hence, $\mathrm{Im}\, \phi(A) = \mathrm{Im}\, A$ for every $A \in \mathscr{E}_n(c)$.

We have proved that the first condition in our theorem implies the second one. The other direction is easy. $\square$

4. Maps on $\mathscr{P}_n^1$

In this section we will characterize bijective maps ϕ acting on $\mathscr{P}_n^1$ satisfying the property

$$\mathrm{tr}\,(PQ) = c^2 \iff \mathrm{tr}\,(\phi(P)\phi(Q)) = c^2$$

under the assumptions that $\mathbb{F} = \mathbb{R}$, $n \geqslant 5$, and $\frac{1}{\sqrt{2}} \leqslant c < 1$.

We can translate this problem into the language of projective geometry. For $x, y \in \mathbb{R}^n \setminus \{0\}$ let $P = \frac{1}{x^t x} xx^t$ and $Q = \frac{1}{y^t y} yy^t$ be projections onto lines $[x]$ and $[y]$. Then $\mathrm{tr}\,(PQ) = \left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle^2$. For arbitrary $[x], [y] \in \mathbb{PR}^n$ denote $\{[x], [y]\} = \left| \left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle \right|$, which is obviously well-defined. So ϕ induces a bijective map ψ which acts on $\mathbb{PR}^n$ and satisfies the property

$$\{[x], [y]\} = c \iff \{\psi([x]), \psi([y])\} = c. \tag{15}$$

Note that for an orthogonal transformation $O : \mathbb{R}^n \to \mathbb{R}^n$ and a projection $P = \frac{1}{x^t x} xx^t$ we have $\phi(P) = OPO^t$ if and only if $\psi([x]) = [Ox]$.

Remark that we can represent $\{[x], [y]\}$ as the cosine of the angle between lines $[x]$ and $[y]$, so in the language of projective geometry condition (15) means that ψ preserves a fixed angle $\varphi \in \left(0, \frac{\pi}{4}\right]$ in a projective space.

For an arbitrary subset $\mathscr{S} \subset \mathbb{PR}^n$ and $a \in [0,1]$ denote

$$\mathscr{S}^a = \{[z] \in \mathbb{PR}^n : \{[x],[z]\} = a \text{ for every } [x] \in \mathscr{S}\}$$

and for $[x] \in \mathbb{PR}^n$ let $[x]^a := \{[x]\}^a$.

From now on, every time we choose a vector x, which spans a line $[x]$, we will suppose that $\|x\| = 1$.

We start with some technical lemmas.

LEMMA 4.1. *Let n be an integer larger than 3, $\frac{1}{\sqrt{2}} \leqslant c < 1$ and $[x],[y] \in \mathbb{PR}^n$.*

- *If $c > \frac{1}{\sqrt{2}}$, then*

$$\{[x],[y]\} = 2c^2 - 1 \iff \#([x]^c \cap [y]^c) = 1.$$

- *If $c = \frac{1}{\sqrt{2}}$, then*

$$\{[x],[y]\} = 2c^2 - 1 = 0 \iff \#([x]^c \cap [y]^c) = 2.$$

REMARK 4.2. If we set $c = \cos\varphi$ for a suitable $\varphi \in \left(0, \frac{\pi}{4}\right]$, then $2c^2 - 1$ is exactly $\cos(2\varphi)$. Although it is possible to give a computational proof of the Lemma, we believe that we can omit tedious computations as the statement is geometrically evident.

COROLLARY 4.3. *Let n be an integer larger than 3, $\frac{1}{\sqrt{2}} \leqslant c < 1$ and $\psi : \mathbb{PR}^n \to \mathbb{PR}^n$ a bijective map such that*

$$\{[x],[y]\} = c \iff \{\psi([x]),\psi([y])\} = c, \quad [x],[y] \in \mathbb{PR}^n.$$

Then

$$\{[x],[y]\} = 2c^2 - 1 \iff \{\psi([x]),\psi([y])\} = 2c^2 - 1, \quad [x],[y] \in \mathbb{PR}^n.$$

Denote the unit sphere in $\mathbb{R}^n$ by S^{n-1}.

LEMMA 4.4. *Let n be an integer larger than 2, $d \in [-1,1]$, and $\tau : S^{n-1} \to S^{n-1}$ a bijective map such that*

$$x \perp y \iff \langle \tau(x), \tau(y) \rangle = d, \quad x,y \in S^{n-1}. \tag{16}$$

Then $d = 0$.

Proof. Obviously, the possibility $d = 1$ contradicts the bijectivity assumption. So, we may assume that $-1 \leqslant d < 1$. Let $m < n$ and $e_1, \dots, e_m$ be a collection of pairwise orthogonal vectors in S^{n-1}. Then we can find at least two vectors $e_{m+1} \in S^{n-1}$ with the property that $e_{m+1} \perp e_j$ for $j = 1, \dots, m$. If $m = n-1$, then there are exactly two such vectors e_n (say e and $-e$). Furthermore let $e_1,\dots,e_n$ be an orthonormal system and denote $f_j=-e_j$ for $j=1,\dots,n$, so the vectors $e_1,\dots,e_{j-1},f_j,e_{j+1},\dots,e_n$ are pairwise orthogonal. Then the vectors $f_1,\dots,f_n$ are also pairwise orthogonal.

When treating the collections of vectors e_j with the property that $\langle e_j,e_k\rangle=d$, whenever $k\neq j$, we will distinguish a few cases. First suppose that $d>-\frac{1}{n-1}$. Set

$$e_1=(u_1,\mathbf{0})^t,$$

$$e_2=(d_1,u_2,\mathbf{0})^t,$$

$$e_3=(d_1,d_2,u_3,\mathbf{0})^t,$$

$$\dots$$

$$e_n=(d_1,d_2,\dots,d_{n-1},u_n)^t,$$

where

$$d_k=d\sqrt{\frac{1-d}{(1+(k-2)d)(1+(k-1)d)}},\quad k=1,\dots,n-1$$

and

$$u_k=\sqrt{\frac{(1-d)(1+(k-1)d)}{1+(k-2)d}},\quad k=1,\dots,n.$$

Then

$$\langle e_j,e_k\rangle=\begin{cases} d, \text{ if } k\neq j,\\ 1, \text{ if } k=j.\end{cases}$$

Observe that for any $j\in\{1,\dots,n\}$ we can find exactly one vector $f_j\in S^{n-1}$ such that $f_j\neq e_j$ and $\langle f_j,e_k\rangle=d$, whenever $k\neq j$. One can verify that the condition

$$\langle f_j,f_k\rangle=d \ \text{ whenever } \ k\neq j$$

can be fulfilled only if $d=0$.

Suppose that $e_1,\dots,e_n\in S^{n-1}$ and $f_1,\dots,f_n$ are as in the second paragraph of the proof. Then the $\tau^{-1}(e_j)$'s and $\tau^{-1}(f_j)$'s are vectors as in the first paragraph. In particular, $\tau^{-1}(f_1),\dots,\tau^{-1}(f_n)$ is an orthonormal set which yields that $\langle f_j,f_k\rangle=d$ whenever $k\neq j$. Consequently, $d=0$, as desired.

Assume now that $d=-\frac{1}{m}$ for some $m\in\{1,\dots,n-1\}$. If $e_1,\dots,e_m$ are vectors as in the second paragraph, then there exists exactly one $e\in S^{n-1}$ such that $\langle e_j,e\rangle=-\frac{1}{m}$ for $j=1,\dots,m$. This contradicts (16).

Turn now to the remaining case, that is $-\frac{1}{m-1}<d<-\frac{1}{m}$ for some integer $2\leqslant m\leqslant n-1$. Then there does not exist a vector $e\in S^{n-1}$ with the property $\langle e_j,e\rangle=d$ for $j=1,\dots,m$. We are again in a contradiction with (16). $\square$

LEMMA 4.5. *Let n be an integer larger than 2, $x,y\in\mathbb{R}^n\setminus\{0\}$, $\lambda,\mu\in\mathbb{R}$ and $d>0$ such that $|\lambda|<d\|x\|$. If*

$$\{z\in\mathbb{R}^n : \|z\|=d,\ \langle z,x\rangle=\lambda\}\subseteq\{z\in\mathbb{R}^n : \|z\|=d,\ \langle z,y\rangle=\mu\},$$

then x and y are linearly dependent.

Proof. Denote

$$X = \{z \in \mathbb{R}^n : \|z\| = d, \langle z, x\rangle = \lambda\},$$

$$Y = \{z \in \mathbb{R}^n : \|z\| = d, \langle z, y\rangle = \mu\}.$$

First suppose that $\lambda \neq 0$. Since $|\lambda| < d\|x\|$, X is an intersection of a sphere in $\mathbb{R}^n$ and $(n-1)$-dimensional affine space, which does not contain 0. Hence, $\operatorname{span} X = \mathbb{R}^n$. It follows from $X \subseteq Y$ that $\langle z, \mu x - \lambda y\rangle = 0$ for every $z \in X$ and consequently, $\mu x - \lambda y = 0$.

If $\lambda = 0$, then it must be $\mu = 0$ as well. Otherwise for any nonzero $u \in [x]^\perp$ we would have $\frac{d}{\|u\|}u \in X \subseteq Y$ and consequently $\langle u, y\rangle = \frac{\|u\|}{d}\left\langle \frac{d}{\|u\|}u, y\right\rangle = \frac{\|u\|}{d}\mu \neq 0$. Hence, $[x]^\perp \cap [y]^\perp = \{0\}$, contradicting the fact that $n \geqslant 3$. It follows from $\mu = 0$ that $[x]^\perp \subseteq [y]^\perp$ and x, y are linearly dependent. $\square$

For a given integer n larger than 2 and a real number $0 < c < 1$ denote

$$\mathscr{C} = \left\{[c, \mathbf{v}] \in \mathbb{PR}^n : \mathbf{v} \in \mathbb{R}^{n-1}, \|\mathbf{v}\| = \sqrt{1-c^2}\right\}.$$

Note that here we have simplified the notation. However, we believe it is clear that $[c, \mathbf{v}]$ denotes the one-dimensional span of the vector, whose first coordinate is c and the other coordinates coincide with the coordinates of $\mathbf{v}$.

LEMMA 4.6. *Let n be an integer larger than 4 and $\frac{1}{\sqrt{2}} < c < 1$. Suppose that $\psi : \mathbb{PR}^n \to \mathbb{PR}^n$ is a bijective map, $u \in \left(0, \sqrt{\frac{(1+2c)(1-c)}{1+c}}\right)$, and the following conditions hold:*

- $\{[x], [y]\} = c \iff \{\psi([x]), \psi([y])\} = c$,
- $\psi([1, 0, \ldots, 0]) = [1, 0, \ldots, 0]$,
- $\psi([c, u, \mathbf{u}]) = [c, u, \mathbf{u}]$ *for every* $\mathbf{u} \in \mathbb{R}^{n-2}$ *of norm* $\sqrt{1-c^2-u^2}$.

Then $\psi([c, t, \mathbf{t}]) = [c, t, \mathbf{t}]$ for any $[c, t, \mathbf{t}] \in \mathscr{C}$, for which

$$0 \leqslant t < \frac{\sqrt{(1+2c)(1-c^2-u^2)} - cu}{1+c}. \tag{17}$$

REMARK 4.7. One can easily verify that $\sqrt{\frac{(1+2c)(1-c)}{1+c}} < \sqrt{1-c^2}$. Consequently, $c^2 + u^2 < 1$. Further, note that

$$\sqrt{(1+2c)(1-c^2-u^2)} > cu \tag{18}$$

follows from $0 < u < \sqrt{\frac{(1+2c)(1-c)}{1+c}}$.

Proof. Let us start with some observations. The obvious one is that $\psi(\mathscr{C}) = \mathscr{C}$, which follows from the first two assumptions in Lemma. Next, we see by definition that for $[c,\mathbf{v}_0], [c,\mathbf{v}] \in \mathscr{C}$ we have $\{[c,\mathbf{v}_0],[c,\mathbf{v}]\} = c$ if and only if $\langle \mathbf{v}_0,\mathbf{v}\rangle = \pm c - c^2$. But $\pm$ on the right-hand side must be $+$. Otherwise we would have $c + c^2 = |\langle \mathbf{v}_0,\mathbf{v}\rangle| \leqslant 1 - c^2$, contradicting $c > \frac{1}{\sqrt{2}}$. Hence,

$$\{[c,\mathbf{v}_0],[c,\mathbf{v}]\} = c \iff \langle \mathbf{v}_0,\mathbf{v}\rangle = c - c^2. \tag{19}$$

We will show that if $[c,\mathbf{v}] \in \mathscr{C}$ and $\psi([c,\mathbf{v}]) = [c,\mathbf{w}]$, then

$$\psi([c,-\mathbf{v}]) = [c,-\mathbf{w}]. \tag{20}$$

In order to prove this we recall Corollary 4.3. So, all we need to do is to show that

$$[c,\mathbf{v}]^{2c^2-1} \cap \mathscr{C} = \{[c,-\mathbf{v}]\}.$$

Checking $[c,-\mathbf{v}] \in [c,\mathbf{v}]^{2c^2-1} \cap \mathscr{C}$ is trivial. So, assume that $[c,\mathbf{z}] \in [c,\mathbf{v}]^{2c^2-1} \cap \mathscr{C}$. Then either

$$\langle \mathbf{z},\mathbf{v}\rangle = c^2 - 1,$$

or

$$\langle \mathbf{z},\mathbf{v}\rangle = 1 - 3c^2.$$

We further know that $|\langle \mathbf{z},\mathbf{v}\rangle| \leqslant 1 - c^2$. So, it is clear that in the first case we have $\mathbf{z} = -\mathbf{v}$, as desired. It remains to show that the second possibility cannot occur. This is indeed so, as $\left|1-3c^2\right| = 3c^2 - 1 > 1 - c^2$.

In particular, if we set

$$A = \left\{[c,u,\mathbf{u}] \in \mathbb{PR}^n \,:\, \mathbf{u} \in \mathbb{R}^{n-2}, \|\mathbf{u}\| = \sqrt{1-c^2-u^2}\right\} \subset \mathscr{C},$$

and

$$B = \left\{[c,-u,\mathbf{u}] \in \mathbb{PR}^n \,:\, \mathbf{u} \in \mathbb{R}^{n-2}, \|\mathbf{u}\| = \sqrt{1-c^2-u^2}\right\} \subset \mathscr{C},$$

then because ψ acts like the identity on A, it acts like the identity on B as well.

Now fix t, for which (17) holds (one can verify that then $c^2 + t^2 < 1$) and introduce the set

$$D = \left\{[c,t,\mathbf{t}] \in \mathbb{PR}^n \,:\, \mathbf{t} \in \mathbb{R}^{n-2}, \|\mathbf{t}\| = \sqrt{1-c^2-t^2}\right\} \subset \mathscr{C}.$$

Our aim is to show that ψ maps every element from D into itself. This follows easily once we prove that for every $[c,t,\mathbf{t}] \in D$ we have

$$\left([c,t,\mathbf{t}]^c \cap (A \cup B)\right)^c \cap \mathscr{C} = \{[c,t,\mathbf{t}]\}. \tag{21}$$

Clearly, $[c,t,\mathbf{t}] \in \left([c,t,\mathbf{t}]^c \cap (A\cup B)\right)^c \cap \mathscr{C}$.

To prove the other inclusion we first show that both $[c,t,\mathbf{t}]^c \cap A$ and $[c,t,\mathbf{t}]^c \cap B$ are infinite sets. Let $[c,\pm u,\mathbf{u}] \in A \cup B$. Then, by (19), $[c,\pm u,\mathbf{u}] \in [c,t,\mathbf{t}]^c$ if and only if

$$\langle \mathbf{t},\mathbf{u}\rangle = c - c^2 \mp tu. \tag{22}$$

All we need to verify is that

$$\left|c-c^2\mp tu\right| < \|\mathbf{t}\|\,\|\mathbf{u}\| = \sqrt{(1-c^2-t^2)\,(1-c^2-u^2)}.$$

By squaring this inequality we get

$$\left(1-c^2\right)t^2 \mp 2c(1-c)ut + \left(1-c^2\right)u^2 - (1-c)^2(1+2c) < 0,$$

which is (after dividing by $1-c$) equivalent to

$$\frac{\pm cu-\sqrt{(1+2c)\,(1-c^2-u^2)}}{1+c} < t < \frac{\pm cu+\sqrt{(1+2c)\,(1-c^2-u^2)}}{1+c}.$$

This is indeed true as the left-hand side is always negative (see (18)), and the second inequality follows from (17).

Let further

$$[c,z,\mathbf{z}] \in \left([c,t,\mathbf{t}]^c \cap (A\cup B)\right)^c = \left([c,t,\mathbf{t}]^c \cap A\right)^c \cap \left([c,t,\mathbf{t}]^c \cap B\right)^c,$$

where $|z| \leqslant \sqrt{1-c^2}$ and $\|\mathbf{z}\| = \sqrt{1-c^2-z^2}$. It follows from $[c,z,\mathbf{z}] \in \left([c,t,\mathbf{t}]^c \cap A\right)^c$ and (22) that for every $\mathbf{u} \in \mathbb{R}^{n-2}$ of norm $\sqrt{1-c^2-u^2}$, for which

$$\langle \mathbf{t},\mathbf{u}\rangle = c-c^2-tu,$$

we have

$$\langle \mathbf{z},\mathbf{u}\rangle = c-c^2-zu. \tag{23}$$

Applying Lemma 4.5 we conclude that $\mathbf{z} = a\mathbf{t}$ for some $a\in\mathbb{R}$.

But since $[c,z,\mathbf{z}] \in \left([c,t,\mathbf{t}]^c \cap B\right)^c$ as well, it follows that

$$\langle \mathbf{z},\mathbf{w}\rangle = c-c^2+zu \tag{24}$$

for every $\mathbf{w} \in \mathbb{R}^{n-2}$ with $\|\mathbf{w}\| = \sqrt{1-c^2-u^2}$ and

$$\langle \mathbf{t},\mathbf{w}\rangle = c-c^2+tu.$$

The equation $\mathbf{z} = a\mathbf{t}$ with (23) and (24) yield

$$a\left(c-c^2-tu\right) = c-c^2-zu$$

and

$$a\left(c-c^2+tu\right) = c-c^2+zu.$$

Hence, $a=1$ and $z=t$, and the proof is completed. $\square$

LEMMA 4.8. *Let n be an integer larger than 3, $\frac{1}{\sqrt{2}} < c < 1$ and $\psi : \mathbb{PR}^n \to \mathbb{PR}^n$ a bijective map such that the following conditions hold:*

- $\{[x],[y]\} = c \iff \{\psi([x]),\psi([y])\} = c,$

- $\psi([1,\mathbf{0}]) = [1,\mathbf{0}]$,
- $\psi([c,\mathbf{v}]) = [c,\mathbf{v}]$ *for any* $[c,\mathbf{v}] \in \mathscr{C}$.

Then $\psi([x]) = [x]$ *for every* $[x] \in \mathbb{PR}^n$.

Proof. Set $c = \cos\varphi$, $0 < \varphi < \frac{\pi}{4}$. For any natural number $k \geqslant 2$ introduce the set

$$A_k = \left\{[\cos\alpha, \mathbf{x}] \in \mathbb{PR}^n : 0 \leqslant \alpha < k\varphi,\ \mathbf{x} \in \mathbb{R}^{n-1},\ \|\mathbf{x}\| = |\sin\alpha|\right\}.$$

Note that if $k\varphi > \frac{\pi}{2}$, then $A_k = \mathbb{PR}^n$. We will show by induction that ϕ acts like the identity on A_k for every $k \geqslant 2$.

Start with $k = 2$. Choose an arbitrary element $[\cos\alpha, \mathbf{x}]$ of A_2, different from $[1,\mathbf{0}]$, that is $0 < \alpha < 2\varphi$. The induction basis will be proven once we show that

$$\left([\cos\alpha, \mathbf{x}]^{\cos\varphi} \cap \mathscr{C}\right)^{\cos\varphi} = \{[\cos\alpha, \mathbf{x}], [1,\mathbf{0}]\}.$$

Show first that $[\cos\alpha, \mathbf{x}]^{\cos\varphi} \cap \mathscr{C}$ is an infinite set. To this end take $[\cos\varphi, \mathbf{v}] \in \mathscr{C}$. Then $[\cos\varphi, \mathbf{v}] \in [\cos\alpha, \mathbf{x}]^{\cos\varphi}$ if and only if

$$\cos\varphi\cos\alpha + \langle \mathbf{v}, \mathbf{x}\rangle = \pm\cos\varphi. \tag{25}$$

Because $-\varphi < \alpha - \varphi < \varphi$, we have $\cos\varphi < \cos(\alpha - \varphi)$, or equivalently $\cos\varphi - \cos\varphi\cos\alpha < \sin\varphi\sin\alpha = \|\mathbf{v}\|\,\|\mathbf{x}\|$. We proved in particular that the set

$$V = \{\mathbf{v} \in \mathbb{R}^{n-1} : \|\mathbf{v}\| = \sin\varphi,\ \langle \mathbf{v}, \mathbf{x}\rangle = \cos\varphi(1 - \cos\alpha)\}$$

is infinite.

It is clear that the set $\left([\cos\alpha, \mathbf{x}]^{\cos\varphi} \cap \mathscr{C}\right)^{\cos\varphi}$ contains $[\cos\alpha, \mathbf{x}]$ and $[1,\mathbf{0}]$. Let now $[\cos\beta, \mathbf{z}] \in \left([\cos\alpha, \mathbf{x}]^{\cos\varphi} \cap \mathscr{C}\right)^{\cos\varphi}$, where $0 < \beta \leqslant \frac{\pi}{2}$ and $\mathbf{z}$ is a vector from $\mathbb{R}^{n-1}$ of norm $\sin\beta$. Then $\cos\varphi\cos\beta + \langle \mathbf{v}, \mathbf{z}\rangle = \pm\cos\varphi$ for any $\mathbf{v} \in V$. But actually, there must be $+$ on the right-hand side of this equation. Otherwise we would have

$$|\langle \mathbf{v}, \mathbf{z}\rangle| = \cos\varphi(1 + \cos\beta) \geqslant \cos\varphi$$

and since $0 < \varphi < \frac{\pi}{4}$,

$$\cos\varphi > \sin\varphi \geqslant \sin\varphi\sin\beta = \|\mathbf{v}\|\,\|\mathbf{z}\|,$$

a contradiction. Hence,

$$\langle \mathbf{v}, \mathbf{z}\rangle = \cos\varphi(1 - \cos\beta), \quad \mathbf{v} \in V. \tag{26}$$

By Lemma 4.5 there exists $a \in \mathbb{R}$ such that $\mathbf{z} = a\mathbf{x}$. The equation (26) now implies that

$$a(1 - \cos\alpha) = 1 - \cos\beta,$$

which together with $\sin\beta = \|\mathbf{z}\| = |a|\|\mathbf{x}\| = a\sin\alpha$ yields $\beta = \alpha$ and $a = 1$.

We continue with the induction step. Let $k \geqslant 2$. We may assume that $k\varphi \leqslant \frac{\pi}{2}$ (otherwise $A_k = A_{k+1} = \mathbb{PR}^n$). Suppose that $\psi([x]) = [x]$ for any $[x] \in A_k$. Our aim is to show that $\psi([y]) = [y]$ for every $[y] \in A_{k+1}$.

Let $[\cos\alpha, \mathbf{x}] \in A_{k+1} \setminus A_k$, that is $k\varphi \leqslant \alpha < (k+1)\varphi$. In order to complete the proof we need to show that

$$([\cos\alpha, \mathbf{x}]^{\cos\varphi} \cap A_k)^{\cos\varphi} = \{[\cos\alpha, \mathbf{x}]\}.$$

As before we first show that $[\cos\alpha, \mathbf{x}]^{\cos\varphi} \cap A_k$ is an infinite set. Introduce the sets

$$Z' = \left\{ \begin{pmatrix} \cos\gamma \\ \mathbf{y} \end{pmatrix} \in S^{n-1} \,:\, 0 \leqslant \gamma < k\varphi,\ \left\langle \begin{pmatrix} \cos\gamma \\ \mathbf{y} \end{pmatrix}, \begin{pmatrix} \cos\alpha \\ \mathbf{x} \end{pmatrix} \right\rangle = \cos\varphi \right\},$$

$$Z = \left\{ \begin{pmatrix} \cos\gamma \\ \mathbf{y} \end{pmatrix} \in Z' \,:\, \gamma \geqslant \alpha - \varphi \right\}.$$

We claim that $Z = Z'$. Note that for $\begin{pmatrix} \cos\gamma \\ \mathbf{y} \end{pmatrix} \in Z'$ we have

$$0 = k\varphi - k\varphi < \alpha - \gamma \leqslant \alpha < 2k\varphi \leqslant \pi. \tag{27}$$

From

$$\cos\varphi - \cos\alpha\cos\gamma = \langle \mathbf{x}, \mathbf{y} \rangle \leqslant \|\mathbf{x}\|\,\|\mathbf{y}\| = \sin\alpha\sin\gamma$$

we get $\cos\varphi \leqslant \cos(\alpha-\gamma)$ and because (27) yields that $0 < \varphi, \alpha-\gamma < \pi$, we have $\gamma \geqslant \alpha - \varphi$. Hence, $Z' = Z$. We are now ready to check that Z is infinite. Let $\alpha - \varphi < \gamma < k\varphi$ and set

$$\begin{aligned} Y_\gamma &= \left\{ \mathbf{y} \in \mathbb{R}^{n-1} \,:\, \begin{pmatrix} \cos\gamma \\ \mathbf{y} \end{pmatrix} \in Z \right\} \\ &= \left\{ \mathbf{y} \in \mathbb{R}^{n-1} \,:\, \|\mathbf{y}\| = \sin\gamma,\ \langle \mathbf{x}, \mathbf{y} \rangle = \cos\varphi - \cos\alpha\cos\gamma \right\}. \end{aligned}$$

We have $\cos\varphi < \cos(\alpha-\gamma)$ and therefore, $\cos\varphi - \cos\alpha\cos\gamma < \sin\alpha\sin\gamma$. Thus, each Y_γ is infinite. Consequently, Z is infinite and therefore $[\cos\alpha, \mathbf{x}]^{\cos\varphi} \cap A_k$ is infinite as well.

Let $[\cos\beta, \mathbf{z}] \in ([\cos\alpha, \mathbf{x}]^{\cos\varphi} \cap A_k)^{\cos\varphi}$, where $0 \leqslant \beta \leqslant \pi$, and $\|\mathbf{z}\| = \sin\beta$. Then

$$\left\langle \begin{pmatrix} \cos\gamma \\ \mathbf{y} \end{pmatrix}, \begin{pmatrix} \cos\beta \\ \mathbf{z} \end{pmatrix} \right\rangle = \pm\cos\varphi \tag{28}$$

for every $\begin{pmatrix} \cos\gamma \\ \mathbf{y} \end{pmatrix} \in Z$. Define the map $f : Z \to \{\cos\varphi, -\cos\varphi\}$ by

$$f\begin{pmatrix} \cos\gamma \\ \mathbf{y} \end{pmatrix} = \left\langle \begin{pmatrix} \cos\gamma \\ \mathbf{y} \end{pmatrix}, \begin{pmatrix} \cos\beta \\ \mathbf{z} \end{pmatrix} \right\rangle.$$

Observe that $Z' = Z$ is the intersection of S^{n-1}, $(\cos(k\varphi), \infty) \times \mathbb{R}^{n-1}$ and an $(n-1)$-dimensional affine subspace in $\mathbb{R}^n$, hence it is connected. Since f is continuous, f

must be constant. In particular, if we fix $\gamma_0 \in (\alpha - \varphi, k\varphi)$, then the expression $\langle \mathbf{y}, \mathbf{z} \rangle$ is constant, when we vary $\mathbf{y} \in Y_{\gamma_0}$. By Lemma 4.5 we have $\mathbf{z} = a\mathbf{x}$ for some $a \in \mathbb{R}$.

Let again γ be an arbitrary element of $(\alpha - \varphi, k\varphi)$. Then (28) yields that

$$\cos\gamma(\cos\beta - a\cos\alpha) = (\pm 1 - a)\cos\varphi$$

is independent of γ. Hence, $\cos\beta = a\cos\alpha$ and $a = \pm 1$, so $[\cos\beta, \mathbf{z}] = [\cos\alpha, \mathbf{x}]$. The proof is completed. $\square$

We are now ready to prove the main result of this section.

THEOREM 4.9. *Let n be an integer larger than 4, $\mathbb{F} = \mathbb{R}$, $\frac{1}{\sqrt{2}} \leqslant c < 1$ and $\phi : \mathscr{P}_n^1 \to \mathscr{P}_n^1$ a map. Then the following two statements are equivalent:*

- *ϕ is bijective and for every pair $P, Q \in \mathscr{P}_n^1$ we have*

$$\operatorname{tr}(PQ) = c^2 \iff \operatorname{tr}(\phi(P)\phi(Q)) = c^2,$$

- *there exists an orthogonal transformation $O : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\phi(P) = OPO^t$$

  for any $P \in \mathscr{P}_n^1$.

Proof. The second condition trivially implies the first one, so we will prove only the other direction. Again, we will deal with the map $\psi : \mathbb{PR}^n \to \mathbb{PR}^n$ induced by ϕ.

First discuss the case when $c = \frac{1}{\sqrt{2}}$. Then Corollary 4.3 tells that

$$\{[x], [y]\} = 0 \iff \{\psi([x]), \psi([y])\} = 0, \quad [x], [y] \in \mathbb{PR}^n.$$

The conclusion now follows from Uhlhorn's theorem. Continue with the more interesting case, that is $c > \frac{1}{\sqrt{2}}$.

We may assume with no loss of generality that

$$\psi([1,0,0,\ldots,0]) = [1,0,0,\ldots,0]. \tag{29}$$

Indeed, we can achieve this by composing ψ with a suitable map induced by an orthogonal transformation. It follows that $\psi(\mathscr{C}) = \mathscr{C}$.

In the next step we will find scalars c_1, c_2, c_3, c_4 such that ψ maps any element of $\mathbb{PR}^n$ of the form $[c_1, c_2, *]$ into an element of the form $[c_3, c_4, *]$. Set $\psi\big([2c-1, 2\sqrt{c(1-c)}, 0, \ldots, 0]\big) = [a_1, \ldots, a_n]$. There is no loss of generality in assuming that $a_1 \geqslant 0$. Moreover, after composing ψ with yet another map induced by an orthogonal transformation, we may, and we will assume that

$$\psi\left(\left[2c-1, 2\sqrt{c(1-c)}, 0, \ldots, 0\right]\right) = \left[a, \sqrt{1-a^2}, 0, \ldots, 0\right] \tag{30}$$

for some $a \in [0,1)$.

Introduce the set

$$A=\left\{\left[c,\sqrt{c(1-c)},\mathbf{u}\right]\in\mathbb{PR}^n : \mathbf{u}\in\mathbb{R}^{n-2}, \|\mathbf{u}\|=\sqrt{1-c}\right\}\subset\mathscr{C}.$$

Observe that for any $\left[c,\sqrt{c(1-c)},\mathbf{u}\right]\in A$ we have

$$\left\{\left[c,\sqrt{c(1-c)},\mathbf{u}\right],[1,0,\mathbf{0}]\right\}=c$$

and

$$\left\{\left[c,\sqrt{c(1-c)},\mathbf{u}\right],\left[2c-1,2\sqrt{c(1-c)},\mathbf{0}\right]\right\}=c.$$

Applying the first equation together with (15) we see that $\psi\left(\left[c,\sqrt{c(1-c)},\mathbf{u}\right]\right)=[c,b_{\mathbf{u}},\mathbf{v}_{\mathbf{u}}]$ for some $|b_{\mathbf{u}}|\leqslant\sqrt{1-c^2}$ and $\mathbf{v}_{\mathbf{u}}\in\mathbb{R}^{n-2}$ such that $\|\mathbf{v}_{\mathbf{u}}\|=\sqrt{1-c^2-b_{\mathbf{u}}^2}$. The second equation above yields $\left\{[c,b_{\mathbf{u}},\mathbf{v}_{\mathbf{u}}],\left[a,\sqrt{1-a^2},\mathbf{0}\right]\right\}=c$, or equivalently,

$$b_{\mathbf{u}}=c\frac{\pm1-a}{\sqrt{1-a^2}}. \tag{31}$$

From here we get directly

$$b_{\mathbf{u}}^2+c^2=\frac{2c^2}{1\pm a}. \tag{32}$$

Now $c>\frac{1}{\sqrt{2}}$ and $b_{\mathbf{u}}^2+c^2\leqslant 1$ (this follows from the fact that $(c,b_{\mathbf{u}},\mathbf{v}_{\mathbf{u}})^t$ is a vector of norm one) yield together with (32) that $\pm$ is actually $+$, and therefore, (31) tells that $b_{\mathbf{u}}=c\sqrt{\frac{1-a}{1+a}}$ for every $\mathbf{u}\in\mathbb{R}^{n-2}$ with $\|\mathbf{u}\|=\sqrt{1-c}$.

We have thus proved that

$$\psi\left(\left[c,\sqrt{c(1-c)},\mathbf{u}\right]\right)=\left[c,c\sqrt{\frac{1-a}{1+a}},\mathbf{v}_{\mathbf{u}}\right] \tag{33}$$

for any $\mathbf{u}\in\mathbb{R}^{n-2}$ with $\|\mathbf{u}\|=\sqrt{1-c}$. Note that $\|\mathbf{v}_{\mathbf{u}}\|=\sqrt{1-\frac{2c^2}{1+a}}$, which is different from 0. Otherwise we would have $a=2c^2-1$ and consequently,

$$\left\{\left[a,\sqrt{1-a^2},\mathbf{0}\right],[1,0,\mathbf{0}]\right\}=2c^2-1.$$

Then it follows from (29), (30) and Corollary 4.3 that $2c-1=\pm(2c^2-1)$, which is not the case.

Hence, ψ induces a map $\tau: S^{n-3}\to S^{n-3}$ by

$$\tau\left(\frac{1}{\sqrt{1-c}}\mathbf{u}\right)=\frac{1}{\sqrt{1-\frac{2c^2}{1+a}}}\mathbf{v}_{\mathbf{u}}.$$

First observe that τ is bijective. The injectivity is clear since ψ is injective. In order to check the surjectivity let $\mathbf{v}$ be an arbitrary vector from $\mathbb{R}^{n-2}$ of norm $\sqrt{1-\frac{2c^2}{1+a}}$. Since

ψ is surjective and $\psi(\mathscr{C}) = \mathscr{C}$, there exist $b \in \left[-\sqrt{1-c^2}, \sqrt{1-c^2}\right]$ and a vector $\mathbf{u} \in \mathbb{R}^{n-2}$ of norm $\sqrt{1-c^2-b^2}$ such that

$$\psi([c,b,\mathbf{u}]) = \left[c, c\sqrt{\frac{1-a}{1+a}}, \mathbf{v}\right].$$

Now it follows from

$$\left\{\left[c, c\sqrt{\frac{1-a}{1+a}}, \mathbf{v}\right], \left[a, \sqrt{1-a^2}, \mathbf{0}\right]\right\} = |ca + c(1-a)| = c$$

and (30) that $\left\{[c,b,\mathbf{u}], \left[2c-1, 2\sqrt{c(1-c)}, \mathbf{0}\right]\right\} = c$. A direct computation shows that

$$b \in \left\{\sqrt{c(1-c)}, -\sqrt{\frac{c^3}{1-c}}\right\}.$$

It is easy to verify that $\sqrt{\frac{c^3}{1-c}} > \sqrt{1-c^2}$, hence $b = \sqrt{c(1-c)}$. Consequently, τ is surjective.

Next, we will find a bijective map on $\mathbb{PR}^{n-2}$, which preserves orthogonality. The equation (19) yields that $\left\{\left[c, \sqrt{c(1-c)}, \mathbf{u}_1\right], \left[c, \sqrt{c(1-c)}, \mathbf{u}_2\right]\right\} = c$ if and only if $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal. Similarly, for vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ we have that

$$\left\{\left[c, c\sqrt{\frac{1-a}{1+a}}, \mathbf{v}_1\right], \left[c, c\sqrt{\frac{1-a}{1+a}}, \mathbf{v}_2\right]\right\} = c \iff \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = c - \frac{2c^2}{1+a}.$$

Hence,

$$x \perp y \iff \langle \tau(x), \tau(y) \rangle = \frac{c - \frac{2c^2}{1+a}}{1 - \frac{2c^2}{1+a}} = c\frac{1+a-2c}{1+a-2c^2}, \quad x, y \in S^{n-3}.$$

It follows from Lemma 4.4 that $a = 2c-1$ and consequently, τ preserves orthogonality on S^{n-3}. For any $x \in S^{n-3}$ we have $\left([x]^\perp\right)^\perp \cap S^{n-3} = \{x, -x\}$, which implies that if $\tau(x) = y$, then $\tau(-x) = -y$. So, τ induces a bijective map $\sigma : \mathbb{PR}^{n-2} \to \mathbb{PR}^{n-2}$, which preserves orthogonality. By Uhlhorn's theorem there exists an orthogonal transformation $O_1 : \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ such that $\sigma([u]) = [O_1 u]$, $[u] \in \mathbb{PR}^{n-2}$. Applying (33) we see that after composing ψ with a map $[x] \mapsto [(I_2 \oplus O_1^t)x]$ we may assume that

$$\psi\left(\left[c, \sqrt{c(1-c)}, \mathbf{u}\right]\right) = \left[c, \sqrt{c(1-c)}, \pm\mathbf{u}\right], \quad \left[c, \sqrt{c(1-c)}, \mathbf{u}\right] \in A. \tag{34}$$

Observe also that (20) yields that $\psi\left(\left[c, -\sqrt{c(1-c)}, -\mathbf{u}\right]\right) = \left[c, -\sqrt{c(1-c)}, \mp\mathbf{u}\right]$ and

$$\psi\left(\left[c, -\sqrt{c(1-c)}, \mathbf{u}\right]\right) = \left[c, -\sqrt{c(1-c)}, \pm\mathbf{u}\right]$$

(we remark that if vectors $\mathbf{u}$ in the last equation and in (34) coincide, then the sign $\pm$ in those two equations coincide as well). We will show that the sign $\pm$ in these equations is independent of $\mathbf{u}$. When achieving this goal the following set will be helpful:

$$\mathscr{O}=\left\{[c,0,\mathbf{v}]\,:\,\mathbf{v}\in\mathbb{R}^{n-2},\,\|\mathbf{v}\|=\sqrt{1-c^2}\right\}\subset\mathscr{C}.$$

Choose an arbitrary $\mathbf{v}$ from the sphere $S=\left\{\mathbf{v}\in\mathbb{R}^{n-2}\,:\,\|\mathbf{v}\|=\sqrt{1-c^2}\right\}$ and introduce the set

$$U_{\mathbf{v}}=\left\{\mathbf{u}\in\mathbb{R}^{n-2}\,:\,\|\mathbf{u}\|=\sqrt{1-c},\,\langle\mathbf{u},\mathbf{v}\rangle=c(1-c)\right\}.$$

Note that $c(1-c)<\sqrt{(1-c)(1-c^2)}$, hence $U_{\mathbf{v}}$ is infinite. Now fix $\mathbf{u}_0\in U_{\mathbf{v}}$. If we set $\psi([c,0,\mathbf{v}])=[c,w,\mathbf{w}]$, then it follows from

$$\left\{[c,0,\mathbf{v}],\left[c,\sqrt{c(1-c)},\mathbf{u}_0\right]\right\}=\left\{[c,0,\mathbf{v}],\left[c,-\sqrt{c(1-c)},\mathbf{u}_0\right]\right\}=c$$

that

$$\left\{[c,w,\mathbf{w}],\left[c,\sqrt{c(1-c)},\pm\mathbf{u}_0\right]\right\}=\left\{[c,w,\mathbf{w}],\left[c,-\sqrt{c(1-c)},\pm\mathbf{u}_0\right]\right\}=c,$$

which yields that $w=0$. This, together with (34) implies that for any $\mathbf{u}\in U_{\mathbf{v}}$ we have

$$\left\{[c,0,\mathbf{w}],\left[c,\sqrt{c(1-c)},\pm\mathbf{u}\right]\right\}=c,$$

or equivalently

$$\langle\mathbf{w},\mathbf{u}\rangle=\pm c(1-c). \tag{35}$$

Since $U_{\mathbf{v}}$ is an intersection of a sphere and an $(n-3)$-dimensional affine subspace in $\mathbb{R}^{n-2}$, it is connected. So, the expression $\langle\mathbf{w},\mathbf{u}\rangle$ must be constant, when we vary $\mathbf{u}\in U_{\mathbf{v}}$. This means that the sign $\pm$ in (34) is independent of $\mathbf{u}\in U_{\mathbf{v}}$. Observe that the collection $\{U_{\mathbf{v}}\}_{\mathbf{v}\in S}$ covers the sphere $\{\mathbf{u}\in\mathbb{R}^{n-2}\,:\,\|\mathbf{u}\|=\sqrt{1-c}\}$. Thus, the desired conclusion that the sign $\pm$ in (34) is constant will follow once we show that for any other $\mathbf{v}'\in S$ the vectors $\mathbf{u}$ from $U_{\mathbf{v}'}$ give us the same sign as the vectors from $U_{\mathbf{v}}$. This is indeed true, because one can find $\mathbf{z}\in S$ such that $U_{\mathbf{v}}\cap U_{\mathbf{z}}\neq\emptyset$ and $U_{\mathbf{v}'}\cap U_{\mathbf{z}}\neq\emptyset$ (that holds for any $\mathbf{z}$, which is orthogonal to both $\mathbf{v}$ and $\mathbf{v}'$). Finally, we may assume that ψ maps every element of A into itself (otherwise we multiply O_1 by -1).

Our next goal is to show that ψ acts like the identity on the set $\mathscr{C}$. We will reach this goal in a few steps. Introduce the set

$$B=\left\{[c,t,\mathbf{t}]\in\mathscr{C}\,:\,0\leqslant t<\frac{\sqrt{1-c}}{1+c}\left(\sqrt{1+2c}-c\sqrt{c}\right)\right\}.$$

We will show that ψ maps every element of B into itself. By Lemma 4.6 (where we choose $u=\sqrt{c(1-c)}$) we see that it is enough to prove that $\sqrt{c(1-c)}<\sqrt{\frac{(1+2c)(1-c)}{1+c}}$, which is easy to check.

In the next step we show that ψ is the identity on the set

$$E=\left\{\left[c,c\sqrt{\frac{1-c}{1+c}},\mathbf{w}\right]\in\mathbb{PR}^n\,:\,\mathbf{w}\in\mathbb{R}^{n-2},\|\mathbf{w}\|=\sqrt{1-\frac{2c^2}{1+c}}\right\}\subset\mathscr{C}.$$

Let $t\in\mathbb{R}$ be arbitrary such that $0\leqslant t<\frac{\sqrt{1-c}}{1+c}\left(\sqrt{1+2c}-c\sqrt{c}\right)$ and $0<t<\sqrt{\frac{(1+2c)(1-c)}{1+c}}$. Since we will again use Lemma 4.6, it remains to show that we can choose t such that it also satisfies the following inequality

$$c\sqrt{\frac{1-c}{1+c}}<\frac{\sqrt{(1+2c)\left(1-c^2-t^2\right)}-ct}{1+c}.$$

This can be done, since for the continuous function $f:\left[0,\sqrt{1-c^2}\right]\to\mathbb{R}$, given by $f(t)=\frac{\sqrt{(1+2c)\left(1-c^2-t^2\right)}-ct}{1+c}$, we have

$$f(0)=\frac{\sqrt{(1+2c)\left(1-c^2\right)}}{1+c}=\sqrt{\frac{(1+2c)(1-c)}{1+c}}>c\sqrt{\frac{1-c}{1+c}}.$$

We are now ready to treat the whole set $\mathscr{C}$. First let $[c,v,\mathbf{v}]\in E^c\cap\mathscr{C}$. Then $\langle\mathbf{v},\mathbf{w}\rangle=c-c^2-vc\sqrt{\frac{1-c}{1+c}}$ for every $\mathbf{w}\in\mathbb{R}^{n-2}$ of norm $\sqrt{1-\frac{2c^2}{1+c}}$. After choosing $\mathbf{w}\perp\mathbf{v}$ we conclude that $v=\sqrt{1-c^2}$. Hence, $E^c\cap\mathscr{C}=\left\{\left[c,\sqrt{1-c^2},\mathbf{0}\right]\right\}$ and consequently

$$\psi\left(\left[c,\sqrt{1-c^2},\mathbf{0}\right]\right)=\left[c,\sqrt{1-c^2},\mathbf{0}\right].$$

It follows from (20) that

$$\psi\left(\left[c,-\sqrt{1-c^2},\mathbf{0}\right]\right)=\left[c,-\sqrt{1-c^2},\mathbf{0}\right].$$

In order to prove that ψ acts like the identity on $\mathscr{C}$, we choose an arbitrary $[c,v,\mathbf{v}]\in\mathscr{C}\setminus\left\{\left[c,\sqrt{1-c^2},\mathbf{0}\right],\left[c,-\sqrt{1-c^2},\mathbf{0}\right]\right\}$. By (20) we see that there is no loss in generality in assuming that $v\geqslant 0$. We will show that

$$\left([c,v,\mathbf{v}]^c\cap E\right)^c\cap\mathscr{C}=\left\{[c,v,\mathbf{v}],\left[c,\sqrt{1-c^2},\mathbf{0}\right]\right\}.\tag{36}$$

We start by proving that $[c,v,\mathbf{v}]^c\cap E$ is an infinite set. It consists of the elements $\left[c,c\sqrt{\frac{1-c}{1+c}},\mathbf{w}\right]\in E$, for which

$$\langle\mathbf{w},\mathbf{v}\rangle=c-c^2-vc\sqrt{\frac{1-c}{1+c}}=c\sqrt{\frac{1-c}{1+c}}\left(\sqrt{1-c^2}-v\right).\tag{37}$$

Note that $v\geqslant 0$ and $\sqrt{1-c^2}-v>0$. Clearly,

$$c\sqrt{\frac{1-c}{1+c}}\left(\sqrt{1-c^2}-v\right)<\sqrt{\frac{(1+2c)(1-c)}{1+c}\left(\sqrt{1-c^2}-v\right)\left(\sqrt{1-c^2}+v\right)}$$

$$= \sqrt{\left(1 - \frac{2c^2}{1+c}\right)(1 - c^2 - v^2)} = \|\mathbf{w}\| \, \|\mathbf{v}\|.$$

Hence, there exist infinitely many vectors $\mathbf{w}$, satisfying (37).

Let $[c, z, \mathbf{z}] \in ([c, v, \mathbf{v}]^c \cap E)^c \cap \mathscr{C}$. Then

$$\langle \mathbf{w}, \mathbf{z} \rangle = c\sqrt{\frac{1-c}{1+c}}\left(\sqrt{1-c^2} - z\right) \tag{38}$$

for every $\mathbf{w} \in \mathbb{R}^{n-2}$ of norm $\sqrt{1 - \frac{2c^2}{1+c}}$, for which (37) holds. Now Lemma 4.5 tells that $\mathbf{z} = a\mathbf{v}$ for some $a \in \mathbb{R}$. The equations (37) and (38) yield

$$\sqrt{1-c^2} - z = a\left(\sqrt{1-c^2} - v\right)$$

and consequently $a \geqslant 0$. Thus, $a = \frac{\|\mathbf{z}\|}{\|\mathbf{v}\|} = \sqrt{\frac{1-c^2-z^2}{1-c^2-v^2}}$, which implies that

$$\sqrt{1-c^2} - z = \sqrt{\frac{\left(\sqrt{1-c^2} - z\right)\left(\sqrt{1-c^2} + z\right)\left(\sqrt{1-c^2} - v\right)}{\sqrt{1-c^2} + v}},$$

from where we easily get that $z = \sqrt{1-c^2}$ or $z = v$. In the first case we get $a = 0$, while in the other $a = 1$.

We proved (36), hence $\psi([c, \mathbf{v}]) = [c, \mathbf{v}]$ for every $[c, \mathbf{v}] \in \mathscr{C}$. Now we use Lemma 4.8 to complete the proof. $\square$

REFERENCES

[1] J. T. CHAN, C. K. LI, AND N. K. SZE, *Mappings on matrices: Invariance of functional values of matrix products*, J. Aust. Math. Soc. **81** (2006), 165–184.

[2] J. T. CHAN, C. K. LI, AND N. K. SZE, *Mappings preserving spectra of products of matrices*, Proc. Amer. Math. Soc. **135** (2007), 977–986.

[3] S. CLARK, C. K. LI, AND L. RODMAN, *Spectral radius preservers of products of nonnegative matrices*, Banach J. Math. Anal. **2** (2008), 107–120.

[4] M. DOBOVIŠEK, B. KUZMA, G. LEŠNJAK, C. K. LI, AND T. PETEK, *Mappings that preserve pairs of operators with zero triple Jordan product*, Linear Algebra Appl. **426** (2007), 255–279.

[5] V. FORSTALL, A. HERMAN, C. K. LI, N. S. SZE, AND V. YANNELLO, *Preservers of eigenvalue inclusion sets of matrix products*, Linear Algebra Appl. **434** (2011), 285–293.

[6] J. HARTMAN, A. HERMAN, AND C. K. LI, *Preservers of eigenvalue inclusion sets*, Linear Algebra Appl. **433** (2010), 1038–1051.

[7] J. C. HOU, C. K. LI, AND N. C. WONG, *Jordan isomorphisms and maps preserving spectra of certain operator products*, Studia Math. **184** (2008), 31–47.

[8] J. C. HOU, C. K. LI, AND N. C. WONG, *Maps preserving the spectrum of generalized Jordan product of operators*, Linear Algebra Appl. **432** (2010), 1049–1069.

[9] C. K. LI, E. POON, AND N. S. SZE, *Preservers for norms of Lie product*, Operators and Matrices **3** (2009), 187–203.

[10] C. K. LI AND L. RODMAN, *Preservers of spectral radius, numerical radius, or spectral norm of the sum on nonnegative matrices*, Linear Algebra Appl. **430** (2009), 1739–1761.

[11] C. K. LI, P. ŠEMRL, AND N. K. SZE, *Maps preserving the nilpotency of products of operators*, Linear Algebra Appl. **424** (2007), 222–239.

[12] L. MOLNÁR, *Selected preserver problems on algebraic structures of linear operators and on function spaces*, Springer-Verlag, Berlin, Heidelberg, 2007.
[13] U. UHLHORN, *Representation of symmetry transformations in quantum mechanics*, Ark. Fysik **23** (1963), 307–340.
[14] Z.-X. WAN, *Geometry of matrices*, World Scientific, Singapore, New Jersey, London, Hong Kong, 1996.

(Received November 5, 2010)

Chi-Kwong Li
Department of Mathematics
College of William and Mary
Williamsburg, Virginia 23187-8795
USA
e-mail: ckli@math.wm.edu

Lucijan Plevnik
Institute of Mathematics, Physics, and Mechanics
Jadranska 19
SI-1000 Ljubljana
Slovenia
e-mail: lucijan.plevnik@fmf.uni-lj.si

Peter Šemrl
Faculty of Mathematics and Physics
University of Ljubljana
Jadranska 19
SI-1000 Ljubljana
Slovenia
e-mail: peter.semrl@fmf.uni-lj.si